PAGES MISSING
IN THE BOOK

AMERICAN
MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS

Volume XXXII

AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS
VOLUME XXXII

TOPOLOGY OF MANIFOLDS

BY
RAYMOND LOUIS WILDER

PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
190 Hope Street, Providence, Rhode Island
1949

1963 EDITION

LIBRARY OF CONGRESS CATALOG CARD NUMBER 49–6722

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA
1963

PREFACE

The historical background of this work is sketched in Chapter I, section and need not be repeated here. It should, however, be complemented by cert remarks of a more personal nature, particularly as regards the author's indebt ness to his mathematical colleagues.

It has become more or less apparent to students of cultural evolution that genesis of a line of thought cannot be fixed either in chronological fashion bibliographically. If proper evidence were on record, an idea which seems emanate at a fixed date or in a particular work would be found upon anal to be only the end product of a collection of prior ideas; the "originator" of idea being only the medium through which these latter ideas achieve th synthesis. Even the particular individuality of the "originator" is proba not of paramount importance; of importance is the perennial presence of "creative" mind, ready to receive the stimuli. Can anyone doubt that calculus would have evolved even though Leibnitz and Newton had taken farming instead of science? Simultaneous announcement of "discoveries" contemporaries, often widely separated, is not a rare occurrence.

It is fitting, then, for an author to attempt to place his work in its pro setting amongst past and contemporary influences. This is the object of historical remarks in Chapter I. But these formal remarks only partially fil the picture. On the more personal side, I wish to express my indebtedness Professor R. L. Moore, under whose tutelage I received a thorough grounc in point set theory. It was during my early contacts with him that I came realize the vacuum in our knowledge of the set-theoretic structure of the n- particularly the lack of a topological characterization. Later, through pers contacts with Professor Paul Alexandroff in 1928, I became convinced (a viction which he obviously shared) that the problem of the n-cell demanded tools, especially the extension to general spaces of the theory of connecti (homology). Acknowledgements are also due to Professor Eduard Čech (wl theory of general homology is used herein), who visited the United States in 1 35 and from whom I gained much stimulation and personal encouragement am also grateful to the Institute for Advanced Study for making possible a ye uninterrupted research in 1933–34, during which the present investigation manifolds were initiated; and to the John Simon Guggenheim Memorial Fou tion for the grant of a fellowship in 1940–41. It was during the latter pe that the euclidean form of many of the results given in Chapters X–XII found.

As regards the end result—the book itself—it cannot be emphasized strongly that what is presented herewith is only a beginning. It is only t properties of manifolds that can be handled by set-theoretic and homo

tools that are developed, and even these are not completely treated. Problems concerning homotopy, mappings of manifolds, applications to the study of group manifolds, etc., are all awaiting attention. But I hope that what is done here will serve as a useful basis for an attack on such problems.

The delay in publishing has been due to several factors. Since my delivery of the Colloquium Lectures on "Topology of Manifolds" at Vassar in September, 1942, in which the general outlines of this work were presented, the major part of a war has been fought, and a teacher in American universities need not be told what the attending demands, and the heavy post-war university enrollment of veterans, have done to the time that can be devoted to research. Also, most of the results in the later chapters, published here for the first time, were worked out with the euclidean n-space as locale. Resetting these in the generalized manifolds required not only revamping of proofs but taking advantage of the parallel advances in algebraic topology. New and more powerful tools were developing, such as the theory of cohomology and chain products, whose incorporation necessitated much revision but which justified themselves by the greater simplicity made possible in proofs. In many cases, proofs involving homology which were long and difficult became much simplified through the device of reverting to cohomology.

It also became apparent that the work would have to be topologically self-containing; the reader could not be expected to have previously read works on point set theory, topology of polyhedrals (combinatorial topology) and the newer algebraic topology. On the other hand, it was not possible to write a complete exposition of all these aspects of topology. The plan finally adopted was to develop the program from its simplest elements to its more complicated stages while simultaneously introducing the tools needed. Starting at first with general spaces, sufficient topological properties are introduced to characterize the basic 1-dimensional configurations (arc, 1-sphere). As a consequence, Chapter I is quite elementary. Some of the Schoenflies results in two dimensions are then given as well as some of the more modern plane point set theory—partly to furnish a natural basis and motive for the n-dimensional case and partly to present a unified treatment which takes advantage of the newer methods.

Algebraic topology is not introduced until needed—some topology of polyhedrals enters incidental to the material on the euclidean n-sphere in Chapter II, the more recent algebraic topology not being introduced until Chapter V. Although the treatment of these topics obviously could not be made in such general and complete fashion as in the companion volume by Lefschetz [L] in this series, enough is given to carry through the later chapters. The discerning reader will see many algebraic problems to be solved. Throughout the later chapters only an algebraic field is used as coefficient group, since, for example, the geometric form of the Alexander-Pontrjagin duality forms an important tool (three coefficient groups are usually involved—one to define the manifold, and one each for the homology theory of a subset M and for the complement of M). However, it is impossible to do more in a work of this size than to sketch in the general

picture; the author hopes that other writers will fill in some of the gaps and bring the picture into sharper focus.

In an Appendix, I have pointed out some unsolved problems. Some of these may have very simple solutions; others (as for instance 1.1) are probably quite difficult. Such well-known (and difficult) classical problems as the classification of manifolds, conditions under which the S^2 in S^3 bounds a 3-cell, etc., are omitted.

References to the bibliography are enclosed in brackets, those involving capital letters such as [V] or [Mo] referring to books on topology, and those involving only lower case letters such as [a], [c] referring to miscellanea, mainly journal articles. Page numbers, etc., may be included, as in [a; 20] referring to page 20 of the article cited. Cross-references to items in the text are generally made by citing chapter and section; thus "V 12.2" refers to Chapter V, section 12.2. When a section number alone occurs, such as "12.2", the reference is to the chapter in which the citation occurs. References to formulae are enclosed in parentheses.

Along with the index of terms, there is included for easy reference an index of symbols. Certain symbols which refer to analogous concepts might easily be confused. The latter remark applies particularly to the symbols for homology and cohomology groups. The problem of symbolizing the various types of these groups which are encountered in the present work, and the corresponding Betti numbers, proved a serious one, and it is questionable if it has been satisfactorily solved!

I am grateful to those who have lent their advice, read some of the chapters or assisted in reading proofs; particularly to Professors Miriam C. Ayer, E. G. Begle, S. Kaplan, P. A. White and Gail S. Young; also to Dr. K. E. Butcher, Dr. E.H. Larguier and Messrs. M. L. Curtis and L. F. Hsieh. Aid in preparation of the manuscript was received from the Alexander Ziwet Fund, administered by the Executive Board of the Rackham School of Graduate Studies of the University of Michigan.

I wish to thank the American Mathematical Society for the honor and privilege of publishing this volume in its Colloquium series.

Ann Arbor, Michigan
December, 1948

TABLE OF CONTENTS

INTRODUCTION TO THE 1963 EDITION

This edition represents primarily a reprinting of the original book published in 1949; however, there have been some corrections made in the text and a list of errata has been added at the end of the book. In addition the NOTES which follow this Introduction have been added to this edition. For calling errors to my attention, as well as for assistance with the Notes, I am indebted to both colleagues and former students.

NOTES TO THE 1963 EDITION

Page 193; 7.2 Theorem. The "$(P, Q)_{n+1}$" condition may be replaced by the weaker condition "$(P, Q, \smile)_{n+1}$" defined on page 327. For a much simpler proof of this theorem see Theorem VI.2 on page 227 of my paper *A certain class of topological properties*, Bull. Amer. Math. Soc., vol. 66 (1960), pp. 205–239.

Chapters VII, VIII. Material in these chapters may be greatly simplified by the use of exact sequences; for example, Theorem 2.19, p. 208 (Mayer-Vietoris sequence); Theorems 3.9, 3.10, p. 215; Theorem 9.1 on p. 241 (Mayer-Vietoris sequence); and Lemmas 6.2, 6.3 on p. 262. Sheaf theory has been applied by F. A. Raymond, A. Borel and others to simplifying and extending dualities to other coefficient domains; see, e.g., A. Borel, *The Poincaré duality in generalized manifolds*, Mich. Math. Jour., vol. 4 (1957), pp. 227–239; F. A. Raymond, *Poincaré duality in homology manifolds*, Dissertation, University of Michigan, 1958; A. Borel and J. C. Moore, *Homology theory for locally compact spaces*, Mich. Math. Jour., vol. 7 (1960), pp. 137–159; and A. Borel, "Seminar on Transformation Groups," Princeton, N. J., Annals of Math. Studies, No. 46, 1960. In connection with Theorem 9.1, p. 269, attention should be called to K. Sitnikov, *The duality law for non-closed sets*, Doklady Akad. Nauk SSSR, vol. 81 (1951), pp. 359–362; and in the same connection, but with reference also to dualities in general, see Frank Raymond, *Local cohomology groups with closed supports*, Math. Zeit., vol. 76 (1961), pp. 31–41.

Page 257. After the proof of 5.8 Lemma, insert:
"As a consequence of Theorem V 18.31, Theorem 1.1, and Lemmas 5.6 and 5.8, we can show

5.8a Lemma. *With P and Q as before,*

$$H_{r+1}(S\colon Q, 0; P, 0) = h^{n-r-1}(S\colon Q, P).$$

(This Lemma is needed in 7.2, for instance)"

Page 316; 1.1 Theorem. This theorem is valid for any orientable n-gcm (See my paper *A certain class of topological properties*, loc. cit., especially Theorem II.5 thereof and the "Remark" following it.) A similar observation holds with regard to the following items in Chapter XI:

Page 319; 1.4 Theorem and 2.2 Lemma
Page 319; 1.5 Theorem
Page 320; 2.1 Theorem
Page 321; 2.3 Theorem
Page 325; 2.19 Theorem (although Theorem V.1 of the paper cited above is more general)

Page 326; 2.20 Corollary

Page 326; 2.21 Corollary (this holds for any orientable closed locally euclidean n-manifold S such that $p_1(S) = 0$, and M need be only 0-lc and have property $(P, Q, \sim)^{n-2}$. See Corollary V.1 of the paper cited above)

Page 326; 2.22 Theorem and Corollaries (see Theorem V.2 of the paper cited above)

Page 329; 3.5 Theorem (see Theorem II.4 of the paper cited above)

Page 339; 5.12 Theorem (valid for D any domain such that $p^{n-1}(D)$ is finite, in an orientable n-gcm; a like remark holds for Corollary 5.13)

Page 340; 5.15 Theorem

Page 340; 5.16 Theorem

Page 343; 5.26 Theorem

Page 344; all Corollaries 5.27, 5.28, 5.29 and 5.31

Page 345; 6.5 Theorem.

Pages 327, 328; replace 3.3 Theorem and 3.4 Theorem, respectively, by Theorems II.1 and II.2 of the paper cited above.

Page 366; 2.14 Theorem. The weaker condition "$(P, Q, \sim)^r$" may be substituted for the lc^r" condition in the hypothesis.

Page 381; Problem 1.1. Solved affirmatively by Mary Ellen Estill, *A primitive dispersion set of the plane*, Duke Math. Jour., vol. 19 (1952), pp. 323–328.

Page 381; Problems 1.2 and 2.3. See M. Łubański, *An example of an absolute neighborhood retract, which is the common boundary of three regions in the 3-dimensional euclidean space*, Fund. Math., vol. 40 (1953), pp. 29–38. (Notice also footnote 2), ibid., concerning an unpublished result of Gruba in 1937.)

Page 381; Problem 2.1. Solved affirmatively for the case where there exists a well-ordered (by inclusion) basis for the open neighborhoods of B. See my paper *Some consequences of a method of proof of J. H. C. Whitehead*, Mich. Math. Jour., vol. 4 (1957), pp. 27–31.

Page 382; Problems 3.1 and 3.2. For solutions for certain types of "homology n-manifolds" over the reals mod 1 or a principal ideal ring, see, respectively, C. T. Yang, *Transformation groups on a homological manifold*, Trans. Amer. Math. Soc., vol. 87 (1958), pp. 261–283; and F. A. Raymond, *Poincaré duality in homology manifolds*, Dissertation, University of Michigan, 1958.

Page 383; Problem 4.6. Solved affirmatively by R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Annals of Math., vol. 56 (1952), pp. 354–362.

CHAPTER I

ELEMENTARY CONCEPTS; CHARACTERIZATIONS OF $\bar{E}^1$ AND S^1

We shall describe in this chapter some elementary types of topological spaces, as well as some of the properties such as connectedness, separation by cut-points, etc., that are needed in characterizations of the simplest types of manifolds. Several modes of recognition of the simplest manifold of all, the 1-sphere, are given in this chapter and the next, partly as an application of the concepts introduced, and partly as a key to subsequent generalizations. We begin with the logical notion of class or collection and the related symbols and operations.

1. **Sets.** We use the term *set* as synonymous with the logical notion of class or collection. That an individual x is an element of a set M we denote by the relation $x \in M$. More generally, that individuals $x, y, \cdots, w$ are elements of M will be denoted by the single relation $x, y, \cdots, w \in M$. We usually denote sets by capital italic letters, their elements by lower case italic letters. When it becomes necessary to employ three levels in the hierarchy of sets, we shall use capital German letters such as $\mathfrak{U}$ to denote the highest level. Identity between sets is denoted by the equality sign "$=$".

If A and B are sets such that every element of A is also an element of B—that is, $x \in A$ implies $x \in B$—then A is called a *subset* of B; this relationship is denoted by $A \subset B$ or $B \supset A$. In particular, the *null set*—the set which has no elements—is denoted by 0 and is a subset of every set. Negations of $\subset$, $\supset$, $\in$ are indicated by $\not\subset$, $\not\supset$, $\notin$ respectively. If $A \subset B$ and there exists x such that $x \in B$ and $x \notin A$, then we call A a *proper subset* of B.

By the *union* (or *join*) of two sets A and B, symbolized $A \cup B$, we mean the set of all x's such that at least one of the relations $x \in A$, $x \in B$ holds. For several sets $A, B, \cdots, N$, we write $A \cup B \cup \cdots \cup N$, meaning the set of all x's such that at least one of the relations $x \in A$, $x \in B$, $\cdots$, $x \in N$ holds. The *difference*, $A - B$, is the set of all x's such that $x \in A$ and $x \notin B$. For example, we may define A to be a proper subset of B by the relations $A \subset B$ and $B - A \neq 0$. If $A \subset B$, then we may call $B - A$ the *complement* of A in B.

The *intersection* ("meet") of two sets A and B, symbolized $A \cap B$, is the "common part" of A and B; that is, the set of all x's such that both relations $x \in A$, $x \in B$ hold. For example, $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$. If $A \cap B = 0$, we say that A and B are *disjoint* sets.

Use of the symbols $\cup$ and $\cap$ instead of the classical $+$ and $\cdot$ is due to the desire to preserve the latter for use in the algebraic portions of topology. It will also be of advantage to introduce the following device for set definitions: If A is the set of all elements x having a certain property P, we may indicate

the fact by writing:

(1) $$A = \{x \mid x \text{ has the property } P\}.$$

We may also find convenient the symbols & for "and", and $\vee$ for "or" (more precisely, "and/or"). For example, $A \cap B = \{x \mid (x \in A)\&(x \in B)\}$; $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$; and $B - A = \{x \mid (x \in B)\&(x \notin A)\}$.

Generally, we use the braces $\{\ \}$ to indicate a collection whose elements are generically denoted by the symbol within the braces. Thus, $A = \{x\}$ means a collection any one of whose elements we may denote by the symbol x. Usually the elements will be indexed in some fashion, the index denoted by a subscript. For example, $A = \{x_n\}$ will usually denote a set of elements which are indexed by the natural numbers $1, 2, \cdots, n, \cdots$. The elements of a set may, and frequently will be, sets themselves. And if $\{A_i\}$ is a collection of sets A_i, $\bigcup_i A_i$, or simply $\bigcup A_i$, will mean the set $\{x \mid x \in A_i \text{ for at least one } i\}$. And by $\bigcap_i A_i$, or $\bigcap A_i$, we denote the set $\{x \mid x \in A_i \text{ for all } i\}$.

We shall have frequent need of combinations like $\bigcup A_i \cup \bigcup B_i$, meaning $(\bigcup A_i) \cup (\bigcup B_i)$, for example; or such as $\bigcup A_i \cap \bigcup B_i$, meaning $(\bigcup A_i) \cap (\bigcup B_i)$. Wherever confusion might result, however, parentheses will be employed.

A set $\neq 0$ is called *nonempty* or *nonvacuous*. If a set has more than one element, it is called *nondegenerate*.

2. Spaces. A *space* is a specialized form of set. In general it is a set in which for each subset A, a set A' of "limit points" has been assigned. The means by which this is accomplished are various and dependent in general upon one's purposes. Usually we want the sets A' to be assigned so as to satisfy a certain minimum set of conditions which may be stated in the form of axioms. In our treatment we prefer to assign to each point x a nonempty collection of certain special subsets, to be called *neighborhoods* of x, *which contain* x (this corresponds to the first axiom of Hausdorff for a topological space; Hausdorff [H; 213]) and in terms of which it is determinate whether x is a "limit point" of any given set or not. For example, in the case (2a) of the cartesian plane, E^2, we usually stipulate that a neighborhood of a point is the set of all points within any circle having that point as center. In assigning neighborhoods for the plane in this manner, we have in mind its special character as a set of points constituting a "plane" in the cartesian sense.

When neighborhoods have been assigned for the points of a set S, we say that an element, or *point*, x of S is a *limit point* of a set $A \subset S$ if for every neighborhood U of x it is true that $(U - x) \cap A \neq 0$. Thus in (2a) above, every point of the set S of all points in the cartesian plane is a limit point of S; and the origin $(0, 0)$ is a limit point of the set, A, of all points of the form $(1/n, 0)$, where n is any natural number. Note that

2.1. *Whether a point x is a limit point of a set A is not in any way dependent upon whether $x \in A$ or $x \notin A$.*

Also we note that it follows immediately from the definition that

2.2. *If x is a limit point of a set A, and M is a set such that $A \subset M$, then x is a limit point of M.*

We shall sometimes symbolize the statement "x is a limit point of M" by "x lp M". A neighborhood of x will often be indicated by symbols such as $U(x)$, $V(x)$, etc.

EXAMPLE (2b). Let R^1 denote the set of all real numbers, and for each $x \in R^1$ such that $x \neq 0$ and each positive number ϵ, let a neighborhood consist of all numbers $y \in R^1$ such that $| x - y | < \epsilon$. And denoting by A the set of all numbers $1/n$, n a natural number, let a neighborhood of 0 be the set of all numbers $y \in R^1$ which are not in A and for which $| y | < \epsilon$. With neighborhoods so defined the set R^1 becomes a space S_b.

Now when we speak of the space R^1 of real numbers, we do not mean the space S_b, since in the former space we always assign neighborhoods for 0 exactly like those assigned in S_b for all numbers except 0. Thus while S_b and R^1 are identical as *sets*, they are different as *spaces*, although they may be said to differ only "at" the one point, 0. For in R^1, 0 is a limit point of the set A, whereas in S_b, 0 is not such a limit point.

Another way of looking at this is the following: Suppose M is a subset of a space S—meaning that M is a subset of the *set* S—and that for $x \in M$ and neighborhood $U(x)$ in S, we let $V(x) = M \cap U(x)$ be a neighborhood of x in M. With neighborhoods defined in this way for M, we call M a *subspace* of S. In the above examples S_b is *not* a subspace of R^1.

It is frequently convenient to replace the given collection, usually called *system*, of neighborhoods of a space, hereafter to be called the *defining system*, by a different system which is equivalent to the defining system in the following sense: Two systems of neighborhoods $\mathfrak{N}_1$ and $\mathfrak{N}_2$ of a space S are called *equivalent* if for every $x \in S$ and $U(x) \in \mathfrak{N}_1(\mathfrak{N}_2)$ there exists a $V(x) \in \mathfrak{N}_2$ $(\mathfrak{N}_1)$ such that $V(x) \subset U(x)$. Thus in Example (2a) the defining system is clearly equivalent to the system obtained by letting each neighborhood of the defining system also be a neighborhood of any point which it contains.

The reader may verify that

2.3 *If $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are equivalent neighborhood systems of a space S, and x lp M in terms of $\mathfrak{N}_1$, then x lp M in terms of $\mathfrak{N}_2$*; conversely, if x lp M in terms of $\mathfrak{N}_1$ implies x lp M in terms of $\mathfrak{N}_2$ and vice versa, then $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are equivalent.

Thus equivalent neighborhood systems give the same special character to a set in the sense that the relations x lp M are identical for the two systems.

Going back to Example (2a) again, the set of points on the x-axis is a subset E^1 of E^2. The neighborhood system obtained by considering E^1 as a subspace of E^2 is equivalent to the neighborhood system obtained by considering E^1 as identical with the space R^1 of real numbers.

It is to be noted that any set S whatsoever can be turned into a space in a

trivial way, namely by letting each point be its own neighborhood. In such a space no point would be a limit point of any set.

3. **Metric spaces.** A simple type of space is that whose character is determined by *distances* between points. If S is a set, then a *distance function* or *metric over* S is a single-valued real function $\rho(x, y)$ which is defined for all $x, y \in S$ and which satisfies the following conditions: (1) $\rho(x, y) = 0$ if and only if x and y are the same element of S; (2) if $x, y, z \in S$, then $\rho(x, y) \leqq \rho(x, z) + \rho(y, z)$. It follows easily that $\rho(x, y)$ is necessarily nonnegative and symmetric. In order to assign neighborhoods in S, for each $x \in S$ and positive number ϵ, let $S(x, \epsilon) = \{y \mid (y \in S)\&[\rho(x, y) < \epsilon]\}$. We call such sets $S(x, \epsilon)$ *spherical neighborhoods*, and they constitute the natural neighborhoods of the metric space S_ρ whose distance function is ρ. The number ϵ is called the "radius" of the neighborhood $S(x, \epsilon)$. In terms of their usual metrics, the euclidean spaces are metric spaces, and we have already, in Example (2a), set up the spherical neighborhoods for the special case of the euclidean plane.

According to the definition of limit point in §2, x lp M in a metric space if for every neighborhood $S(x, \epsilon)$, $[S(x, \epsilon) - x] \cap M \neq 0$. In metric terms, this means that x lp M if for every $\epsilon > 0$ there is a point y of M such that $0 < \rho(x, y) < \epsilon$—that is, there are points of M distinct from x which are "as near to x as we please."

The reader may verify that

3.1 *If M is a metric space, then the neighborhood system obtained by letting each $S(x, \epsilon)$ be a neighborhood of every $y \in S(x, \epsilon)$ is equivalent to the defining system.*

Any set whatsoever can be turned into a metric space by defining the distance function $\rho(x, y) = 1$ for every pair of distinct elements x, y of the set. This is the metric space which is spacially identical with the space mentioned at the end of §2.

Of special importance among the metric spaces are the so-called "complete" spaces:

3.2 Definition. Let a sequence $\{x_n\}$ of points of a metric space S have the property that for arbitrary $\epsilon > 0$ there exists a natural number $n(\epsilon)$ such that for $k > n(\epsilon)$ and $m > n(\epsilon)$, $\rho(x_k, x_m) < \epsilon$; such a sequence is called a *Cauchy sequence* (of points of S). If for a sequence $\{x_n\}$ there exists $x \in S$ such that for arbitrary $\eta > 0$ there exists $n(\eta)$ such that for $n > n(\eta)$, $\rho(x_n, x) < \eta$, then the sequence $\{x_n\}$ is called convergent.[1] Then a metric space in which every Cauchy sequence is convergent is called *complete.*

The reader will recognize the relation to the ordinary Cauchy sequences of real numbers;[2] the space R^1 furnishes an elementary example of a complete space.

[1] Compare III 1.19.

[2] See 5.12 below.

4. **Closed and open subsets of a space.** We pointed out in 2.1 that whether a point x is a limit point of a set M in a space S is not in any way dependent upon whether $x \in M$ or $x \notin M$. Now if M is a subset of a space S such that M contains all its limit points in S, then we call M *closed*. Inasmuch as the notion is a relative one, we may frequently, in order to avoid confusion, speak of M as closed *relative to* S (rel. S) or closed *in* S.

We may also define this notion as follows: With each subset M of a space S is associated its *closure*, $\overline{M}$, which is the set consisting of M and all limit points of M in S. Then a set M is closed if and only if $M = \overline{M}$. In particular, S is itself closed. (Obviously, if S is a subspace of a space T, however, S may fail to be closed rel. T.) The set A of Example (2b) is closed since it has no limit points and consequently $A = \overline{A}$.

If F is a closed point set in a space S, then the complement of F in S is called *open*; that is, a set U is open if and only if $S - U$ is closed. (Here again we may, to avoid confusion, stipulate "rel. S" or "in S".) In particular, since S is closed, the null set 0 is open. Also, since we stipulated in §2 that a set becomes a space only upon the assignment of neighborhoods to *all* its points, we can conclude that the null set is closed, and hence S is open.

4.1 *In every space S, each of the sets S, 0 is both closed and open.* Incidentally, 4.1 emphasizes the fact that as used in topology, the terms "closed" and "open" are not logically disjunctive; "open" does *not* mean "not closed." From the definition of "open" we have

4.2 *In order that $U \subset S$ be open, it is necessary and sufficient that $x \in U$ imply that there exists a neighborhood N of x such that $N \subset U$.*

In 2.2 we stated that if x lp A, and $A \subset M$, then x lp M. From this follows easily that

4.3 *In any space S, if $\{F_\nu\}$ is a collection of closed point sets F_ν, then $\bigcap F_\nu$ is a closed point set.*

And since in any set S, if $\{M_\nu\}$ is a collection of sets M_ν, then $\bigcup (S - M_\nu) = S - \bigcap M_\nu$, we may infer from 4.3 that

4.4 *In any space S, if $\{U_\nu\}$ is a collection of open point sets U_ν, then $\bigcup U_\nu$ is an open point set.*

Now if we had the following axiom,

4.5 *If $x \in S$, and $U(x)$, $V(x)$ are neighborhoods of x in S, then there exists a neighborhood $W(x)$ of x such that $W(x) \subset U(x) \cap V(x)$*, we could prove that

4.6 *The union of a finite number of closed point sets is closed; and the intersection of a finite number of open sets is open.* Axiom 4.5 is the so-called "2nd Hausdorff axiom". Its necessity in proving 4.6 is shown by the following example:

4a Let S be the set of all real numbers, with neighborhoods defined as in Example (2b) except for 0; for 0, let the neighborhoods be the sets $\{x \mid 0 \leqq x < 1/n\}$ and $\{x \mid -1/n < x \leqq 0\}$ for all natural numbers n. Then if $A = \{x \mid x < 0\}$ and $B = \{x \mid 0 < x\}$, 0 is a limit point of $A \cup B$, but not a limit point of either A or B. Consequently A and B are closed, but $A \cup B$ is not closed.

A candidate for a neighborhood system of a space is the set of all open point sets, each open set to be a neighborhood of each of the points that it contains. However, without imposing further requirements, the space defined by the open sets may be different from the original. For example, consider the example (4b) of a space T having only three points a, b, c. Denoting a neighborhood of x by $U(x)$, let $U(a) = a \cup b$, $U(b) = b \cup c$, $U(c) = c \cup a$, these being the only neighborhoods.[3] Then $\bar{a} = c \cup a$, $\bar{b} = a \cup b$, $\bar{c} = b \cup c$; and hence no subset of S consisting of one point is closed, so that no point-pair forms an open set. Also, $\overline{c \cup a} = \overline{a \cup b} = \overline{b \cup c} = a \cup b \cup c$, so that no point-pair is closed and hence no single point constitutes an open set. Thus the only possible open set which can be used as neighborhood is S itself. Consequently if the open sets of S were to be used as neighborhoods, we should have $\bar{a} = a \cup b \cup c$, whereas actually, $\bar{a} = a \cup c$ as originally defined. Thus the space obtained from S by using open sets as neighborhoods is topologically different from the old. Note that both the 1st (see §2) and 2nd Hausdorff axioms are true of S. The following axiom, the 3rd Hausdorff axiom, would allow the use of open sets as neighborhoods without changing the character of the space:

4.7 *If $y \in U(x)$, then there exists $U(y)$ such that $U(y) \subset U(x)$.* Evidently 4.7 implies that every neighborhood is an open set, and

4.8 *Every space which satisfies* 4.7 *has the property that its defining neighborhood system is equivalent to the system of all open sets, each open set being a neighborhood of each point that it contains.*

In view of 4.8, it has become customary, in dealing with any space which satisfies 4.7, to use the system of all open sets as neighborhood system.

An important consequence of 4.7 is:

4.9 *In a space satisfying* 4.7, *the closure of a point set is closed.*

4.10 Frequently, in later chapters, we shall wish to indicate that the closure of a point set B is a subset of a set A; to do this, we write $B \Subset A$, or $A \Supset B$.

5. **Mappings; homeomorphisms.** If S and S' are (the same or different) sets and f is a correspondence which makes correspond to each $x \in S$ a unique

[3]The set which consists of the point a alone should be denoted by a new symbol, such as (a), and the above relations written $U(a) = (a) \cup (b)$, for example. But in accordance with the usual custom, when no confusion might result, we shall abbreviate by using the same symbol for the set whose sole element is x as for the element itself.

element $x' = f(x)$ of S', then we call f a *mapping* of S *into* S'. A mapping f of S into S' may be denoted variously by $f : S \to S'$ or $f(S) \subset S'$. If every element of S' corresponds to some $x \in S$, then we call f a mapping of S *onto* S', and write $f(S) = S'$. If $M' \subset S'$, then by $f^{-1}(M')$ we denote the set $\{x \mid (x \in S)\& [f(x) \in M']\}$. Generally, the same element x' of S' may correspond to any number of elements of S—i.e., although $f(x)$ is unique, the inverse, $f^{-1}(x')$ may not be unique. When $f^{-1}(x')$ is unique, the mapping f is called (1-1).

In practice, S and S' may be spaces, vector spaces, groups, etc., and usually f will have further properties in addition to those mentioned above. In particular, f may be required to "preserve" certain relationships between elements or sets. Thus, if A, B are subsets of S, and r is a binary relationship between sets or elements, of significance for S and S', then we may require of f that for all A, B such that ArB, the relation $f(A)rf(B)$ hold in S'. Suppose, for instance, r is the relationship of nonidentity between elements—$\neq$. Then to require that f preserve $\neq$ (i.e., $x \neq y$ implies $f(x) \neq f(y)$) is another way to impose the (1-1) character upon the mapping.

Of more significance to us, however, is the case where S and S' are spaces and the relationship ArB is that of $x \in \overline{M}$. If we require that $x \in \overline{M}$ imply $f(x) \in \overline{f(M)}$, then we say that the mapping *preserves limits*.

5.1 Definition. *If S and S' are spaces and $f : S \to S'$ a mapping which preserves limits, then f is called continuous.*

5.1a There are many properties which are equivalent to the continuity property defined in 5.1. An obvious one is that *if $x' \in S'$ and $U(x')$ a neighborhood of x' in S', then there exists a neighborhood $V(x)$ of x in S such that $f[V(x)] \subset U(x')$.* Another is that *if U' is an open (closed) subset of S', then $f^{-1}(U')$ is open (closed) in S.* (The equivalence in the latter case requires that S' satisfy 4.7.)

The most familiar example is that of real single-valued functions. Every such function is a mapping $f : R^1 \to R^1$, where R^1 is the space of real numbers. Continuity of such a function in the ordinary sense is equivalent to continuity of the mapping f as defined above.

(5a) Example. In the set R^1 of all real numbers, let $A = \{x \mid (0 < x < 1) \vee (x = -1) \vee (x = 2)\}$; $B = \{x \mid (0 \leqq x < 1) \vee (x = 2)\}$, $C = \{x \mid 0 \leqq x \leqq 1\}$. Then there exist continuous mappings $f(A) = B$, $g(B) = C$.

Along with any class of mappings one may study the associated invariants. One of the most important invariants associated with the class of all continuous mappings of one space into another is that of connectedness:

5.2 Definition. A space S is *connected* if it is not the union of two nonempty sets A and B such that $\overline{A} \cap B = 0 = A \cap \overline{B}$.

5.2a We leave to the reader the proof of the invariance of the connectedness property defined in 5.2. The most familiar and fundamental of the connected spaces is the real number continuum with the usual topology.

For handling spaces that are not connected, the notion of separate sets is very useful: Two non-empty point sets A and B are called *separate* (or *separated*) if $\bar{A} \cap B = 0 = A \cap \bar{B}$. The fact that a set M is the union of two such sets may be expressed by a relation

$$M = A \cup B \text{ separate.} \tag{5.2a}$$

If a space S is not connected, then clearly it is the union of separate sets.

Another invariant of continuous mappings, for which we shall have important use in this chapter, is that of countable compactness:

5.3 Definition. A space S is *countably compact* if every infinite subset of S has a limit point in S.[4]

For example, any closed interval of the real number continuum is countably compact. Such an interval loses its countable compactness, however, if one of its end points is deleted.

But to return to the general mapping f of a space S into a space S': If such a mapping is (1-1), then f is called *bicontinuous* if both f and f^{-1} are continuous.

5.4 A mapping $f : S \rightarrow S'$ which is (1-1) and bicontinuous as well as "onto" is called a *topological mapping* of S onto S', or a *homeomorphism* between S and S'.

5.5 Definition. If between two spaces S and S' there exists a homeomorphism, then S and S' are called *topologically equivalent* or *homeomorphic*.[5]

Properties which are invariant under topological mappings are called *topological invariants*, and the latter of course form a larger class than the invariants of mappings that are merely continuous.

5.6 As a branch of geometry, Topology may be defined as the study of topological invariants of a space.[6]

The reader may verify the following theorem:

5.7 Theorem. *If $f : S \rightarrow S'$ is a* (1-1) *mapping of S onto S', then a necessary and sufficient condition that f be a homeomorphism is that in each of the spaces S, S', the defining system of neighborhoods be equivalent to the system formed by the images of the neighborhoods of the other space.*

We shall have occasion to use the notion of imbedding:

---

[4]The notion defined here is that which Fréchet called "compact". The reason for our use of the qualifying "countably" will be apparent later.

[5]The reader will not confuse "homeomorphic" with the group-theoretic "homomorphic".

[6]Although this definition is perhaps adequate for the scope of the present work, it is certainly no longer valid to confine the meaning of the term "Topology" within the framework of the Klein classification. Indeed, to attempt a formal definition of Topology during the present period of rapid evolution would be as fruitless as to propose a definition of mathematics itself.

5.8 DEFINITION. If A and B are spaces, and a subspace A' of B is homeomorphic with A, then we say variously that *A' is A topologically imbedded in B*, or simply *A' is A imbedded in B; also, that A' is imbedded in B.*

We recall that (§2) as a space, A' has for its defining system of neighborhoods the "overlappings" with A' of neighborhoods of B. On the other hand, A has its own defining system entirely independent of B or A'. The homeomorphism between A' and A may be considered a result of such an equivalence as stated in Theorem 5.7. Frequently the term "imbedded" is implied. For example, it is customary to speak of "an S^1 in S^2," meaning "an S^1 imbedded in S^2."

5.9 DEFINITION. If A and B are spaces, then we say that *A can be imbedded in B* if there exists $A' \subset B$ such that A' is an A imbedded in B.

For example, as pointed out below, if A is the set of points within a square in the plane together with the boundary, and B is an interval of a straight line, then A cannot be imbedded in B. As another example, it is trivial that the identity mapping $f(x) = x$ of a space S into itself is a homeomorphism between S and itself. However, if S has neighborhoods that do not satisfy the 3d Hausdorff axiom (§4.7), and S' is the space obtained by taking open sets in S as neighborhoods, then the identity mapping may not be a homeomorphism—as for instance, if S is the space T of example (4b), §4.6, above where each point is an open set and therefore a neighborhood of itself in S'. As a matter of fact, one can prove the following:

5.10 THEOREM. *If S is any space, and S' is the space obtained from S by using open sets as neighborhoods, then the identity mapping $x = f(x)$ of S onto S' is continuous; and if the neighborhoods of S satisfy the 3d Hausdorff axiom, then the identity mapping is a homeomorphism.*

One might ask, in view of the Bernstein equivalence theorem in transfinite number theory, whether the existence of (1-1) continuous mappings $f(S) = S'$ and $g(S') = S$ implies the existence of a homeomorphism between S and S'.[7] The following example shows the answer is negative: Let A, B and C be the spaces of Example (5a), §5.1. Let S consist of an infinite sequence of disjoint spaces $A_1, A_2, \cdots, A_n, \cdots$ each homeomorphic with A, and an infinite sequence of disjoint spaces $C_1, C_2, \cdots, C_n, \cdots$, each homeomorphic with C. Let $A_i \cap C_j = 0$ for all i, j and let the only neighborhoods of S be those already defined in the A_i's and C_j's. Let S' be formed by a space S_1' homeomorphic with S but with the addition of a set B (as of example (5a)), such that $S_1' \cap B = 0$ and the only neighborhoods are those already defined in S_1' and B.

We define continuous mappings $f(S)$ and $g(S')$ as follows: Denoting the homeomorphs of A_i and C_j under a fixed homeomorphism $h(S) = S_1'$ by A_i'

[7]See Fundamenta Mathematicae, vol. 1 (1920), p. 223, Prob. 1 by W. Sierpinski; and C. Kuratowski, *Solution d'un problème concernent les images continues d'ensembles de points*, ibid., vol. 2 (1921), pp. 158-160.

and C_i' respectively, and noting that $S - A_1$, S_1', $S - C_1$ are homeomorphic, let $f(A_1) = B$ (cf. example (5a)), and let $f(S - A_1) = S_1'$ constitute a homeomorphism; let $g(B) = C_1$, and let $g(S' - B) = g(S_1') = S - C_1$ also constitute a homeomorphism. Then $f(S) = S'$ and $g(S') = S$ are continuous mappings.

However, the sets S and S' are not homeomorphic. For the set $B' = B - b$, where b is the point corresponding to the number 2 in Example (5a), being connected, would have to correspond to one of the sets C_i of S or else to one of the sets homeomorphic to the set of real numbers $0 < x < 1$. Now no two of the sets $A' : 0 < x < 1$; $B' : 0 \leqq x < 1$; $C' : 0 \leqq x \leqq 1$ are homeomorphic. That B' and C' are not homeomorphic follows from the fact that B' is not countably compact and C' is countably compact. An easy way to show that A' and B' are not homeomorphic is to note (1) that A' contains only cut points (see Definition 5.11 below) and B' has one non-cut point, (2) that the property of being a cut point is topologically invariant.

5.11 Definition. If S is connected and $x \in S$ such that $S - x = X \cup Y$ separate, then x is called a *cut point* of S. If $S - x$ is connected, x is called a *non-cut point* of S.

In general, if S is connected and $M \subset S$ is such that $S - M$ is not connected, then M may be said to *disconnect* or *separate* S, or we may say "S is disconnected by the omission of M." (If $S - M = A \cup B$ separate, we may say that M separates S into A and B, and if $a \in A$, $b \in B$, that M separates a and b in S; or, more generally, if $S - M$ is the union of multi-wise separate[8] sets A_ν, we may say that M separates S into the sets A_ν.) Obviously a cut point of a connected space is a point which disconnects the space.

If D is the set of points within a square in the euclidean plane, then no homeomorphism exists between C' and $\bar{D}$ since C' has cut points and $\bar{D}$ has none. In this case, too, continuous mappings $f(C') = D$ (see Theorem III 2.5) and $g(D) = C'$ exist, although neither is (1-1).

A classical example of imbedding is that of the space of real rational fractions in the complete space of real numbers; indeed, by use of the Cauchy-Cantor-Meray-Hausdorff process one may show:

5.12 Theorem. *Every metric space can be imbedded in a complete metric space.*

6. **Historical remarks.** The term "topology" apparently originated in Listing's *Vorstudien zur Topologie*, which was published in 1847.[9] The synonymous terms "Analysis situs", "Geometria situs" were used even earlier; thus, Gauss employed the latter in 1833 in connection with the presentation of his classical "linking integral". (See [A-H, 497-498]). As one might expect, theorems which are clearly topological in nature appear here and there in works which

[8]See § 9.4 below.

[9]Göttingen Studien, pp. 811-875.

are by no means topological. A classical example of this is the Euler polyhedral formula. The reasons for this are not always to be found in an inability to recognize the topological nature of a theorem, but are partially due to the generality and fundamental character of topological ideas.

As is so commonly the case with a new branch of mathematics, one finds the beginnings of topology in the applications to already existing fields of mathematics, as well as to physical theories. In Riemann's investigations of functions which arise from the integration of total differentials, we find the basic ideas of homology theory emerging from the necessity for distinguishing between the various connectivities of the related surfaces.[10] And Poincaré's extensive work, which formed the real basis for the so-called *combinatorial method* in topology, was instigated by his interest in the classification of algebraic surfaces (although he had earlier employed topological ideas in his work in analysis). The first work on linear graphs (1-dimensional complexes) was published by the physicist Kirchhoff, who applied them to the theory of electric circuits; and Tait's fundamental work on the theory of knots was inspired by one of the molecular theories then in vogue in physics.

The *theory of sets* was developed by Cantor with an eye to its use in the clarification and solution of problems in function theory and analysis; it would be difficult to imagine a function theory without such basic topological ideas as are embodied in the classical *covering* theorems, for example. And in the early part of the twentieth century the evolution of the theory of sets into the theory of abstract spaces was motivated by the needs of functional and general analysis.

The rapid development of topology as a self-sufficient branch of mathematics during the first quarter of the twentieth century took place generally along two lines: the *combinatorial* and the *set-theoretic*. The former was the natural development of the ideas of Riemann and Poincaré, and was distinguished by its *finite character*. The basic configuration was not a point set, but a *polyhedral* or *complex* consisting of a finite number of *faces* of various dimensions, and the manner in which the latter were joined together or *incident* to one another determined the complexity of the configuration—the sort of "holes", and how many such it might have, for instance. The situation *locally*, in the neighborhood of a point, was not of special interest, since the local situation was either simply euclidean or a finite combination of simple euclidean surfaces joined at a point to form a singularity. Such a set-up led inevitably to algebraic analysis. The incidence between faces, or *cells* as they were called, could be displayed by finite matrices, whose elements were 0 (to indicate "not incident to"), 1 or -1 (according as the incidence "agreed" with the *orientations* assigned to the cells or not). The *connectivity* of the polyhedral, described by numbers called the

[10] See Weber's edition of Riemann's *Werke*, in which will be found, besides the detailed treatment of the 2-dimensional surfaces, a previously unpublished *Fragment* containing the essential notions concerning "connectivity numbers" which were later published by Betti (who evidently had no knowledge of this Fragment).

Betti numbers or *connectivity numbers*, could be calculated from the numbers of cells and the ranks of these matrices. (See [V], for instance.) The surface, or portion of a surface bounded by closed curves, treated by Riemann, came to be represented by a polynomial in the symbols for the cells, with integral coefficients, called a *chain*, and the boundary of a chain led to the concept of bounding *cycle*, and more generally to that of *cycle*. When a cycle occurred which bounded no chain, the presence of a "hole" was indicated—that is, the surface was not simply connected.

Although the real significance of these algebraic tools was only gradually realized, the ink was not long dry on Poincaré's publications before chains came into use in which the integral coefficients were replaced by integers modulo 2 (Tietze [a], Veblen-Alexander [a]). And although these seemed to be a device to avoid orientation, for cases where the latter was not of significance or perhaps undesirable, it opened the way for the use of other types of coefficients. During the 1920's the integers mod p, where p is any integer, were introduced as coefficients by Alexander [b], and the rational numbers by Lefschetz [a, b]. The latter type of coefficient seemed intuitively difficult to grasp, since the geometrically minded were prone to ask, what can "one-half a cell" mean? Only the full realization that a chain was an algebraic entity, not a geometric object, could remove the difficulty, and although the distinction may now seem trite, it formed one of the "stone walls" which had to be hurdled. This accomplished, the way was open for the introduction of group-theoretic methods; and the study of homology groups through the introduction of chains with coefficients in an optional abelian group led rapidly to an algebraic topology rich in problems unrealized in the older topology of complexes.

A growth parallel to the developments just mentioned was the study of topological properties by the *set-theoretic method*. Here the basic configuration was a point set, and whereas in the combinatorial approach the properties of the configuration in the large were the center of interest, in the set-theoretic approach the local properties—the situation in the neighborhood of a point—were those naturally studied. Ignoring for the moment the work of Schoenflies during the first decade of the century, one of the most fruitful notions of the set-theoretical topology, namely that of *local connectedness*, was introduced during the second decade for the purposes of an easier recognition, and topological analysis, of *continuous curves* (originally defined analytically by C. Jordan [a]). The space-filling curve problem—*can a continuous curve in the plane contain all the points inside a circle?*—had been solved by Peano in 1890, but the satisfactory solution of the really inherent problem—*what can a continuous curve in any dimension look like, and what type of configuration is excluded?*—was not solved until Hahn and Mazurkiewicz independently showed, about 1914, that the notion of continuous curve in separable spaces is identical with that of *locally connected continuum*. With the introduction of this topological characterization, topologists—particularly of the Polish and American

schools—undertook a thorough-going structural study of the continuous curve, which gradually came to be called *Peano space* or *Peano continuum*.

At the same time the various types of *abstract spaces* were coming to be an object of inquiry; the metrization problem is an excellent illustration—*what topological properties characterize the metric spaces?* Here the coverings of a space, which were ultimately to form the connecting link between the local and the large properties of a space, were of fundamental importance—particularly the existence of a denumerable set of neighborhoods equivalent to the defining system. For a resumé of the work on the metrization problem, the reader is referred to Chittenden [a].

The line of thought which eventually showed the way to the merging of these two approaches to topology can be traced back to the investigations of Schoenflies on what we shall call *positional invariants*: *Given two configurations A and B such that B is imbedded in A, what can be said about the relationships existing between A and B as B is subjected to topological transformations within A?* The classic example—one whose antedating of Schoenflies' work incidentally shows the difficulty in pointing to any one mathematical event as the genesis of a line of thought—is the Jordan Curve Theorem: *If A is the euclidean plane and B is homeomorphic to the circle $x^2 + y^2 = 1$ $(A \supset B)$ then the set $\underline{A} - B$ is the union of two connected, separated point sets X and Y such that $\overline{X} \cap \overline{Y} = B$.* Although this theorem appears in Schoenflies' 1908 work *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten* [S], its proof would hardly accord him any special distinction since proofs which precede his work are nearly as numerous as those which succeed it. But Schoenflies not only proved a converse theorem (see Theorem II 5.38 below); he considered many other cases, such as the case where B is a *totally disconnected closed point set,* and the case where B is a *general closed curve*—a configuration which contains the homeomorph of the circle as a special case. And one of his most important contributions was the case where B is a *continuous curve*; in this case he found positional properties, such as *accessibility,* which were sufficient completely to characterize the continuous curve in the plane.

Although this work of Schoenflies did not, in the opinion of the present writer, attract the attention it deserved,[11] it did not escape the attention of Brouwer [a], who wrote some critical material concerning Schoenflies' investigations, and later contributed many fundamental ideas of his own (whose influence, also, was not at first fully felt), such as the notion of linking coefficient. The particular item which we wish to mention in Brouwer's work resulted from his discovery of certain incorrect assumptions made by Schoenflies concerning the closed curve (evidently due to the inexperienced intuition of the pioneer).

[11] A like opinion seems to be held by Professor R. L. Moore; see 463 of [Mo]. One is moved to wonder, for instance, how much of the fumbling use of "continuous curves" which may be encountered in some texts on complex function theory, might have been avoided by a familiarity with Schoenflies' work on plane point sets.

In 1912, Brouwer published a proof [b] of the theorem (lacking in Schoenflies' work) that the property of being a closed curve in the plane is invariant—i.e., *a point set in the plane which is the boundary of all its complementary domains*[12] *will still have this property after any topological transformation, and, furthermore, the number of the domains will remain invariant.* The methods which Brouwer used to prove this theorem contained the germ of the modern application of the combinatorial method to general spaces. The possibility of such an extension was apparently known to Brouwer himself, who communicated his ideas to Vietoris [a; 454 footnote]. The latter, in 1927, published his homology theory of metric spaces, with an application to the study of continuous transformations. Almost simultaneously there appeared works of Alexandroff [c] and Frankl [a] embodying ideas of a similar nature, and it is noteworthy that the latter authors made applications to positional properties.

It is the general intent of the present work to carry on the researches begun by Schoenflies in the positional properties of configurations in euclidean spaces. The topologists of the Polish and American schools, in connection with their investigations of continuous curve spaces—Peano spaces—settled rather conclusively the problems concerning the positional properties of continuous curves in the plane, begun by Schoenflies, as well as the general properties of plane closed point sets.[13] But the case of three and higher dimensions was virtually untouched. A notable exception was the extension of the Jordan Curve Theorem: Denoting the n-dimensional euclidean sphere by S^n, we may say that the Jordan Curve Theorem treats the case of the S^1 topologically imbedded in S^2. Brouwer [c] was the first to prove that *if K is an S^{n-1} topologically imbedded in S^n, then K separates S^n into two connected sets A and B whose common boundary is K.* And in 1922, J. W. Alexander [a], using combinatorial methods in conjunction with certain limiting processes, extended this result, making it a special case of *a general duality relating the Betti numbers* (mod 2) *of a complex topologically imbedded in S^n to the Betti numbers of its complement in S^n.* The unification of the set-theoretic and combinatorial methods in topology could not be far in the offing.

That Schoenflies could foresee this unification, at least to a certain extent, cannot be denied. In discussing the material included in his book [S] he says, ". . . one can easily distinguish two main groups of theorems. . . A first group is formed by the general theorems on point sets; they represent the set-theoretic foundation. A second group is formed by the simple theorems on straight lines, polygons, and polyhedrals, which I assume as given without a closer axiomatic analysis. In these is the conception of form, the intuitively accessible foundation, contained." At the end of Chapter 5 he remarks: "I have chosen the methods of proof in this chapter in such a way that they permit an extension to configurations in space. Nevertheless the proofs are directly applicable only in a certain part. Firstly, in three-space there must be taken into consideration

[12]By *domain* (in any space) is meant an open, connected point set.

[13]For non-closed plane point sets there remain many unsolved problems.

the contrast between curve and surface, and secondly, the connectivity number[14] plays an important role in the theorems and proofs. The connectivity number is an invariant of topology, but without a knowledge of it only a part of the developments of this chapter can be directly extended. Yet one should not consider this a defect in the methods used. For *the consideration of the connectivity number in three-space is unavoidable*; *every method which ignores it would yield only a part of that which is to be proved.*"[15]

Evidently Schoenflies was aware, then, that in discussing properties in the large of configurations in higher dimensions, one cannot avoid the use of some such invariant as the connectivity numbers. And today, as we shall see later, it is not necessary even to consider the polyhedral case in order to introduce these numbers (or, more generally, the homology groups); one can revise the historical development by first introducing them for the general space and then applying them to the polyhedral by considering the latter as a *point set* rather than a complex of cells.[16]

Now inasmuch as it is no longer necessary to have a complex in order to have a homology theory, the problem presents itself of generalizing the topology of euclidean spaces or, more generally, the theory of manifolds, especially with regard to the duality theorems and positional invariants, to a class of more general configurations among the abstract spaces. This problem was attacked by Čech [b] and Lefschetz [c] in 1933. Both proposed generalized manifolds defined axiomatically in terms of their homology properties. About the same time, having succeeded in extending a portion of the Schoenflies program to three-space, especially as regards the converse of the Jordan-Brouwer separation theorem, the present author published in 1934 [n] a characterization, by intrinsic homology properties, of those point sets in n-space which bounded domains having the same sort of smooth properties which the domains bounded by 2-manifolds in 3-space were found to possess. It soon became apparent that, for the compact metric case, the point sets thus characterized were identical with the Čech-Lefschetz manifolds.

The question then arose: Assuming that the generalized manifolds (not necessarily metric) are the natural extensions to abstract spaces of the classical manifolds, would it not be more fitting to extend the Schoenflies program to positional properties in the new manifolds than to limit one's investigations to the euclidean spaces? For the latter, as pointed out in the preface, there has not been found any suitable topological characterization among the abstract spaces, and their topology must be based on analytical considerations foreign

[14]Here Schoenflies was referring to the Riemann connectivity numbers.

[15]The italics are the present author's.

[16]It is nevertheless convenient to *calculate* the Betti numbers of a polyhedral by means of a cellular subdivision, however. But the invariance of these numbers, and of the corresponding homology groups, is more easily proved by use of the fact that the latter are isomorphic with the respective homology groups of the polyhedral considered as a point set, the latter being obviously invariant.

to the spirit of topology. Moreover, a beautiful symmetry might be attained if, for instance, the Jordan-Brouwer separation theorem and its converse concerning the separation of the n-sphere by an $(n - 1)$-manifold could be presented as a special case of a theorem on the separation of the generalized n-manifold by the generalized $(n - 1)$-manifold.

It is to this program—extension of the Schoenflies program to the generalized n-dimensional manifold—that the present work is devoted.

7. Connected spaces. In §5.2 we defined connectedness of a space and cited it as an example of a property invariant under all continuous mappings. Up to now we have used solely the definition of *space* given in §2; namely, *a set S in which to each point there is assigned a nonempty collection of subsets of S containing the point, called neighborhoods of the point, and in terms of which limit points are determined.*

With so general a notion of space one might expect not to be able to proceed far, and we have already stated above certain axioms of Hausdorff that may be used to avoid situations that are intuitively undesirable. However, we shall not, in the present section, use any of the latter unless it is specifically mentioned. That is, all sets mentioned in the present section are assumed to be imbedded in a space S as defined in the preceding paragraph. Thus, when we speak of a connected set M, it may be assumed that M is imbedded in such a space S, and therefore that the limit points of M, etc., are determined by the neighborhoods of S. (M and S may be identical, of course, unless the contrary is specifically stated.)

7.1 THEOREM. *If M is a point set and $M = A \cup B$ separate, then a connected subset of M must be either a subset of A or a subset of B.*

7.2 THEOREM. *If M is a connected point set and L is a point set such that $M \subset L \subset \overline{M}$, then L is connected.*

7.3 THEOREM. *If A and B are connected point sets and $\overline{A} \cap B \neq 0$, then $A \cup B$ is connected.*

Theorem 7.3 is the most frequently used special case of the following theorem:

7.3a THEOREM. *If $A_1, A_2, \cdots, A_\alpha, \cdots$ is a well-ordered collection of connected point sets A_α such that no A_α is separated from the union of those that precede it, then the totality of sets A_α forms a connected set ($= \bigcup A_\alpha$).*

7.3b THEOREM. *The union of any number of connected point sets, no two of which are separated, is connected.*

7.4 THEOREM. *If M is a point set, and $x, y \in M$ implies that there exists $A(x, y) \subset M$ such that $A(x, y)$ is connected, then M is connected.*

7.5 DEFINITION. If $M \subset S$, then a point x such that every neighborhood of x contains at least one point of M and a point of $S - M$ is called a *boundary*

point of M (and of $S - M$). The set of all boundary points of M is called the *boundary* of M; this set will frequently be denoted by $F(M)$ in the sequel. Evidently if M is a neighborhood of a point x, then x is not a boundary point of M.

A boundary point of a set may or may not belong to the set. Thus, if M is $\{x \mid 0 \leqq x < 1\}$ in R^1, then both 0, 1 are boundary points of M. The notion is clearly a relative one; if the set M just defined is considered as a subset of the cartesian plane, then every point of M is a boundary point of M. If $x \in M$, and x is not a boundary point of M, then x is called an *interior point* of M, and the set of all such points may be called the *interior* of M. Evidently from 4.2 we have

7.6 *In order that a set of points should be open, it is necessary and sufficient that it be identical with its interior.*

7.7 *If M is an open set, then the boundary of M is a subset of $S - M$; moreover, it is identical with the set of points $\overline{M} \cap (S - M)$.*

The following theorem will be of frequent use in the sequel:

7.8 THEOREM. *If $A \subset S$, and M is a connected set such that $A \cap M \neq 0 \neq (S - A) \cap M$, then M contains a boundary point of A.*

From the definition 5.2 it follows that in every space S the null set and the sets consisting of single points are connected. That a space consisting of any finite number of points may be connected is shown by spaces modelled after the idea of Example (4b) of §4, which is an example of a connected space consisting of three points.

7.9 EXAMPLE. Let S consist of n points $x_1, \cdots, x_n$, and for each $i < n$ let the only neighborhood of x_i be the set $x_i \cup x_{i+1}$, and the only neighborhood of x_n be $x_1 \cup x_n$. Then S is connected.

As noted in connection with Example (4b), such spaces fail to satisfy the 3rd Hausdorff axiom. However, consider the following:

7.10 EXAMPLE. Let S consist of two points x, y, and let the only neighborhoods of x and y be $x \cup y$ and y, respectively. Then S satisfies the 1st, 2nd and 3rd Hausdorff axioms and is connected.

It is evident, then, that if one is to attain a type of space in which "connected" is to imply something more in accord with the intuitive connotation of the word—and for a nondegenerate connected space to be finite seems intuitively undesirable—one needs more than the three Hausdorff axioms. It will be noted that in each of the above examples there is a point that does not constitute a closed point set. This suggests the following axiom:

7.11 WEAK SEPARATION AXIOM. *Every point constitutes a closed point set.*

One can now prove:

7.12 THEOREM. *In a space satisfying the weak separation axiom, every point*

of a nondegenerate connected set M is a limit point of M; and M contains at least three points.

That a space S can be connected, satisfy 7.11, and yet consist of exactly three points is shown by the following example:

7.13 Example. Let S consist of three points x, y, z, and let neighborhoods be defined as follows: $U_1(x) = x \cup y$; $U_2(x) = x \cup z$; $U_1(y) = y \cup x$; $U_2(y) = y \cup z$; $U_1(z) = z \cup x$; $U_2(z) = z \cup y$.

However, Example 7.13 does not satisfy the 2nd Hausdorff axiom. As a matter of fact one can prove:

7.14 Theorem. *In a space satisfying the 2nd Hausdorff axiom and the weak separation axiom, no finite set of points has a limit point.*

7.15 Theorem. *In a space satisfying the 2nd Hausdorff axiom and the weak separation axiom, every nondegenerate connected point set contains an infinite number of points.*

We do not go further in this direction—for instance, as to whether a connected space might be denumerably infinite. We refer the reader to the paper of Urysohn [b]. It is interesting to note that the study of this problem led Urysohn to his famous lemma, to which we make reference later on (III 1.14). The reader will note that it follows immediately from Theorem 7.8 that in a metric space every nondegenerate connected point set is of at least the cardinal number of the continuum; and that hence, in particular, nondegenerate connected subsets of euclidean spaces are of exactly the cardinal number c.

8. Components; quasi-components.

8.1 Definition. Two points x and y of a space S are called *c-equivalent* (in S) if there exists a connected subset of S which contains them.

8.2 Theorem. *The relation of c-equivalence in a space S is reflexive, symmetric and transitive.*

As a consequence, a space may be decomposed into classes of c-equivalent points:

8.3 Definition. The classes of c-equivalent points of a space S are called *components* of S.

8.4 Theorem. *The components of a space are both closed and connected.*

Intuitively, then, the components of a space are the "largest" connected subsets of the space. In particular, if a space is itself connected, it has only one component, and conversely.

8.5 Definition. Two points x and y of a space S are called *q-equivalent* if there does not exist any decomposition $S = A \cup B$ separate such that $x \in A$,

8.6 THEOREM. *The relation of q-equivalence in a space S is reflexive, symmetric and transitive.*

8.7 DEFINITION. The classes of q-equivalent points of a space S are called *quasi-components* of S.

8.8 THEOREM. *The quasi-components of a space are closed.*

8.9 EXAMPLE. In the cartesian plane, let M consist of all points $(1/n, y)$, $n = 1, 2, 3, \cdots, -1 \leqq y \leqq 1$, together with the points $a = (0, -1)$, $b = (0, 1)$. Then, considering M as a space (subspace of the cartesian plane), a and b are q-equivalent in M but not c-equivalent.

Thus, two points may be q-equivalent without being c-equivalent. That the converse cannot be, follows from Theorem 7.1:

8.10 THEOREM. *A component of a space S lies in a single quasi-component of S.*

Regarding the invariants of the above equivalences, we can state (cf. §§5.1, 5.2):

8.11 THEOREM. *Both c- and q-equivalence are invariant under continuous mappings.*

Of course it does not follow that the components or quasi-components of a space are invariant under continuous mappings, since non-c- and non-q-equivalence are not invariant. However, we have:

8.12 THEOREM. *The components and quasi-components of a space are invariant under topological mappings of the space.*

9. Connected spaces satisfying the 2nd Hausdorff axiom and the weak separation axiom. We saw in §7 that in order that the connectedness of a space should imply (in the nondegenerate case) an infinity of points in the space, it was necessary to restrict the character of the space by imposition of the 2nd Hausdorff axiom and the weak separation axiom. In the present section we assume that all point sets are imbedded in a space satisfying both these axioms.

9.1 THEOREM. *If a point is a limit point of the union of a finite number of point sets, then it is a limit point of at least one of them.*

9.2 COROLLARY. *If $M_1, \cdots, M_n$ are point sets, finite in number, such that M_1 and M_i are separated for $i = 2, \cdots, n$, then M_1 and $\bigcup_{i=2}^{n} M_i$ are separated.*

9.3 COROLLARY. *If the number of components of a point set is finite, then its components are also quasi-components; moreover, if the number of quasi-components is finite, then quasi-components and components are identical.*

9.4 DEFINITION. If $\{M_\nu\}$ is a collection of point sets M_ν such that $M_{\nu'}$

and $M_{\nu''}$ are separated for every pair of different indices ν', ν'', then we say that the sets M_ν are *pairwise separated.* If for every pair $\bigcup_{\nu'} M_\nu$, $\bigcup_{\nu''} M_\nu$ of disjoint unions of these sets, $\bigcup_{\nu'} M_\nu$ and $\bigcup_{\nu''} M_\nu$ are separated, then we say the sets M_ν are *multiwise separated.*

9.5 COROLLARY. *If a point set M has only a finite number n (>1) of components, then for any natural number k such that $1 < k \leqq n$, M is the union of separated sets $M_1, \cdots, M_k$ such that $M_1, \cdots, M_{k-1}$ are arbitrary components of M, and we may write $M = \bigcup_{i=1}^k M_i$ with the understanding that the sets M_i are multiwise separated.*

9.6 COROLLARY. *If the point sets $M_1, \cdots, M_n$, finite in number, are pairwise separated, then they are multiwise separated.*

9.7a THEOREM. *If a point set M has at least n (>1) components, then it is the union of n separated sets.*

For the case of quasi-components, the analogue of Theorem 9.7a is as follows:

9.7b THEOREM. *If $x_1, x_2, \cdots, x_k$ are a finite number of points of a space S, no two of which lie in the same quasi-component of S, then S is the union of pairwise separated sets S_i, $i = 1, 2, \cdots, k$, such that $x_i \in S_i$.*

(True for $k = 2$ by definition of quasi-component, and for general k by mathematical induction.)

REMARK. That Theorem 9.7b fails to hold if "component" is substituted for "quasi-component" is shown by Example 8.9 above.

9.8 THEOREM. *If C is a connected subset of a connected point set M such that $M - C = A \cup B$ separated, then both $C \cup A$ and $C \cup B$ are connected.*

PROOF. Suppose $C \cup A$ is not connected. Then

$$C \cup A = E \cup F \text{ separated.} \tag{9.8a}$$

From Theorem 7.1 we know that $C \subset F$, say. Hence by (9.8a), $E \subset A$, and consequently E and B are separated. Accordingly we may write $M = (C \cup A) \cup B = E \cup F \cup B = E \cup (F \cup B)$ separated (by Corollary 9.2), contradicting the fact that M is connected.

The same type of argument shows that:

9.9 THEOREM. *If C is a connected subset of a connected point set M such that $M - C$ is the union of n (>1) pairwise separated sets M_i, then $M_i \cup C$ is connected for $i = 1, \cdots, n$.*

9.10 THEOREM. *If M is a nondegenerate connected point set, then M is the union of two nondegenerate, proper, connected subsets (which are not necessarily disjoint, however).*[17]

[17]For the existence of nondegenerate connected point sets that are not unions of disjoint nondegenerate, proper connected subsets, see B. Knaster and C. Kuratowski [a; § 5], Wilder [f].

Proof. Let $x \in M$, and consider the set $M - x$. The latter is either (1) connected or (2) not connected. If (2) holds, the proof concludes on the basis of Theorem 9.8 with $C = x$. If (1) holds, let $y \in M - x$. If $M - y$ is not connected, we proceed as in case (2). If $M - y$ is connected, then $M = (M - x) \cup (M - y)$.

9.11 Theorem. *If C is a connected subset of a connected point set M and A is a component of $M - C$, then $M - A$ is connected.*

Proof. Suppose that $M - A$ is not connected. Then

(9.11a) $$M - A = E \cup F \text{ separated.}$$

Since $A \subset M - C$, the sets A and C are disjoint. Accordingly, $C \subset M - A$. Hence by (9.11a), $C \subset E \cup F$ and by Theorem 7.1, $C \subset E$ or $C \subset F$; say

(9.11b) $$C \subset F.$$

From (9.11a, b) we have that $E \cup A \subset M - C$. But by Theorem 9.8, $E \cup A$ is connected. Hence if $x \in E$, $y \in A$, x and y are c-equivalent in $M - C$ and A cannot be a component of $M - C$, contrary to hypothesis.

10. **Spaces irreducibly connected about a subset.** In this section we continue our study of connected spaces (*continuing to assume the 2nd Hausdorff and weak separation axioms*), but with attention fixed upon the additional property of being irreducibly connected about a subset.

10.1 Definition. If S is connected, and A is a subset of S such that no proper connected subset of S contains A, then S is said to be *irreducibly connected about A*. In particular, it is trivial that every connected space is irreducibly connected about itself. An important special case is that where A consists of two points a and b; in this case we frequently say that S is *irreducibly connected from a to b*. The following are examples of sets of the latter type:

10.2 Example. This is the subspace $\bar{E}^1$ of the space R^1 of all real numbers x consisting of all x such that $0 \leqq x \leqq 1$; a, b are 0, 1 respectively.

10.3 Example. In the cartesian plane, $S = (0, 0) \cup \{(x, y) \mid (0 < x \leqq 1/\pi) \& (y = \sin 1/x)\}$; a, b are $(0, 0)$, $(1/\pi, 0)$, respectively.

10.4 Example. In the cartesian plane, E^2, let $S = \{(x, y) \mid (0 \leqq x \leqq 1) \& (0 \leqq y \leqq 1)\}$; but this time we do not consider S as a subspace of the cartesian plane. Rather, we define neighborhoods for S by first ordering its points as follows: If $x, y \in S$ such that in E^2, abscissa $x <$ abscissa y, let $x < y$ in S; if abscissa $x =$ abscissa y and ordinate $x <$ ordinate y in E^2, let $x < y$ in S. The points of S are simply ordered by this definition, and the neighborhoods of S may be taken as open intervals. Let $a = (0, 0)$, $b = (1, 1)$.

10.5 Theorem. *In order that a connected space S should be irreducibly connected about $A \subset S$, it is necessary and sufficient that if $x \in S - A$, then $S - x$*

is the union of two separated sets each of which contains at least one point of A.[18]

The proof of the necessity is based on Theorem 9.8.

As for the sufficiency: Suppose a proper connected subset M of S contains A; let $x \in S - M$. Then by hypothesis, $S - x$ is the union of separated sets C and D each of which contains points of A and hence points of M. This contradicts Theorem 7.1.

By the same method of proof we have:

10.6 THEOREM. (This is the same as Theorem 10.5 but with "x" replaced by "X", where X is a connected subset of $S - A$.)

10.7 COROLLARY. *If S is irreducibly connected about $A \subset S$, then A contains all the non-cut points of S.*

10.8 COROLLARY. *In order that a connected space S should be irreducibly connected from a to b ($a, b \in S$), it is necessary and sufficient that if $x \in S - a - b$, then $S - x = A \cup B$ separated where $a \in A$ and $b \in B$.*

10.9 THEOREM. *In order that a connected space S should be irreducibly connected about $A \subset S$, where A consists of exactly n (>1) points, it is necessary and sufficient that if $x \in S - A$, then $S - x = \bigcup_{i=1}^{k} M_i$ where $2 \leqq k \leqq n$, and the sets M_i are components of $S - x$ such that $M_i \cap A \neq 0$.*

PROOF OF NECESSITY. By Theorem 10.5, $S - x$ is not connected. If on the other hand $S - x$ has more than n components, at least one of these, C, contains no point of A. But by Theorem 9.11, $S - C$ is connected, and thus S is not irreducibly connected about A. Hence if k is the number of components of $S - x$, then $2 \leqq k \leqq n$. That each of these meets A is shown as above.

10.10 THEOREM. (This is the same as Theorem 10.9 but with "x" replaced by "X", where X is a connected subset of $S - A$.)

10.11 COROLLARY. *In order that a space S should be irreducibly connected from a to b, $a, b \in S$, it is necessary and sufficient that if $x \in S - a - b$, or more generally if x is a connected subset of $S - a - b$, then $S - x$ consists of exactly two components each of which contains either a or b.*

The following are examples of spaces S irreducibly connected about subsets A containing more than two points:

10.12 EXAMPLE. S consists of those points (x, y) of the cartesian plane such that (1) $-1 \leqq x \leqq 1$, $y = 0$; (2) $x = \pm 1$, $-1 \leqq y \leqq 1$; A consists of the four points $(-1, -1)$, $(-1, 1)$, $(1, -1)$, $(1, 1)$. (Theorems 10.5-10.10 are well illustrated in this example.)

10.13 EXAMPLE. In the polar coordinate plane, S is the subspace con-

[18]In view of Corollary 9.6, the word "two" may be replaced by "a finite number of pairwise".

sisting of all points (ρ, θ) such that (1) $\rho = 1$, θ arbitrary; (2) $\rho = (\theta - 1)/\theta$, $1 \leqq \theta < \infty$; A consists of $(0, 0)$ and all points for which $\rho = 1$.

10.14 THEOREM. *If S is irreducibly connected about $A \subset S$ and x is a point of S such that $S - x = M_1 \cup \cdots \cup M_n$ separated, $n > 1$, then the set $M_i \cup x$ is irreducibly connected about $A_i \cup x$, where $A_i = M_i \cap A$ and $i = 1, \cdots, n$.*

PROOF. The sets $M_i \cup x$ are connected by Theorem 9.9. Suppose a set $M_i \cup x$, say $M_k \cup x$, has a proper connected subset C containing $A_k \cup x$. But then the union of the sets C, $M_i \cup x$ for $i \neq k$, is a proper connected (Theorem 7.3b) subset of S containing A.

10.15 COROLLARY. *If S is irreducibly connected from a to b, $a, b \in S$, $x \in S - a - b$, and $A'(x)$, $B'(x)$ are the components of $S - x$ containing a, b respectively, then $A(x) = A'(x) \cup x$ is irreducibly connected from a to x and $B(x) = B'(x) \cup x$ is irreducibly connected from x to b.*

10.16 DEFINITION. If S is irreducibly connected about $A \subset S$, but S is not irreducibly connected about any proper subset of A, then A is called a *basic set* about which S is irreducibly connected.

Thus, if S is irreducibly connected from a to b, $a, b \in S$, then $a \cup b$ is a basic set about which S is irreducibly connected. In Examples 10.12, 10.13, the sets A are basic. A space S may have a subset A about which it is irreducibly connected, yet have no basic set:

10.17 EXAMPLE. Let S consist of the subspace of the real numbers $\{x\}$ such that $0 < x < 1$. Then S is irreducibly connected about the set A consisting of the rational numbers in S, but has no basic set about which it is irreducibly connected (see Theorem 10.18 below).

A trivial case of a basic set is that where S is connected and has no proper subset about which it is irreducibly connected; in this case S is its own basic set, and by virtue of Theorem 10.18 below has no cut points. Thus the set of all points on a circle has itself as basic set.

10.18 THEOREM. *If S has a basic set, B, about which it is irreducibly connected, then B is the set of non-cut points of S; and if S is irreducibly connected about $A \subset S$, then $A \supset B$.*

PROOF. Let N be the set of non-cut points of S Then $B \supset N$ by Corollary 10.7. Suppose $x \in B - N$. Then $S - x = M_1 \cup M_2$ separated. Let $B \cap M_i = B_i$, $i = 1, 2$. Neither B_1 nor B_2 is empty, else one of the sets $M_i \cup x$ is a proper connected (Theorem 9.8) subset of S containing B.

If $B' = B_1 \cup B_2 = B - x$, then S has a proper connected subset C containing B', since B is a basic set. But $C \cap M_i \neq 0$, $i = 1, 2$, and hence $x \in C$ by Theorem 7.8. But then C is a proper connected subset of S containing B. Consequently $B - N = 0$ and B is the set of non-cut points of S.

Finally, if A is a set about which S is irreducibly connected, $A \supset B$ by Corollary 10.7.

A space S may be irreducibly connected about subsets A, B such that $A \cap B = 0$; for instance, in Example 10.17, let B be the set of irrational numbers. However, from Theorem 10.18 we have

10.19 COROLLARY. *A space can have only one basic set about which it is irreducibly connected.*

10.20 COROLLARY. *A necessary and sufficient condition that a connected space S be irreducibly connected about itself as basic set* (i.e., have no proper subset about which it is irreducibly connected) *is that it consist entirely of non-cut points.*

PROOF. The necessity is of course an immediate consequence of Theorem 10.18. As for the sufficiency, if S has no cut points, then by Corollary 10.7, S has no proper subset about which it is irreducibly connected.

A connected space S is not always irreducibly connected about its set, N, of non-cut points—as for instance in the trivial case where $N = 0$ as in Example 10.17. For a nontrivial case, consider the space S' obtained from the space S of Example 10.3 by deleting the point b, but inserting a new point $b' = (0, 1)$. S' is connected, $a \cup b'$ is its set of non-cut points, but S' is not only not irreducibly connected from a to b', but has no subset whatsoever that is.

We have seen that a connected space S may or may not have a proper subset about which it is irreducibly connected; and that even if it is irreducibly connected about a subset, it may not have a basic set. We next inquire whether there exist special conditions under which the existence of a set about which the space is irreducibly connected leads to the existence of a basic set. We shall establish such conditions in Theorems 10.24-10.26 below. In the proofs we shall need the following fundamental lemma (Cantor Product Theorem):

10.21 LEMMA. *In order that a space S should be countably compact, it is necessary and sufficient that if $M_1, M_2, \cdots, M_n, \cdots$ is a sequence of nonempty closed point sets such that for each n, $M_n \supset M_{n+1}$, then $\bigcap_{n=1}^{\infty} M_n \neq 0$.*

PROOF. For each n, let $x_n \in M_n$. If, for infinitely many values n_i of n, the x_{n_i} all represent the same point, then this point is in $\bigcap_{n=1}^{\infty} M_n$. Otherwise, the set $M = \bigcup_{n=1}^{\infty} x_n$ is an infinite set, and if S is countably compact, has a limit point x. Consider any fixed value k of n, and let $A = \bigcup_{n=1}^{k} x_n$, $B = \bigcup_{n=k+1}^{\infty} x_n$. Then $M = A \cup B$, and by Theorem 9.1, x is a limit point of either A or B, and since by Theorem 7.14 x cannot be a limit point of A, we have $x \operatorname{lp} B \subset M_k$, and hence by 2.2, $x \operatorname{lp} M_k$. Since M_k is closed, $x \in M_k$. Consequently $x \in \bigcap_{n=1}^{\infty} M_n$.

Conversely, suppose $x_1, x_2, \cdots, x_n, \cdots$ an infinite set of distinct points such that $M = \bigcup_{n=1}^{\infty} x_n$ has no limit point. Then M is closed, and each of the sets $M_k = \bigcup_{n=k}^{\infty} x_n$ is closed. But clearly $\bigcap_{k=1}^{\infty} M_k$ is empty.

10.22 DEFINITION. A space S is called *separable* if it has a countable subset D such that $\overline{D} = S$.

The space of real numbers is separable (the set of rational numbers, for instance, forms a set D), but the space of Example 10.4 is not. Consider also the following example:

10.23 EXAMPLE. Let M denote the well-ordered set of ordinal numbers of the first and second classes. Between each two successive ordinals α, $\alpha + 1$, insert a space S_α which is ordinally similar to the space of Example 10.17. The set consisting of M and $\bigcup S_\alpha$ is ordered in the natural fashion, and the space S resulting from assigning the usual open interval neighborhoods is countably compact, but not separable.

From Theorem 10.5 it follows that

10.24 THEOREM. *If a space S is irreducibly connected about its set, N, of non-cut points, then N is a basic set about which S is irreducibly connected.*

10.25 THEOREM. *A necessary and sufficient condition that a connected space S have a basic set about which it is irreducibly connected is that S be irreducibly connected about its set of non-cut points.*

Theorem 10.25 is a reformulation of the combined theorems 10.18 and 10.24.

10.26 THEOREM. *If a space S is irreducibly connected about a closed and countably compact set A, and $A - N$ is separable (where N is the set of non-cut points of S), then the set N is a basic set about which S is irreducibly connected.*

PROOF. By Corollary 10.7, $A \supset N$. If $A = N$, or S is irreducibly connected about N, the theorem follows from Theorem 10.24.

Suppose $A - N \neq 0$ and S not irreducibly connected about N. Then S has a proper connected subset $C \supset N$ (C may be empty if N is empty). By hypothesis, $A - N$ has a countable subset $X = \bigcup_{n=1}^{\infty} x_n$ such that $\overline{X} \supset A - N$.

Not all points of $A - N$ are in C, else $C \supset A$ and S is not irreducibly connected about A. Let $x \in (A - N) \cap (S - C)$. As x is not a point of N, $S - x = B \cup D$ separated. By Theorem 7.1, $C \subset B$ say. Then the set D contains points of $A - N$; otherwise, since $N \subset C$, $B \cup x$ is a connected (Theorem 9.8) proper subset of S containing A. And as no point of D is a limit point of $B \cup x$, there are points of X in D; let x_{n_1} be the first such point in the sequential order of the points of X indicated above.

Then $S - x_{n_1} = B_1 \cup D_1$ separated, where $B_1 \supset B \cup x \supset C \supset N$. Arguing as before, we let x_{n_2} be the first point of X in D_1, and then $S - x_{n_2} = B_2 \cup D_2$ separated, where $B_2 \supset B_1 \cup x_{n_1} \supset B \cup x \supset C \supset N$. The inductive definition of sets B_i, D_i, x_{n_i}, $i = 3, \cdots$, should be clear.

Let $D_i \cap A = A_i$ ($i = 1, 2, 3, \cdots$). Then $A_i \cup x_{n_i}$ is a closed subset of $A - N$, and since A is countably compact, $A_i \cup x_{n_i}$ is countably compact. Also, for each i, $A_i \cup x_{n_i} \supset A_{i+1} \cup x_{n_{i+1}}$, and hence $\bigcap_{i=1}^{\infty} (A_i \cup x_{n_i})$ contains at least one point p, by Lemma 10.21. As $p \in A - N$, $S - p = B_\omega \cup D_\omega$ separate, where $x \in B_\omega$. As $p \in A_i \subset D_i$ for all i, evidently $B_\omega \supset B_i \cup x_{n_i}$. Hence $X \subset B_\omega$. But the set D_ω must contain points of $A - N$, else $B_\omega \cup p$ is a proper connected subset of S containing A. But this is impossible since

$\overline{X} \cap D_\omega = 0$. This contradiction shows that either $A = N$ or S is irreducibly connected about N and the theorem follows.

An important corollary of Theorem 10.26 follows:

10.27 THEOREM. *Every countably compact, separable, connected space S which is nondegenerate has at least two non-cut points.*

PROOF. Suppose S has either no non-cut points, or only one non-cut point x. Then in Theorem 10.26, $N = 0$ or $N = x$, and S is irreducibly connected about the closed and countably compact set $A = S$, with $A - N = S$ or $A - N = S - x$. Space S is separable, hence $S - x$ is separable, and by the conclusion of Theorem 10.26, S has N as basic set about which it is irreducibly connected. But this is impossible, since no nondegenerate connected set is irreducibly connected about a single point or the null set.

REMARK. The necessity of the compactness assumption in the above theorem is shown by the space of Example 10.17; and the necessity of the separability assumption is shown by Example 10.23, which has only one non-cut point.

By the same method of reasoning we can prove the following, actually more general theorem.

10.28 THEOREM. *If a nondegenerate space S is irreducibly connected about a separable, countably compact and closed point set, then S has at least two non-cut points.*

The following theorem is another corollary of Theorem 10.26:

10.29 THEOREM. *A space which is irreducibly connected about a finite set of points is irreducibly connected about its set of non-cut points.*

10.30 THEOREM. *Every countably compact, separable, connected space is irreducibly connected about its set of non-cut points.*[19]

PROOF. Let S be a countably compact, separable, connected space and let N be the set of non-cut points of S.[20] If $N = S$ the theorem is trivial.

Let $x \in S - N$. Then $S - x = A \cup B$ separate. The sets $A \cup x$, $B \cup x$ are connected (Theorem 9.8), countably compact and separable; hence each has at least two non-cut points of itself by Theorem 10.27. The latter points are easily shown to yield a non-cut point of S in each of the sets A, B; i.e., $A \cap N \neq 0 \neq B \cap N$. Hence S is irreducibly connected about N by Theorem 10.5.

10.31 COROLLARY. *Every countably compact, separable, connected space has its set of non-cut points as basic set about which it is irreducibly connected.*

This corollary follows from Theorem 10.30 and Theorem 10.24.

---

[19] Compare H. M. Gehman [a, Theorem 1]; Gehman's theorem follows from Theorem 10.26 above.

[20] Note that if we knew $S - N$ were separable, the theorem would follow at once from Theorem 10.26; this would be the case if S were metric, for example.

10.32 Corollary. *A necessary and sufficient condition that a countably compact, separable, connected space S be irreducibly connected about a subset, A, is that A contain all the non-cut points of S.*[21]

11. The simple arc and the 1-sphere. In this section we consider the position, in the above order of ideas, of the euclidean straight line interval and the circle (1-sphere). These two configurations evidently occupy extreme positions in regard to basic sets about which they are irreducibly connected, the former having a basic set of only two points, the latter being its own basic set. We shall show, first, that among the countably compact, separable spaces, and indeed among a wider class (the "locally peripherally countably compact"—see below), the homeomorphs of the straight line interval are characterized by the above property.

Throughout this section we shall assume the 2nd Hausdorff and weak separation axioms. And I will denote a space that is irreducibly connected between two points a and b.

11.1 Definition. A space which is homeomorphic with the euclidean straight line interval (or what amounts to the same thing, the subspace $\overline{E}^1$ (see 10.2) of the real numbers) will be called an *arc*. If a and b are the points of the arc that correspond to the real numbers 0 and 1, then a and b are called the *end points* of the arc, and we shall often speak of the arc as an *arc from a to b*. By "arc ab" will be meant an arc with end points a and b, and by the symbol $\langle ab\rangle$ will be denoted the *open arc* $ab - a - b$.

11.2 Theorem. *If C is a connected subset of I which contains a or b, then $I - C$ is connected.*

Proof. Were $I - C = A \cup B$ separate, then one of the connected (Theorem 9.8) sets $A \cup C$, $B \cup C$ would be a proper connected subset of I containing $a \cup b$.

11.3 Theorem. *If M and N are connected subsets of I each of which contains a, then either $M \subset N$ or $N \subset M$.*

Proof.[22] If $N \subset M$ the theorem is proved. If $N \not\subset M$, then

$$(I - M) \cap N \neq 0. \tag{11.3a}$$

By Theorem 11.2, $I - M$ is connected. Also, $I - M$ contains b, else $b \in M$ and $M = I \supset N$. Consider

$$(I - M) \cup N = I. \tag{11.3b}$$

Relation (11.3b) must hold since, by (11.3a), $I - M$ and N have a common

[21] Compare Gehman, loc. cit., Theorem 2.

[22] See Knaster and Kuratowski [a; 218].

point, and consequently their union is a connected (Theorem 7.3) set containing $a \cup b$.

However, relation (11.3b) shows that the part of I which is deleted to give $I - M$, namely M, is supplied by N; that is, $M \subset N$.

In Corollary 10.15 we saw that if for $x \in I - a - b$ we let $A'(x)$, $B'(x)$ denote respectively the components of $I - x$ containing a, b, then the sets $A(x) = A'(x) \cup x$, $B(x) = B'(x) \cup x$ are irreducibly connected about $a \cup x$, $b \cup x$ respectively. We retain the symbols $A(x)$, $B(x)$ below, extending their meanings so that $A(a) = a$, $A(b) = I$.

11.4 Corollary. *The set $A(x)$ is the only subset of I that is irreducibly connected from a to x.*

11.5 Theorem. *If for $x, y \in I$ we let $x < y$ indicate that $x \neq y$* and *$A(x) \subset A(y)$, then the set I is simply ordered by the relation $<$.*

Proof. It follows from Theorem 11.3 and Corollary 10.15 that if $x \neq y$, then $x < y$ or $y < x$; and it follows from the definition of the $<$ relation given in the theorem that if $x < y$, then $x \neq y$. Hence it is only necessary to establish the transitivity of the relation $<$.

The relations $A(x) \subset A(y) \subset A(z)$ imply $A(x) \subset A(z)$, and we have only to show that $x \neq z$. Now if $x = z$, then $A(x) = A(z)$ (Corollary 10.15), and hence $A(x) = A(y)$. But the relation $A(x) = A(y)$ implies $x = y$. For $A(x)$ is irreducibly connected from a to x (Corollary 10.15) and a, x are non-cut points of $A(x)$ by Theorem 11.2. Hence by Corollary 10.7, $a \cup x$ is the complete set of non-cut points of $A(x)$ and, since $A(y)$ is irreducibly connected from a to y, we must have $a \cup y \supset a \cup x$ (again by Corollary 10.7). Hence (by symmetry), $x = y$, contradicting the supposed relation $x < y$.

11.6 Theorem. *For every $x \in I$, $A(x)$ is identical with the set of all $y \in I$ such that $y \leqq x$.*

Proof. If $y < x$, then $A(y) \subset A(x)$ and $y \in A(x)$ by definitions; and if $y = x$, $y \in A(x)$. Hence $y \leqq x$ implies $y \in A(x)$.

Conversely, let $y \in A(x)$. As $A(x)$ is irreducibly connected from a to x, it follows from Corollary 10.15 that $A(x)$ contains a set irreducibly connected from a to y, and hence by Corollary 11.4, $A(x) \supset A(y)$. Thus $y \leqq x$.

11.7 Theorem. *If $p, q \in I$ such that $p < q$ and R is the set of all $x \in I$ such that $p < x < q$, then R is an open set.*

Proof. As R is the set of all $x \in I$ such that $p < x < q$, the set $I - R$ must consist of all points y such that $y \leqq p$ or $q \leqq y$; let $A = \{y \mid y \leqq p\}$ and $B = \{y \mid q \leqq y\}$.

By Theorem 11.6, $A = A(p)$, and by definition, $A(p) = A'(p) \cup p$, where $I - p = A'(p) \cup B'(p)$ separate. By Theorems 7.14 and 9.1, a limit point of $A(p)$ is also a limit point of $A'(p)$, and as the latter has no limit points in $B'(p)$,

$A(p)$ is therefore a closed set. Similarly, $B = B(q)$, since $A(q) - q$ is the set of all z such that $z < q$ by Theorem 11.6. Hence B is closed. Thus $A \cup B$ is closed by Theorem 9.1, and R is therefore open.

11.8 THEOREM. *As ordered by the relation* $<$, *I satisfies the Dedekind Cut Axiom.* (See Index).

PROOF. Let $I = A \cup B$ such that $A < B$ in the sense that $x \in A$, $y \in B$ imply $x < y$ as well as that $A \neq 0 \neq B$. Suppose A has no last element and that B has no first element. Consider $x \in A$. Then there exists $c \in A$ such that $x < c$. If $a < x$, then by Theorem 11.7 the set R of all y such that $a < y < c$ is an open set. But $R \subset A$. Hence A is open if a is not a limit point of B—and the latter is easily shown. Similarly B is open. Then $I = A \cup B$ separate, contradicting the fact that I is connected.

11.9 THEOREM. *If I is separable, then I is ordinally similar to the set of real numbers x such that* $0 \leqq x \leqq 1$.

PROOF. Let X be a countable set such that $\overline{X} \supset$ I. If $p, q \in I$ and R is defined as in Theorem 11.7, then by the latter theorem R is open, and since $R \neq 0$, we must have $X \cap R \neq 0$. That is, I contains a countable separating set—i.e., a set X such that if $p, q \in I$, $p < q$, there exists $x \in X$ such that $p < x < q$.

Since the order type of a closed real number interval is characterized by (1) simple order, (2) validity of the Dedekind cut axiom, (3) existence of a countable separating set, and (4) existence of nonidentical first and last elements, the theorem now follows with the aid of Theorems 11.5 and 11.8.

11.10 DEFINITION. A space S is called *locally compact* if for $x \in S$ and neighborhood U of x, there exists a neighborhood V of x such that $U \supset \overline{V}$ and $\overline{V}$ is compact (12.6). A space S is called *locally peripherally countably compact* if for $x \in S$ and U a neighborhood of x, there exists a neighborhood V of x such that $U \supset V$ and such that the boundary (§7.5) of V is a closed, countably compact set.

11.11 THEOREM. *If I is locally peripherally countably compact, then I is countably compact.*

PROOF. Suppose I contains an infinite sequence $x_1, x_2, x_3, \cdots$, such that the set $X = \bigcup_{n=1}^{\infty} x_n$ has no limit point. As I is simply ordered by the relation $<$, we may assume $x_n < x_{n+1}$ for all n without loss of generality. We decompose I into disjoint sets A, B, such that $A < B$, in the following manner: $x \in A$ if there exists $x_n \in X$ such that $x < x_n$; otherwise $x \in B$. Evidently $X \subset A$, $b \in B$. By Theorem 11.8, either A has a last point or B has a first point. Evidently the latter must be the case; let p denote the first point of B.

Since p is not a limit point of X, there is a neighborhood U of p containing no points of X, and by hypothesis we may assume that the boundary, F, of U is a closed, countably compact set.

Let $R_n = \{x \mid x_n \leqq x \leqq p\}$. Then R_n is a connected set. For $B(x_n)$ is irreducibly connected from x_n to b by Corollary 10.15, and contains R_n by Theorem 11.6; and R_n is identically the set "$A(p)$" in $B(x_n)$, hence connected by Corollary 10.15. Consequently by Theorem 7.8, $R_n \cap F \neq 0$. Let $q_n \in R_n \cap F$. Clearly $q_n \neq p$ (7.5), and therefore $q_n < p$. Also, for fixed n, there will exist by definition a natural number k such that $q_n < x_k$, hence a q_k such that $q_n < q_k$. We may assume without loss of generality that the q_n are all distinct.

Let $Q = \bigcup_{n=1}^{\infty} q_n$. As $Q \subset F$, Q has a limit point q. Now $q \neq p$, and $q \notin B$ since $A'(p)$, $B'(p)$ are separated and $Q \subset A'(p)$, $B'(p) = B - p$. But neither can $q \in A$. For suppose $q \in A$. Then $q < p$, and there exists x_n such that $q < x_n$. Let $Q_1 = Q \cap A'(x_n)$, $Q_2 = Q \cap B'(x_n)$. Then $Q = Q_1 \cup Q_2$ separate. But Q_1 is finite, hence q is a limit point of Q_2 (Theorem 9.1). This is impossible, since $q \in A'(x_n)$.

Thus the assumption of the existence of an infinite subset of I that has no limit point leads to contradiction, and I must be countably compact.

11.12 THEOREM. *If I is separable and countably compact, then I is an arc from a to b.*

PROOF. By Theorem 11.9, I has the same order type as the set $\bar{E}^1$ of real numbers x such that $0 \leqq x \leqq 1$. To establish the homeomorphism asserted above, we use Theorem 5.7. In I, let $\mathfrak{U}$ denote the defining system of neighborhoods, and $\mathfrak{R}$ denote the system obtained from the images of the open interval neighborhoods of $\bar{E}^1$. With the aid of Theorem 11.7 we see that for $x \in I$ and an $\mathfrak{R}$-neighborhood of x, the latter contains a $\mathfrak{U}$-neighborhood of x.

Let U be a $\mathfrak{U}$-neighborhood of x, and suppose that no $\mathfrak{R}$-neighborhood of x lies in U. Now there is a monotonic sequence $\{R_n\}$ of $\mathfrak{R}$-neighborhoods such that $\bigcap_{n=1}^{\infty} R_n = x$; let $q_n \in R_n \cap (I - U)$. As I is countably compact, the set $Q = \bigcup_{n=1}^{\infty} q_n$ has a limit point $q \neq x$. Considerations such as those used in the proof of Theorem 11.11, however, show that q cannot exist.

REMARK. Concerning the proof of Theorem 11.12: In proving that every $\mathfrak{U}$-neighborhood contains an $\mathfrak{R}$-neighborhood, the separability hypothesis is used. The question might be raised as to whether it is not the case that the countable compactness alone is sufficient for this. More specifically, if an I is countably compact, are not its $\mathfrak{R}$-neighborhoods equivalent to the defining system? The following example shows this not to be the case.

11.13 EXAMPLE. To the space of Example 10.23 let us adjoin ω_1, the first ordinal of the 3rd class. But instead of using neighborhoods of ω_1 as defined in terms of order, suppose that we define neighborhoods as follows: Let α be any ordinal $<\omega_1$. For every ordinal $\beta > \alpha$, delete an open interval in the segment $(\beta, \beta + 1)$—for example the points corresponding to the real numbers $1/4 < x < 3/4$ in the homeomorphism between the closed interval $[\beta, \beta + 1]$ and $\bar{E}^1$. The residue of the half-open interval $(\alpha, \omega_1]$, after all such deletions, is a neighborhood of ω_1.

From Theorems 11.11 and 11.12 we have

11.14 Theorem. *If S is a separable, locally peripherally countably compact space, irreducibly connected from a to b and satisfying the 2nd Hausdorff axiom and the weak separation axiom, then S is an arc.*

11.15 Theorem. *Let S be a countably compact, separable, connected space satisfying the 2nd Hausdorff axiom and the weak separation axiom, and let a, $b \in S$, $a \neq b$, such that if $x \in S - (a \cup b)$, then $S - x$ is not connected; then S is an arc from a to b.*

Proof. By Theorem 10.27, S has at least two non-cut points. These must be the points a, b. By Theorem 10.30, S is irreducibly connected from a to b. The theorem now follows from Theorem 11.12.

Remark. Another way of stating Theorem 11.15 is as follows:

11.15′ Theorem. *A countably compact, separable, connected space satisfying the 2nd Hausdorff axiom and the weak separation axiom, which has at most two non-cut points, is an arc.*

11.16 Definition. If a space S is homeomorphic with the n-sphere $\{(x_1, x_2, \cdots, x_{n+1}) \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$, of cartesian $(n+1)$-space, then we call S itself an *n-sphere.* We shall always denote a topological space which is an n-sphere by the symbol S^n. The space S^1 is also frequently called the *simple closed curve* or *closed Jordan curve.* A space homeomorphic with the set $\{(x_1, x_2, \cdots, x_n) \mid x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$ of cartesian n-space is called an *n-cell*,[23] and is denoted by E^n, while a space homeomorphic with the set of points defined by $x_1^2 + \cdots + x_n^2 \leqq 1$ is called a *closed n-cell* and is denoted by $\bar{E}^n$; the boundary of E^n in $\bar{E}^n$ is called the *boundary $(n-1)$-sphere* of the set $\bar{E}^n$. Clearly an E^n is also homeomorphic with the entire cartesian n-space, so that the latter and its topological images are also denoted by the symbol E^n. In the above terminology, an arc is a closed 1-cell, and its end points constitute its boundary 0-sphere.

11.17 Theorem. *A necessary and sufficient condition that a countably compact, or locally peripherally countably compact, separable space S should be an S^1 is that it contain two distinct points a, b such that $S = I_1 \cup I_2$ where I_i $(i = 1, 2)$ is irreducibly connected from a to b and the sets $I_i - (a \cup b)$ are separated.*

Proof. The necessity is obvious, and we need only show for the sufficiency that I_1 is an arc from a to b. We do this on the basis of Theorems 11.12 and 11.14. Evidently I_1 is closed by Theorems 7.14, 9.1.

Suppose S countably compact. Then every infinite subset of I_1 has a limit point in S; this point is in I_1, since I_1 is closed. Similarly, if S is locally peripherally countably compact and $x \in I_1$, then for any neighborhood U of x, there

[23] This use of the term *n-cell* coincides with that of the same term as employed in the classical combinatorial ("polyhedral") topology. It should not be confused with the "n-cell" of Chapter V, which has a quite different meaning but bears the same relation to the abstract theory set up therein as the n-cell of the classical theory bore to the latter.

is in U a neighborhood V of x such that the boundary F of V is closed and countably compact. Let $F \cap I_1 = F_1$. The set F_1 is closed (§4.3). As F is countably compact and closed, any infinite subset of F_1 has a limit point in F. Such a point cannot be in $S - I_1$, hence is in F_1. Thus F_1 is countably compact, and I_1 is locally peripherally countably compact.

The set I_1 is separable. For let X be a countable set such that $\overline{X} = S$, and let $X \cap I_1 = X_1$. Evidently $\overline{X}_1 \supset I_1 - (a \cup b)$, and if we let $X_1' = X_1 \cup a \cup b$, then X_1' is a countable subset of I_1 such that $\overline{X_1'} \supset I_1$.

The theorem now follows from Theorems 11.12 and 11.14.

11.18 Definition. If a space S contains two distinct points a and b such that $S = I_1 \cup I_2$, where I_i $(i = 1, 2)$ is irreducibly connected from a to b and the sets $I_i - (a \cup b)$ are separated, then S is called a *quasi-closed curve.*

11.19 Lemma. *If S is a quasi-closed curve, then every two distinct points a and b of S satisfy the conditions stated in Definition* 11.18.

This lemma follows from Theorem 10.5 and Corollary 10.15.

In terms of quasi-closed curves, Theorem 11.17 is simply stated as follows:

Restatement of Theorem 11.17. *A countably compact, or locally peripherally countably compact, separable quasi-closed curve is an S^1, and conversely.*

11.20 Theorem. *In order that a nondegenerate connected space S should be a quasi-closed curve, it is necessary and sufficient that it consist of non-cut points and be disconnected by the omission of every two distinct points.*

Proof. That the conditions of the theorem are necessary follows from Theorems 7.3, 11.2 and Lemma 11.19.

To prove the sufficiency, let a and b be distinct points of S. The set $S - a$ is connected, but $(S - a) - b = S - (a \cup b) = C \cup D$ separate. Then $C \cup b$ and $D \cup b$ are connected by Theorem 9.8. Similarly, $C \cup a$ and $D \cup a$ are connected. Finally, $C \cup a \cup b$ and $D \cup a \cup b$ are connected by Theorem 7.3.

Suppose C' is a proper subset of C such that $C' \cup a \cup b$ is connected. Let $c \in C - C'$, $d \in D$. Then $S - (c \cup d) = H \cup K$ separated, and $C' \cup a \cup b$, as a connected subset of $H \cup K$, lies in H, say (Theorem 7.1). As shown above, a set such as $K \cup c \cup d$ is connected. This is impossible, since $K \cup c \cup d \subset S - (a \cup b) = C \cup D$ separate, and $c \in C$, $d \in D$ (Theorem 7.1). Hence $C \cup a \cup b$ must be irreducibly connected from a to b.

As a consequence of Theorems 11.17 and 11.20, we have:

11.21 Theorem. *If the nondegenerate connected, separable space S is countably compact, or locally peripherally countably compact, and has no cut points but is disconnected by the omission of every pair of distinct points, then S is a* 1*-sphere.*

11.22 Theorem. *In order that a nondegenerate connected space S should be a quasi-closed curve it is necessary and sufficient that if M is any connected subset of S, then $S - M$ is connected.*

Proof. The necessity is a consequence of the definition of quasi-closed curve and the above theorems on irreducibly connected sets.

For the sufficiency, we use Theorem 11.20. The space S has no cut points by hypothesis. And if $a, b \in S$, $a \neq b$, then $S - (a \cup b)$ is not connected. For if it were connected, then $S - [S - (a \cup b)] = a \cup b$ would be connected by hypothesis, which is impossible by Theorem 7.15.

As a corollary of Theorem 11.22 we have:

11.23 Theorem. *If the nondegenerate connected, separable space S is countably compact, or locally peripherally countably compact, and is not disconnected by the omission of any connected subset, then S is a 1-sphere.*[24]

12. **Some fundamental lemmas.** Thus far, we have got along with a rather scanty amount of topological equipment. And although we do not find it necessary as yet to add to our list of axioms, we shall digress to introduce some notions that are fundamental in the sequel.

In §8 we discussed the quasi-components of a space to some extent. We now characterize them from a point of view that makes some important connections with the later theory.

12.1 Definition. If S is a space and $\mathfrak{U}$ is a collection of subsets of S, then $\mathfrak{U}$ is said to *cover* S if every point of S is in at least one set of the collection $\mathfrak{U}$; i.e., $x \in S$ implies that there exists $U \in \mathfrak{U}$ such that $x \in U$.

12.2 Definition. If $x, y \in S$, then a finite collection of sets $S_1, S_2, \cdots, S_n$ will be said to form a *simple chain* of sets from x to y if (1) S_i contains x if and only if $i = 1$; (2) S_i contains y if and only if $i = n$; (3) $S_i \cap S_j \neq 0$, $i < j$, if and only if $j = i + 1$. The sets S_i may be called the *links* of the chain.

12.3 Simple Chain Theorem. *If S is a space, and $a, b \in S$, then a and b are q-equivalent if and only if every covering of S by open sets contains a simple chain from a to b.*

Proof. Suppose every covering of S by open sets contains a simple chain of these sets from a to b. Then a and b are q-equivalent. For if a is not q-equivalent to b, then by definition $S = A \cup B$ separate, where $a \in A$, $b \in B$. But A and B are open sets constituting a covering of S and containing no simple chain from a to b.

Conversely, suppose a and b are q-equivalent, and let $\mathfrak{U}$ be a covering of S by open sets. Let A consist of all $x \in S$ such that $\mathfrak{U}$ contains a simple chain from a to x. If $b \in A$, the proof is complete. Suppose $b \notin A$. Then $S = A \cup (S - A)$ is a decomposition of S into disjoint sets A, $S - A$, such that $a \in A$, $b \in S - A$. As a is q-equivalent to b, these sets are not separated. Suppose an $x \in A$ is a limit point of $S - A$. If $S_1, S_2, \cdots, S_k$ is a simple chain of open sets of $\mathfrak{U}$ from a to x, then S_k must contain at least one point, y,

[24]Compare J. R. Kline [a] Theorem A.

of $S - A$. But then either the collection $S_1, \cdots, S_{k-1}$ or the collection $S_1, \cdots, S_k$ constitutes a simple chain from a to y, contradicting the definition of A. On the other hand, if $S - A$ contains a limit point, y, of A, let $S' \in \mathfrak{U}$ contain y. Then S', being open, contains a point $x \in A$. As there exists a simple chain $S_1, \cdots, S_k$ of sets of $\mathfrak{U}$ from a to x, the collection $S_1, \cdots, S_k, S'$ is readily shown to contain a simple chain from a to y, again contradicting the definition of A. We must conclude, then, that $b \in A$.

12.4 Corollary. *If S is a space, $a \in S$, then the quasi-component of S which contains a is identical with the set of points which can be simply chained to a in every covering of S by open sets.*

12.5 Corollary. *A space S is connected if and only if, for arbitrary a, $b \in S$ and covering $\mathfrak{U}$ of S by open sets, $\mathfrak{U}$ contains a simple chain from a to b.*

12.6 Definition. A space S is called *compact*[25] if every covering of S by open sets contains a finite covering of S. Specifically, if $\{U_\nu\}$ is a collection of open sets U_ν covering S, then a finite number of the U_ν cover S.

An interesting connection between the notion of compactness and the notion of simple chain may be made as follows:

12.7 Theorem. *If I is defined as in* §11, *and is simply ordered as in Theorem* 11.5, *then every covering of I by open intervals is reducible to a finite covering of I.*

Proof. We showed in Theorem 11.7 that every open interval of I of the type $\{x \mid p < x < q\}$, where p and q are fixed elements of I, is also an open set. Similarly the intervals of type $A'(p)$, $p \neq a$, and $B'(p)$, $p \neq b$, are open (see proof of Theorem 11.7). Thus every covering of I by open intervals is a covering of I by open sets, and by Corollary 12.5 contains a simple chain from a to b. We leave to the reader the proof that such a simple chain is again a covering of I.

An interesting consequence of Theorem 12.7 is that if the topology of a space S which is irreducibly connected about two of its points is defined in terms of the order relation as above, then the compactness of S follows without any appeal to separability. In particular, if the order type of the real number continuum is defined in the usual manner, the compactness is not dependent on the separability assumption. And inasmuch as the higher-dimensional euclidean spaces are definable, as we note below, as product spaces of the real number continuum, the like observation holds for these also.

It is trivial, in view of the above, that *every arc is compact.* For in the case of the arc, the topology is definable in terms of the open interval neighborhoods (see proof of Theorem 11.12).

As regards the relation between the properties compact and countably compact, we note the following theorem, which, incidentally, discloses the motive for the term "countably compact."

---

[25]We use compact here in the sense of the *bicompact* of Alexandroff and Urysohn [d; 12].

12.8 THEOREM. *If a space S satisfies the 2nd Hausdorff axiom and the weak separation axiom, then the following properties are equivalent*: (1) *Countably compact*; (2) *If the subsets M_n ($n = 1, 2, 3, \cdots$) of S are nonempty and closed and $M_n \supset M_{n+1}$ for all n, then $\bigcap_{n=1}^{\infty} M_n \neq 0$*; (3) *Every covering of a closed subset of S by a countable collection of open sets has a finite subcollection covering the set.*

PROOF. The equivalence of (1) and (2) is given in Lemma 10.21. That (2) implies (3) may be seen as follows: If M is a closed set and the collection $\{U_n\}$, $n = 1, 2, 3, \cdots$, of open sets covers M, then some set $M - \bigcup_{n=1}^{k} U_n$ is empty, else, by (2), $M - \bigcup_{n=1}^{\infty} U_n \neq 0$, implying that the collection $\{U_n\}$ fails to cover M. (The set $\bigcup_{n=1}^{k} U_n$ is open by 4.4. And if U is an open subset of S, then $M - U$ is closed, being the intersection of the closed sets $S - U$ and M (4.3).)

That (3) implies (1) may be shown as follows: Suppose S has a sequence of distinct points $x_1, \cdots, x_n, \cdots$ such that $X = \bigcup_{n=1}^{\infty} x_n$ has no limit point. Then the sets $U_k = S - \bigcup_{n=k}^{\infty} x_n$ are open sets, since $\bigcup_{n=k}^{\infty} x_n$ is closed. But the collection $\{U_k\}$ has no finite subcollection which covers S.

12.9 COROLLARY. *If a compact space S satisfies the 2nd Hausdorff axiom and the weak separation axiom, then S is countably compact.*

The proof follows from the lemma:

12.10 LEMMA. *If a space S is compact, M is a closed subset of S, and $\mathfrak{U}$ is a collection of open sets of S covering M, then a finite subcollection of $\mathfrak{U}$ covers M.*

PROOF. Since M is closed, $S - M$ is open. Hence the collection $\mathfrak{U}'$, whose elements are $S - M$ and the elements of $\mathfrak{U}$, is a collection of open sets covering S and, as S is compact, has a finite subcollection $\mathfrak{U}''$ covering S. Evidently those elements of $\mathfrak{U}''$ that belong to $\mathfrak{U}$ cover M.

That a space which is countably compact may fail to be compact is shown by the space of Example 10.23. For in this space, each of the ordinal numbers has a neighborhood consisting of an open interval whose upper boundary point is an ordinal of the first or second class; and the sets S_α are neighborhoods of the points they contain. And in this space, S, the neighborhoods are open, but the covering constituted by them has no finite subset covering S—indeed, no denumerable subset of this covering is a covering of S.

12.11 THEOREM. *A necessary and sufficient condition that a space M be compact is that it be homeomorphic with a closed subset of a compact space S.*

PROOF. The necessity is trivial, since M is closed in itself.

To prove the sufficiency, suppose that M is homeomorphic, under a homeomorphism f, with a closed subset M' of a compact space S, and let $\mathfrak{U} = \{U\}$ be a collection of open sets U covering M. For each U, let $U' = f(U)$. As the sets $M - U$ are closed, the sets $M' - U'$ are closed rel. M' (§5), and as M' is closed in S, the sets $M' - U'$ are closed in S (cf. 2.2). Hence the sets $S - (M' - U')$

are open in S. That is, $(S - M') \cup U'$ is an open subset V' of S such that $V' \cap M' = U'$.

Since S is compact, a finite collection of the sets V' covers S, and consequently a finite set of the sets U' covers M'. As the corresponding U's cover M, the theorem is proved.

It will be noted that in proving Theorem 12.11 we have also proved:

12.12 Lemma. *If U is an open subset of a space M, and M is homeomorphic with a closed subset M' of a space S, then S has an open subset V such that $V \cap M' = U'$, where U' is the image of U in M'.*

The following theorem, of interest in analogy to Theorem 12.8, is of great importance in later chapters:

12.13 Theorem. *In order that a space should be compact, it is necessary and sufficient that if $\{F_\alpha\}$ is a simply ordered series of nonempty closed sets such that for $\alpha_1 < \alpha_2$, $F_{\alpha_1} \supset F_{\alpha_2}$, then $\bigcap F_\alpha \neq 0$.*[26]

Proof. The condition is necessary. For if $\bigcap F_\alpha = 0$, then $S - \bigcap F_\alpha = S$. But since $S - \bigcap F_\alpha = \bigcup (S - F_\alpha)$, and each set $S - F_\alpha$ is open, the space S is covered by a finite number of the sets $S - F_\alpha$. It follows that for some $\alpha = \alpha'$, $S - F_{\alpha'} = S$—i.e., $F_{\alpha'} = 0$, contrary to hypothesis.

The condition is sufficient. Let $\mathfrak{U}$ be an infinite collection of open sets covering S, and let ρ denote the smallest cardinal number such that a subset $\mathfrak{U}'$ of $\mathfrak{U}$ of cardinal number ρ covers S. If ρ is infinite, let $\rho = \aleph_\gamma$, and let $U_1, \cdots, U_\alpha, \cdots$ be a well-ordering of $\mathfrak{U}'$ of order type ω_γ, the latter being the first ordinal of its class. Let $F_\beta = S - \bigcup_{\alpha=1}^{\beta} U_\alpha$ for each $\beta < \omega_\gamma$. Then no F_β can be empty, else the sets U_α of subscript $\leqq \beta$ form a subset of $\mathfrak{U}$ of cardinal number less than ρ covering S. Hence $\bigcap F_\beta \neq 0$. But then no point of $\bigcap F_\beta$ is covered by the collection $\mathfrak{U}'$. We conclude that ρ must be finite.

The following definition is fundamental:

12.14 Definition. By a *continuum* we shall mean a nondegenerate, compact, connected space.

Another interesting example of the application of the notion of compactness, as well as an application of Theorem 12.13, may be made as follows: We showed in Theorem 10.27 that every countably compact, separable, connected space S which is nondegenerate has at least two non-cut points. With the stronger notion of compactness, the separability assumption may be dropped:

---

[26]For a discussion of such properties as are exemplified here, see P. Alexandroff and P. Urysohn [d]. So far as I have been able to determine, Theorem 12.13 was first proved by R. L. Moore in 1919 [a], for a certain class of Fréchet spaces. In this connection, Moore raised the question "whether it is not desirable to substitute, for Fréchet's definition of the word compact, a definition which is, for some spaces, ... more restrictive than that of Fréchet;" and thereupon called "compact in the new sense" a space having the property stated in Theorem 12.13. (Apparently this was the first suggestion that use of the term "compact" in the sense employed above might be desirable.)

12.15 THEOREM. *Every continuum (in which the 2nd Hausdorff and weak separation axioms hold) has at least two non-cut points.*

PROOF. Let S be a continuum and let N be the set of non-cut points of S. Suppose N contains at most one point. Let $x_0 \in S - N$. Then $S - x_0 = A \cup B$ separate, where $B \supset N$.

For each $x \in A$, select a fixed separation $S - x = A(x) \cup B(x)$ separate, where $x_0 \in B(x)$. By Theorem 9.8, $A(x) \cup x$ is connected and as it is a subset of $S - x_0$ must lie wholly in A (Theorem 7.1). Let the sets $A(x)$ be partially ordered by $\supset$. Then there exists a maximal simply ordered subset $\{A(x_\rho)\}$ of $\{A(x)\}$.

Now $\bigcap A(x_\rho) = \bigcap [A(x_\rho) \cup x_\rho]$. For if $x_\eta \in A(x_\rho)$, then $B(x_\rho) \cup x_\rho$, as a connected subset of $S - x_\eta$, must lie wholly in $A(x_\eta)$ or $B(x_\eta)$, and as $x_0 \in B(x_\rho) \cap B(x_\eta)$, x_ρ must lie in $B(x_\eta)$. Then $A(x_\eta) \cup x_\eta$, as a connected subset of $S - x_\rho$, must lie in $A(x_\rho)$.

The set $A(x_\rho) \cup x_\rho$ is compact, being a closed subset of the compact space S (Theorem 12.11), and hence, by Theorem 12.13, $\bigcap [A(x_\rho) \cup x_\rho] \neq 0$. Hence there exists $p \in \bigcap A(x_\rho)$. But if $q \in A(p)$, then, as shown above, $p \notin A(q) \subset \bigcap A(x_\rho)$. Thus the assumption that N contains at most one point leads to contradiction.

12.16 COROLLARY. *Every continuum (in which the 2nd Hausdorff and weak separation axioms hold) is irreducibly connected about the set of its non-cut points.*

(Cf. Proof of Theorem 10.30.)

In the remarks following Theorem 12.7 above we mentioned the definition of the higher-dimensional euclidean spaces as product spaces of the real number continuum. For example, if R_1, R_2 are two spaces homeomorphic with the space R^1 of real numbers, then the cartesian plane is topologically equivalent to the space E^2 whose points are the ordered pairs (x_1, x_2), $x_1 \in R_1$, $x_2 \in R_2$, and whose neighborhoods are the products of neighborhoods of R_1 and R_2. Thus, if (a, b) is a point of the space E^2, and $U(a)$ is a neighborhood of a in R_1 and $U(b)$ is a neighborhood of b in R_2, then the set of all pairs (x_1, x_2), $x_1 \in U(a)$, $x_2 \in U(b)$, constitutes a neighborhood of (a, b) in E^2. The topological structure of three-dimensional cartesian space is similarly defined as the product of three real number spaces, etc.

In general, if $S_1, \cdots, S_n$ are spaces, then the *product space* $S_1 \times S_2 \times \cdots \times S_n$ has as points the sets $(x_1, x_2, \cdots, x_n)$ such that $x_i \in S_i$, and if $(a_1, \cdots, a_n)$ is one such point and $U(a_i)$ a neighborhood of a_i in S_i, then $\{(x_1, x_2, \cdots, x_n) \mid x_i \in U(a_i),\ i = 1, 2, \cdots, n\}$ is a neighborhood of $(a_1, \cdots, a_n)$ in $S_1 \times S_2 \times \cdots \times S_n$.

One of the uses to which the notion of product space may be put is that of defining new types of configurations from the elementary ones. For example, the torus is the product of two S^1's. We are more interested here, however, in the fact that the elementary "building blocks" of the complexes of combinatorial

topology (cf. §6), the *n-cells* defined above (§11.16), are so definable. Thus, the closed 2-cell, $\bar{E}^2$, is the product space of two arcs, and generally the closed n-cell, $\bar{E}^n$, is the product space of n arcs. Consequently, since the arc, as shown above, is recognizable so simply among the general spaces above employed—namely, spaces satisfying the 2nd Hausdorff and the weak separation axiom—the notion of product space readily affords a definition of the n-sphere, S^n, for example. Specifically, if K is a space forming an arc from a to b ($a, b \in K$), and L a space forming an arc from c to d ($c, d \in L$), then $K \times c$, $K \times d$, $a \times L$, $b \times L$ together form a 1-sphere J, the boundary of the $\bar{E}^2$ formed by $K \times L$. And given two such $\bar{E}^2$'s, say $\bar{E}_1^2$, $\bar{E}_2^2$, with boundaries J_1 and J_2, respectively, there exists an obvious homeomorphism h between J_1 and J_2 which we use in order to identify J_1 and J_2 so as to form one S^1 which we call J. That is, if $x_1 \in J_1$, $x_2 \in J_2$, and x_1 and x_2 correspond under the homeomorphism h, then we identify x_1 and x_2 as one point x, but the sets $\bar{E}_1^2 - J_1$, $\bar{E}_2^2 - J_2$ remain disjoint. To complete the process we define neighborhoods of the latter two sets as before; for a point x of J, we combine neighborhoods of $\bar{E}_1^2$ and $\bar{E}_2^2$ which meet J in the same open intervals.

We do not carry out the details here, nor go through the process of showing that the above space is actually the homeomorph of the S^2 as defined in §11.16. We merely point out the general way in which one may now proceed to define the n-sphere topologically. However, this method of definition, while useful for some purposes, is not suitable from the standpoint of general topology. For example, how is one to *recognize* that a given topological space is an n-sphere? Of course, we expect such a space to be connected; also separable and countably compact. If in addition it has no cut points but every pair of points disconnects it, we know by Theorem 11.21 that it is an S^1. But suppose this is not the case, and we suspect the space to be an S^2. How are we going to recognize it as the proper combination of E^2's? We may not even know that it contains an E^2, especially if we are able to recognize the latter only as a product space of two arcs. And we may expect the recognition problem to be even more difficult in the case of higher-dimensional configurations. In short, the fact that a given space is a product space is not readily determinate, as a rule, from its topological structure. We shall see in the sequel, to be sure, that in the case of S^2, for instance, we may give a nice extension to the theorem just quoted above (Theorem 11.21) for recognizing S^1, so that S^2, and, more generally, what are known as the 2-dimensional closed manifolds, are characterized among the general topological spaces by simple topological properties. Further than this we shall be unable to go. That is, no comparable characterization of S^3 is known. We shall, however, give simple definitions of higher-dimensional configurations, which we shall call *generalized manifolds*, which, when metric and compact, reduce to the S^1 in the 1-dimensional case and to the 2-dimensional manifolds in the 2-dimensional case, and which seem to have the properties to be expected of a unified theory of "spherical" or "manifold" spaces.

Bibliographical comment

For a concise summary of fundamental notions regarding sets and topological spaces, see Lefschetz [L, Chapter I].

§5. Regarding the definition of "connected" see Schoenflies [S, 108], Lennes [a, 284; b, 303], and Hausdorff [H, 244].

§9. Theorems 9.8–9.11; see Knaster and Kuratowski [a].

§10. In addition to the paper of Gehman cited above, see Wilder [m]; also see Gehman [b; c]. Regarding Definition 10.16, compare Gehman [b]; and regarding Theorem 10.27, see Moore [e, Theorem C], and Gehman [c, Corollary 2a].

§11. Knaster and Kuratowski [a; §2]. In regard to Theorem 11.15, see Moore [b, Theorem 2]. Regarding quasi-closed curves, see Wilder [b, m].

§12. Some examples of surfaces obtained by the use of product spaces are given in Alexandroff [g]; also see Lefschetz [L; I 12]. In [L; I 24] appears a proof that an arbitrary product of compact spaces is compact. Relative to Theorem 12.15, see Moore [e; §5].

CHAPTER II

LOCALLY CONNECTED SPACES; FUNDAMENTAL PROPERTIES OF THE EUCLIDEAN n-SPHERE

A basic property of the classical manifolds, and in particular of euclidean spaces, is that of local connectedness. In the present chapter we introduce this property in what we might call its "zero-dimensional" aspects.

We shall show, first, that it may be used in place of compactness properties to characterize the elementary configurations (arc, S^1) discussed in Chapter I. We shall then give some important properties of locally connected spaces and investigate in particular the euclidean n-sphere from the point of view of local connectedness. In the latter connection, we establish further motive and justification for much of the material to be given in later chapters.

1. **Local connectedness.** We begin by recalling Examples I 10.3 and I 10.13. In the case of the former, we notice that every circle with center (0, 0) and radius <1 encloses a portion of S that has infinitely many components; likewise, in the case of Example I 10.13, a circle with center (1, 0) and radius <1 encloses a portion of the space in question that has infinitely many components. Each of these spaces fails to have, for certain points, what we shall call "local connectedness." The notion indicated by this term may be defined in a variety of ways, of which we give only three:

1.1 Definition. A space S is called *quasi-locally connected*—abbreviated *lcq*—at $x \in S$ if every neighborhood U of x contains a neighborhood V of x which lies entirely in one quasi-component of U.

1.2 Definition. A space S is called *weakly locally connected*—abbreviated *lcw*—at $x \in S$ if every neighborhood U of x contains a neighborhood V of x which lies entirely in one component of U.

1.3 Definition. A space S is called *locally connected*—abbreviated *lc*—at $x \in S$ if every neighborhood U of x contains a connected open set V such that $x \in V$.

That lcw is stronger than lcq is shown by the following example:

1.4 Example. In the polar coordinate plane of points (ρ, θ), let S_n denote the sector defined by the relations $0 < \rho \leqq 1$, $\pi/n \geqq \theta \geqq \pi/(n + 1)$. In $S_1 \cup S_2$ let $M_1 = \{(\rho, \theta) \mid [\theta = \pi, \rho \geqq 1/2] \vee [\rho = m, m \text{ irrational}, 1/2 < m < 1]\}$. In S_2 let $M_2 = \{(\rho, \theta) \mid \theta = m\pi, m \text{ irrational}, 1/4 \leqq \rho \leqq 1/2\}$. In $S_3 \cup S_4$ let $M_3 = \{(\rho, \theta) \mid \rho = m, m \text{ irrational}, 1/4 < m < 1/3\}$. In S_4 let $M_4 = \{(\rho, \theta) \mid \theta = m\pi, m \text{ irrational}, 1/6 \leqq \rho \leqq 1/4\}$. In general, for any

natural number $n > 1$, in $S_{2n-1} \cup S_{2n}$ let $M_{2n-1} = \{(\rho, \theta) \mid \rho = m, m \text{ irrational}, 1/2n < m < 1/(2n-1)\}$; and in S_{2n} let $M_{2n} = \{(\rho, \theta) \mid \theta = m\pi, m \text{ irrational}, 1/(2n+2) \leqq \rho \leqq 1/2n\}$. Denoting the point (0, 0) by p, let $M = p \cup \bigcup_{n=1}^{\infty} M_n$. Then M is a connected space, and is lcq at p. But M is not lcw at p.

It may also be shown, by example, that (in spaces admitting open sets as neighborhoods) lc at a point is actually stronger than lcw at a point. However, the important fact for our purposes is that these definitions define properties the existence of any one of which at *all* points of a space of quite usual type (see the next theorem) implies the existence of the other two at all points.

1.5 DEFINITION. A space is called *lcq* (*lcw*, *lc*) if it is *lcq* (*lcw*, *lc*) at all its points.

We recall that we have seen (I 4.8) that if a space satisfies the 3rd Hausdorff axiom (I 4.7), then the system of all open sets may be used as a neighborhood system such that each open set is a neighborhood of every point that it contains. *In the present section we shall in general assume that all spaces under consideration satisfy the 2nd and 3rd Hausdorff axioms as well as the weak separation axiom.* Such spaces we call *weak Hausdorff spaces.*

1.6 THEOREM. *If a space S is lcq, then it is lcw.*

PROOF. Suppose there exists $p \in S$ such that S is not lcw at p. Then there exists a neighborhood U of p such that every neighborhood V of p in U contains some point that is not c-equivalent to p in U. Let Q be the quasi-component of U that contains p. Then some $x \in Q$ is a limit point of $U - Q$; otherwise, Q being closed in U (Theorem I 8.8), we would have $U = Q \cup (U - Q)$ separate, Q would be connected, and there would exist a V such that $p \in V \subset Q$. Then x, and U as a neighborhood of x, fail to satisfy the condition stated in Definition (1.1) for lcq at x.

1.7 THEOREM. *If the space S is lcw, then S is lc.*

PROOF. Let $x \in S$, and let U be a neighborhood of x. As S is lcw, there is a neighborhood V of x such that V lies in one component C of U. If C is open, it is an open connected neighborhood of x satisfying the lc condition. That such is the case becomes apparent when we consider that the existence of a $p \in C$ that is a limit point of $U - C$ will constitute a violation of the fact that S is lcw at p.

From Theorems 1.6 and 1.7 we have:

1.8 THEOREM. *A space that is lcq is also lc.*

For applications in the next section, we need the following two lemmas:

1.9 LEMMA. *If a space S is lc and K is a closed subset of S, then $S - K$ is lc.*

1.10 LEMMA. *If a space S is connected and lc, and $x \in S$ such that $S - x = A \cup B$ separate, then $A \cup x$ is connected and lc.*

PROOF. That $A \cup x$ is connected follows from Theorem I 9.8. Let U be an open subset of $A \cup x$ containing x. Then $U \cup B$ is an open subset of S containing x, and accordingly $U \cup B$ contains an open connected subset V of S which contains x. Then $V - x = (V \cap A) \cup (V \cap B)$ separate, and $(V \cap A) \cup x$ is an open connected subset of U—for from the closure of $S - V$ follows the closure of $(A \cup x) - V$, the latter being the complement of $(V \cap A) \cup x$ in $A \cup x$.

2.[1] **Irreducible lc-connexes; recognition of $\overline{E}^1$ and S^1 among lc spaces.** In order to apply the notion of local connectedness to the problem of the characterization of the arc and the S^1, we first make the following definition:

2.1 DEFINITION. A space S will be called an *irreducible lc-connexe about a point set* K if S is connected, lc, and contains K, but has no proper subset having these properties.

2.2 THEOREM. *In order that a space S should be an irreducible lc-connexe about a set K, it is necessary and sufficient that S be lc and irreducibly connected about K.*

PROOF. The sufficiency is obvious. To prove the necessity, we apply Theorem I 10.5. Let $x \in S - K$. Then $S - x = A \cup B$ separate. For $S - x$ is lc by Lemma 1.9, and if it were also connected, then S would not be an irreducible lc-connexe about K.

Furthermore, $A \cap K \neq 0 \neq B \cap K$. For were $K \subset A$, for instance, then by Lemma 1.10, $A \cup x$ would be a connected and lc proper subset of S containing K.

2.3 COROLLARY. *No space S is an irreducible lc-connexe about a connected subset that is not itself lc.*

2.4 THEOREM. *If S is an irreducible lc-connexe about a compact subset A, then S is compact.*

PROOF. Let $\mathfrak{U}$ be any collection of open sets covering S. Then if $x \in S$, there exists $U \in \mathfrak{U}$ such that $x \in U$. As S is lc, there is an open connected set $R(x)$ such that $x \in R(x) \subset U$. For each $x \in S$ let there be assigned such an $R(x)$. The totality $\{R(x)\}$ covers S, and as A is compact, a finite number of the sets $R(x)$, say $R(x_1), \cdots, R(x_k)$, covers A. As S is connected, there exists for each $i > 1$ a simple chain $\mathfrak{C}_i$ of sets $R(x)$ from x_1 to x_i (Theorem I 12.3). Let the sets $R(x)$ present in such chains $\mathfrak{C}_i$ be denoted by $R(x_{k+1}), \cdots, R(x_n)$.

Let $R = \bigcup_{i=1}^{n} R(x_i)$. Since the sets $R(x_i)$ can be rearranged, according to chains $\mathfrak{C}_2, \mathfrak{C}_3, \cdots, \mathfrak{C}_k$, so as to satisfy the hypothesis of Theorem I 7.3a,

[1]The spaces considered in this section need only satisfy the 2nd Hausdorff and weak separation axioms.

their union R is connected. And since, by Theorem 2.2, S is irreducibly connected about A, and $R \supset A$, we must have $R = S$. Accordingly, if we select for each $R(x_i)$ a set $U_i \in \mathfrak{U}$ such that $R(x_i) \subset U_i$, the sets U_i form a finite subset of $\mathfrak{U}$ covering S, and S is therefore compact.

2.5 Corollary. *If S is an irreducible lc-connexe about a finite point set, then S is compact.*

As a consequence of Theorem I 11.12 and Corollary 2.5 we have:

2.6 Corollary. *If a separable space S is an irreducible lc-connexe about a pair of distinct points a, b, then S is an arc from a to b.*

It is well known that not every countably compact, separable connected space is compact. However, we can state:

2.7 Theorem. *If S is a countably compact, separable, lc, connected space, and S has a compact subset A which contains the set of non-cut points of S, then S is itself compact.*

Proof. By Corollary I 10.32, S is irreducibly connected about the set A. Hence S is irreducibly connected about a compact set and by Theorem 2.4, S is itself compact.

2.8 Corollary. *If the set of non-cut points of a countably compact, separable, lc, connected space S is compact, then S is itself compact.*

Corollary 2.6 shows that the arc may be characterized without any compactness assumption whatsoever, by use of the lc notion. In line with the remarks in the latter part of §I 12 we can therefore assert that the n-cell, the n-sphere, etc., may, through the notion of product space, be topologically characterized on the basis of the lc "weak Hausdorff" spaces, without any compactness assumptions whatsoever.

We shall next obtain characterizations of the 1-sphere analogous to those of Theorems I 11.21 and I 11.23, but with the lc condition replacing the compactness assumptions. We need the following lemma and corollary:

2.9 Lemma. *In a space whose neighborhoods are open, the boundary of every point set is closed.*

2.10 Corollary. *A countably compact space whose neighborhoods are open is locally peripherally countably compact.*

2.11 Theorem. *If a separable quasi-closed curve is lc, then it is a* 1*-sphere.*

Proof. By definition, a quasi-closed curve S is the union of two I's (cf. §I 10), say I_1 and I_2, such that $\bar{I}_1 \cap I_2 = I_1 \cap \bar{I}_2 = a \cup b$. By Lemma 1.9, $S - \bar{I}_2$ is lc. If we can show that I_1 is lc at a and b and separable, then I_1 is an arc by Corollary 2.6, and since by symmetry I_2 is likewise an arc, S is then a 1-sphere.

By Theorem I 11.20 and Lemma 1.9, $S - b$ is connected and lc. Application of Lemma 1.10 to $S - b$ (with "a" replacing the "x" of that lemma) shows that $I_1 - b$ is lc; in particular, then, I_1 is lc at a and, by similar reasoning, at b.

Let X be a denumerable subset of S such that $\overline{X} = S$. Since A is open, evidently $\overline{X \cap A} \supset A$, and consequently the closure of $(X \cap A) \cup a \cup b$ contains I_1.

From the above theorem, and Theorems I 11.20 and I 11.22, we have:

2.12 THEOREM. *If a nondegenerate separable, connected space is lc, has no cut points, and is disconnected by the omission of each pair of its points, then it is a* 1-*sphere.*

2.13 THEOREM. *If a nondegenerate, separable, connected space is lc and is not disconnected by the omission of any connected subset, then it is a* 1-*sphere.*

The importance of the above theorems is that they afford characterizations of the 1-sphere that are independent of the fact that the 1-sphere is the union of two arcs; i.e., they do not in any way make use of the definition of the arc. Later when we come to the 2-sphere (or, more generally, the 2-manifold), we shall again obtain characterizations that do not depend on the fact that the configuration is made up of the corresponding euclidean element (the 2-cell).

When we come to the 2-sphere we shall find that one of the most important characterizations makes use of the so-called Jordan Curve Theorem, and for the sake of unity we shall give here its 1-dimensional analogue. The latter makes use of the simple fact that a 0-sphere in a 1-sphere separates the 1-sphere into exactly two components of which it is the common boundary:

2.14 THEOREM. *Let S be a connected*[2] *space containing at least one* 0-*sphere, and such that if S^0 is any* 0-*sphere of S, then $S - S^0$ has just two components. Then S is a quasi-closed curve.*

PROOF. By Corollary I 9.5, every 0-sphere separates S. Hence if we can show that S has no cut points, it will follow from Theorem I 11.20 that S is a quasi-closed curve.

Suppose $x \in S$ such that $S - x = A \cup B$ separate. Then if $a \in A$, $A - a$ is connected. For were $A - a = A_1 \cup A_2$ separate, then would $S - (x \cup a) = (A - a) \cup B$ separate $= A_1 \cup A_2 \cup B$, where the latter sets are multiwise separate, although by hypothesis $S - (x \cup a)$ can have only two components. And since $A \cup x$ is connected (Theorem I 9.8), A has infinitely many points (Theorem I 7.15) and has x as a limit point (Theorem I 7.12). Similarly, B has infinitely many points, and if $b \in B$, $B - b$ is connected and has x as a limit point.

Then $S - (a \cup b) = (A - a) \cup (B - b) \cup x$, where $A - a$, $B - b$ are nonvacuous and connected, and have x as a limit point. It follows that $S -$

[2]As a matter of fact, instead of assuming that S is connected, it is sufficient to assume that the number of points in S is not 4.

$(a \cup b)$ is connected (Theorems I 7.2, I 7.3), and thus the assumption that S has a cut point leads to a contradiction of the hypothesis.

As corollaries of the above theorem and of Theorems I 7.14 and 2.11, we can now state:

2.15 THEOREM. *If the connected, separable space S is locally peripherally countably compact, contains at least one 0-sphere and is separated by every 0-sphere into just two components, then S is a 1-sphere.*

2.16 THEOREM. *If a connected, separable lc space S is nondegenerate and is separated by every 0-sphere into just two components, then S is a 1-sphere.*

3. **Some general properties of lc spaces.** Unless otherwise stated, we shall assume that the spaces considered are weak Hausdorff spaces.

3.1 THEOREM. *In order that a space S should be lc, it is necessary and sufficient that all components of open sets be open.*

PROOF. The condition is necessary. For let C be a component of an open set U, and $x \in C$. As U is open, $x \in U$, and S is lc, there exists by 1.3 a connected open set V such that $x \in V \subset U$. Every point of V is c-equivalent to x in U, and as C is the set of all points of U that are c-equivalent to x in U, we must have $V \subset C$. Accordingly C is open (I 4.2).

The condition is sufficient. For let $x \in S$ and U any open set containing x. Then the component of U containing x is open, and may serve as the set V satisfying the requirement of Definition 1.3.

3.2 COROLLARY. *If S is lc and U an open subset of S, then every union of components of U is open.*

3.3 THEOREM. *If S is lc, B is the boundary of a component C of an open set, and $S - \overline{C} \neq 0$, then $S - B = C \cup (S - \overline{C})$ separate.*

PROOF. By the preceding theorem, C is open. As $S - \overline{C}$ is open (I 4.9), the sets C and $S - \overline{C}$ are separated. All we need to notice, then, is that $S - B = C \cup (S - \overline{C})$ (cf. I 7.7).

3.4 COROLLARY. *If C is a component of an open subset P of a connected and lc-space S, and $C \neq S$, then the boundary of C is a nonempty subset of $S - P$.*

3.5 COROLLARY. *If A is a closed subset of a connected and lc space S, then every component C of $S - A$ has limit points in A.*

3.6 THEOREM. *In an lc space any two points that are q-equivalent in an open set U are also c-equivalent in U.*

PROOF. Let $a, b \in U$ be q-equivalent. As U is open, every point x of U is in a connected open subset V_x of U. By I 12.3, there exists a simple chain of such sets V_x from a to b. It follows that a is c-equivalent to b in U (I 7.3a).

3.7 Definition. Let A and B be disjoint subsets of a point set M. Then by a set $H(A, B)M$ we shall mean a connected subset of $M - (A \cup B)$ that has limit points in both A and B.

3.8 Theorem. *Let A and B be closed, disjoint subsets of a connected and lc space S. Then S contains a set $H(A, B)S$.*

Proof. Consider a component C of $S - A \cup B$. By Corollary 3.5, C has a limit point in at least one of the sets A, B, and if in both, then the theorem is proved. Suppose every such component has limit points in only one of the sets A, B. Then $S = S_A \cup S_B$, where S_A consists of A and all components C of $S - (A \cup B)$ which have limit points in A, and S_B is defined similarly relative to B.

The sets S_A, S_B are disjoint, else some C has limit points in both A and B. Hence, since S is connected, S_A, say, contains a limit point a of S_B. Suppose $a \in S_A - A$. Then a is a point of some set C which has limit points in A. As B is closed and S is lc, there exists a connected neighborhood U of a such that $U \cap (A \cup B) = 0$. But as a is a limit point of S_B, U must contain some $b \in S_B - B$. Evidently b is therefore a point of some set of type C, which we call C', which has limit points in B. Both C and C' contain points (a and b) of a connected subset, U, of $S - A \cup B$, so that necessarily $C = C'$. But this contradicts the supposition that no such component has limit points in both A and B.

Suppose $a \in A$. Let U be a connected neighborhood of a such that $U \cap B = 0$. Then U contains a point b of some component of type C which has limit points in B. (We continue to denote this component by C.) Let $U = U_1 \cup U_2$, where U_1 is the portion of U in C, and $U_2 = U - U_1$. As C is closed in $S - A \cup B$ and has no limit points in A, the set U_1 is closed in U. Hence, as U is connected, U_1 contains a limit point x of U_2. But now the method used in the preceding paragraph can be employed to show that C contains points of U_2, again affording a contradiction.

3.9 Definition. Let A and B be subsets of a point set M. Then by a set $H^*(A, B)M$ we shall mean a set $H(A, B)M$ in case $A \cap B = 0$; but in case $A \cap B \neq 0$, then it is either (1) an $H(A_1, B_1)M$, where $A_1 = A - A \cap B$ and $B_1 = B - A \cap B$, (2) a component of $A \cap B$, or (3) a component of $S - (A \cup B)$ having a limit point in $A \cap B$.

3.10 Theorem. *Let S be an lc space, A and B be closed subsets of S, and N a connected subset of S that meets both A and B. Then some point of N is also a point of an $H^*(A, B)S$.*

Proof. Suppose no $H^*(A, B)S$ contains a point of N. We form two sets A', B' as follows: If $x \in N \cap A$, we place x in A', and if $y \in N \cap B$, we place y in B'. If $x \in N - (A \cup B)$, we consider the component C_x of $S - (A \cup B)$ which contains x. By Corollary 3.5 C_x has limit points in $A \cup B$, but by our supposition C_x can have limit points in only one of the sets A, B. Then if the

limit points of C_x in $A \cup B$ are in A, we place x in A'; otherwise in B'. Then $N = A' \cup B'$, and $A' \cap B' = 0$.

As N is connected, A' and B' are not separate; hence we may suppose that A' contains a limit point, p, of B'. Now $p \notin N - (A \cup B)$, since if it were, a connected neighborhood U_p of p such that $U_p \cap (A \cup B) = 0$, together with a C_x for some $x \in U_p \cap B'$, would enable us to show that C_p has limit points in B. Hence $p \in N \cap A$. But in this case, we select a connected U_p such that $U_p \cap B = 0$, and again let $x \in U_p \cap B'$. Then as C_x is open by Theorem 3.1, $C_x \cup U_p$ is an open connected subset of S, and is lc. Hence by Theorem 3.8, $C_x \cup U_p$ contains a set $H(x, A)S$. Evidently the latter set and C_x together form a set $H(A, B)S$ that contains x, contradicting our initial supposition.

Remark. Evidently Theorem 3.8 can be considered as a special case of Theorem 3.10 (by simply letting $N = S$ in the case of 3.8).

4. "Phragmen-Brouwer properties" and their equivalences in lc spaces.

We now define a set of properties which, as we shall see later, all hold for the euclidean n-sphere S^n. Their equivalence will be demonstrated, however, for the general lc spaces.

4.1 Definitions. For any space S, we define the following properties:

Property I. If A and B are disjoint closed subsets of S, and $x, y \in S$ such that neither A nor B separates (I 5.11) x and y in S, then $A \cup B$ does not separate x and y in S.

Property I′ (Phragmen-Brouwer Property). If neither of the disjoint closed subsets A and B of S separates S, then $A \cup B$ does not separate S.

Property II (Brouwer Property). If M is a closed, connected subset of S and C is a component of $S - M$, then the boundary of C is a closed and connected set.

Property III (Unicoherence). If $S = A \cup B$, where A and B are closed and connected, then $A \cap B$ is connected.

Property IV. If F is a closed subset of S, and C_1, C_2 are disjoint components of $S - F$ which have the same boundary, B, then B is closed and connected.

Property V. If A and B are disjoint closed subsets of S, $a \in A$, $b \in B$, then there exists a closed, connected subset C of $S - (A \cup B)$ which separates a and b.

4.2 Theorem. *In a connected and lc space S, Properties* I *and* I′ *are equivalent.*

Proof. That Property I implies Property I′ is trivial.

Suppose that the connected and lc space S has Property I′ but not Property I. Then S contains two points x, y and two disjoint closed point sets A, B neither of which separates x and y but whose union does. We shall define

sets F_1, F_2 as follows: If C_x is the component of $S - (A \cup B)$ that contains x, then $y \notin C_x$ and if F is the boundary of C_x it follows from Theorem 3.3 that

(1.12a) $$S - F = C_x \cup (S - \overline{C}_x) \text{ separate.}$$

Let $F \cap A = F_1$, $F \cap B = F_2$. Then $F = F_1 \cup F_2$ and $F_1 \neq 0 \neq F_2$; for if $F_1 = 0$ then $F \subset B$, and (1.12a) implies that $S - B = C_x \cup (S - C_x - B)$ separate and B separates x and y, contrary to hypothesis.

Now it follows, by similar reasoning, that the component C_y of $S - F$ that contains y has limit points in both F_1 and F_2. Let $\overline{C}_y \cap F_i = B_i$, $i = 1, 2$, and $B^* = B_1 \cup B_2$. Also, let K_x denote the component of $S - B^*$ which contains C_x. As every point of F is a limit point of C_x, so must every point of B^* be a limit point of K_x. Consequently B^* is the common boundary of the disjoint domains K_x, C_y.

To B_1 let us add every component of $S - B^*$ that has its boundary entirely in B_1 and call the resulting set B_1'. Similarly form a set B_2' relative to B_2.

The set $S - B_1'$ is connected. For let C be a component of $S - B_1'$. Suppose $C \cap B_2 = 0$. Then $C \subset S - B^*$. Let C' be the component of $S - B^*$ that contains C. If the boundary B' of C' were entirely in B_1, then would $C \subset C' \subset B_1'$. If B' were entirely in B_2, then we would have $C \subset C' \subset B_2'$ and consequently, since $B_2' \subset S - B_1'$, we would have $C' \subset S - B_1'$ and $C \cap B_2 \neq 0$. Finally, were $B' \cap B_1 \neq 0 \neq B' \cap B_2$, we would again have $C' \subset S - B_1'$ and hence $C \cap B_2 \neq 0$.

Consequently $C \cap B_2 \neq 0$. But $K_x \cup C_y \cup B_2'$ is a connected subset of $S - B_1'$, and as every component C of $S - B_1'$ has limit points in B_2', the set $S - B_1'$ is connected. Similarly $S - B_2'$ is connected. But $S - (B_1' \cup B_2')$ is not connected, since K_x and C are disjoint components of the latter set. This contradicts the fact that S is supposed to have Property I′.

4.3 THEOREM. *In a connected and lc space S, Properties* I *and* II *are equivalent.*

PROOF. Suppose S has Property II but not Property I. We repeat the first part of the proof that I′ implies I in Theorem 4.2—specifically, through the first paragraph and the first sentence of paragraph two of the proof that I′ implies I. We then conclude as follows: But C_y is a component of $S - \overline{C}_x$, since $\overline{C}_x = C_x \cup F$. And $\overline{C}_x$ is closed and connected. Then the boundary of C_y should be connected since S has Property II. But the boundary of C_y has points in both F_1 and F_2 and is therefore not connected.

Conversely, if S has Property I, then S has Property II. Suppose C a component of $S - M$, where M is a closed, connected subset of S. Then by Theorem 3.3, if B is the boundary of C, and B is not connected, $S - B = C \cup (S - \overline{C})$ separate. (If B is not connected, $M - B \neq 0$ and hence $S - \overline{C} \neq 0$.) Let $B = B_1 \cup B_2$ separate. Some point y of M is in an $H(B_1, B_2)S = C'$, by Theorem 3.10. If $x \in C$, then B_1 does not separate x and y in S (since $C \cup B_2 \cup C' \subset S - B_1$), and the same holds for B_2. But B separates x and y.

4.4 THEOREM. *In a connected and lc space S, Properties* I, III *and* IV *are equivalent.*

PROOF. Property I implies Property III. Suppose S is the union of two closed, connected sets S_1 and S_2, but that $S_1 \cap S_2 = A \cup B$ separate. By Theorem 3.10, some point x of S_2 is in a set $H(A, B)S$, and therefore the component C_1 of $S - S_1$ that contains x has limit points in both A and B. However, $C_1 \subset S_2$, implying $\overline{C}_1 \subset S_2$, and hence $\overline{C}_1 \cap S_1 \subset S_1 \cap S_2 = A \cup B$. But by Theorem 4.3, the boundary of C_1 should be connected.

Property III implies Property IV. Suppose S has Property III but not Property IV. Then S has a closed subset F and two disjoint components C_1, C_2 of $S - F$ with common boundary $B = B_1 \cup B_2$ separate. By Theorem 3.3, $S - B = C_1 \cup (S - \overline{C}_1)$ separate. Hence C_1 is a component of $S - B$. If C is any other component of $S - B$, then C has boundary points in B by Theorem 3.5. Consequently $S - C_1$ is connected, since it consists of the set $\overline{C}_2$ and components of $S - B$ having limit points in $B \subset \overline{C}_2$ (cf. Theorems I 7.3 and I 7.4). But then S is the union of the closed and connected sets $\overline{C}_1$, $S - C_1$, whose intersection, B, is not connected, thus violating Property III.

Property IV implies Property I. If S fails to have Property I, we may proceed as in the first paragraph of the proof that I′ implies I in Theorem 4.2 to find components K_x and C_y of $S - B$ having common boundary $B = B_1 \cup B_2$ separated, violating Property IV.

4.5 THEOREM. *In a connected and lc space S, Property* V *implies Property* III.

PROOF. Suppose S has Property V, but is the union of closed and connected sets S_1, S_2 such that $S_1 \cap S_2 = A \cup B$ separate. If $a \in A$, $b \in B$, then since S has Property V there exists a closed, connected set $K \subset S - (A \cup B)$ which separates a and b. But S_1 and S_2 both contain a and b, hence $S_1 \cap K \neq 0 \neq S_2 \cap K$. As $K \subset S - (A \cup B) = [S_1 - (A \cup B)] \cup [S_2 - (A \cup B)]$, we have $K = \{K \cap [S_1 - (A \cup B)]\} \cup \{K \cap [S_2 - (A \cup B)]\}$ separate, contradicting the fact that K is connected.

4.6 DEFINITION. A space S is called *normal* if for each pair of disjoint closed subsets A, B of S there exist disjoint open subsets of S that contain A and B respectively.

4.7 THEOREM. *If a normal space S is connected and lc, and has Property* II, *then it has Property* V.

PROOF. Let A, B be disjoint closed subsets of S, $a \in A$, $b \in B$. As S is normal, there exist open sets $U \supset A$, $V \supset B$, such that $U \cap V = 0$. Let U_a be the component of U that contains a. Using the symbols $F(\quad)$ to denote boundary, $F(U_a) \subset F(U) \subset S - (A \cup B)$. Let C_b be the component of $S - \overline{U}_a$ that contains b. As S has Property II, $F(C_b)$ is a closed and connected set K. By Theorem 3.3, $S - K = C_b \cup (S - \overline{C}_b)$ separate. As U_a is open

by Theorem 3.1, $K \subset F(U_a) \subset S - (A \cup B)$. Hence K is a closed and connected subset of $S - (A \cup B)$ that separates a and b in S.

REMARK. To see that Properties I—IV do not imply Property V in a space that is connected and lc but not normal, consider the following example:

4.8 EXAMPLE. In the cartesian plane, let M consist of the set $\{(x, 0) \mid 0 \leqq x \leqq 2\}$ together with the set $\{(x, y) \mid (x = 1/n)\&(0 < y \leqq 1), n = 1, 2, 3, \cdots\}$. Let $A = (0, 0)$, $p_n = (1/n, 1)$, $B = \bigcup p_n$, $q_n = (1/n, 0)$, $q = (2, 0)$. Let S be the space obtained from M by the following defining system of neighborhoods: (1) points of type q_n have connected "T-shaped" neighborhoods, nonoverlapping for different values of n; (2) points of type p_n or q have "half-open interval" neighborhoods, not overlapping with the neighborhoods of points q_n; (3) points $(1/n, y)$ such that $0 < y < 1$, or $(x, 0)$ such that $1/(n + 1) < x < 1/n$, have open interval neighborhoods that do not contain points of type p_n, q_n or q; (4) the only neighborhoods of A are those sets U_n such that $U_n = \{(x, y) \mid [(x, y) \in M]\&[x < 1/n, y < 1]\}$. Note that for each n, the boundary of U_n consists of q_n and a subset of B. Also, B is closed, as this set has no limit points in S. Now any set $H \subset S$ such that for all sufficiently great n, H contains a point $(1/n, y)$, $0 < y < 1$, has A as a limit point. It follows that S does not contain disjoint open sets containing A, B respectively, and a fortiori does not have Property V. But S is connected, lc, and has Properties I-IV.

4.9 DEFINITION. A space S is called *completely normal* if for each pair of separated sets A, B of S there exist disjoint open subsets of S that contain A and B respectively.

The following property, which is stronger than V, may be shown equivalent to IV in a completely normal space:

Property V′. If A and B are separated subsets of S, $a \in A$, $b \in B$, then there exists a closed, connected set C of $S - (A \cup B)$ which separates a and b.

4.10 THEOREM. *If a completely normal space S is connected and lc, and has Property* II, *then it has Property* V′; *and conversely.*

PROOF. The proof that II implies V′ is identical with the proof of Theorem 4.7, except that the hypothesis of complete normality replaces that of normality. The converse follows from Theorem 4.5 and the fact that V′ implies V.

In view of the application which we wish to make of the above theorem, we interpolate the following theorem:

4.11 THEOREM. *Every metric space is completely normal.*

PROOF. Let S be a metric space (I; 3), and let A and B be separated subsets of S. For $x \in A$, let $\rho(x, B) = \text{glb}\{\rho(x, y) \mid y \in B\}$; we define $\rho(A, y)$ symmetrically. Both $\rho(x, B)$, $\rho(A, y)$ are positive.

For each $x \in A$ let $U(x) = S(x, \rho(x, B)/2)$, and for each $y \in B$ let $V(y) = S(y, \rho(A, y)/2)$. Then let $U = \bigcup U(x)$, $V = \bigcup V(y)$. Both U and V are

open sets (I 4.4). That $U \cap V = 0$ follows from the "triangle law" of distance—Condition (2) of I 3.

As a consequence of Theorem 4.11 and the preceding theorems of this section we can state:

4.12 THEOREM. *If a metric space is connected and lc, and has any one of the Properties* I, I′, II, III, IV, V, V′, *then it has all of the other properties.*

The following theorem, which embodies a strengthening of Property II in lc spaces, will be found of use:

4.13 THEOREM. *If S is a connected and lc space which has Property* II, *then for any closed subset M of S, and components C, D of M, $S - M$ respectively, the set $C \cap F(D)$ is connected; and as a consequence, $F(D)$ cannot have a greater number of components than M.*

PROOF. We may suppose that $C \cap F(D) \neq 0$. Let M be augmented by all components of $S - M$ except D to form the closed set $M' = S - D$. Then let D be augmented by all components of M' except C' (the component of M' containing C) to form the open set $D' = S - C'$.

The set D' is connected. For consider a component E of M', and suppose that $E \cap F(D) = 0$. Since S is connected and E is closed, the latter set contains a limit point, x, of $S - E$. But since $E \cap \overline{D} = 0$ and S is lc, there exists a connected open set P containing x such that $P \cap D = 0$, implying $P \subset M'$. But then $P \subset E$, since E is a component of M', and x cannot be a limit point of $S - E$. Hence $F(D)$ meets every component of M', and as D' is the union of D and components of M', it is connected by Theorems I 7.3 and I 7.4.

As D' is a domain complementary to the closed and connected set C', and S has Property II, the set $F(D')$ is a closed and connected subset of C'.

Let $y \in F(D')$, and Q any open set containing y. Since S is lc, we may suppose Q is connected. If $Q \cap D = 0$, then $Q \subset S - D = M'$. But as C' is a component of M', this implies $Q \subset C'$ and $y \notin F(D')$. Hence $F(D') \subset F(D)$.

Let $z \in C \cap F(D)$; such points as z exist by hypothesis. Then $z \in C'$; and $z \in F(D')$, since every neighborhood of z contains points of $D \subset D'$. Hence $z \in C' \cap F(D') = F(D')$, and we infer that

$$C \cap F(D) \subset F(D'). \tag{4.13a}$$

But $F(D) = [C \cap F(D)] \cup [F(D) \cap \bigcup C_\nu]$ where the C_ν indicates components of M distinct from C, and hence every connected subset of $F(D)$ that meets $C \cap F(D)$ must lie in the latter set; i.e.,

$$F(D') \subset C \cap F(D). \tag{4.13b}$$

Relations (4.13a) and (4.13b) imply that $F(D') = C \cap F(D)$, and as $F(D')$ is connected, the same is true of $C \cap F(D)$.

5. Some topology of the n-sphere. It was stated in §4 above that Properties I—V′ were all valid for the n-sphere. We shall prove this assertion in the present

section. However, the purpose of the digression to the n-sphere is threefold: We not only (1) provide a proof of the properties mentioned for this basic euclidean case, but we (2) introduce the machinery of chains and cycles of the so-called *algebraic topology* in a natural manner, which should pave the way for its use later in general spaces; and we (3) provide the material whereby we may give a simple proof of the Jordan-Brouwer separation theorem for the separation of the n-sphere by the $(n - 1)$-sphere.

Regarding (2) we should add that we restrict ourselves at present to the so-called modulo 2 topology and follow closely methods due to J. W. Alexander [a]. Regarding (3), the case $n = 2$ is of course the Jordan Curve Theorem (cf. I 6), which we need for the topological characterization of the 2-sphere in the sequel.

5.1 An n-sphere S^n is topologically equivalent to the continuum given by the equation $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$ in $(n + 1)$-dimensional cartesian space E^{n+1} (I 11.16). An n-plane through the origin of E^{n+1} divides S^n into two disjoint domains, which we call *n-cells* (I 11.16), whose common boundary is an S^{n-1}, the latter being the intersection of S^n by the n-plane.[3] The latter S^{n-1} may in turn be subdivided by an n-plane through the origin into a pair of $(n - 1)$-cells whose common boundary is an $(n - 2)$-sphere; and so on down to a 0-sphere consisting of a pair of points which we call 0-cells. The decomposition of S^n obtained in this manner is called an *elementary subdivision* s_0 of S^n. Note that s_0 consists of two i-cells of each dimension $i = 0, 1, \cdots, n$.[4] The $(i - 1)$-cells of the subdivision in the boundary of the open point set which constitutes an i-cell E^i $(i > 0)$ we call the *boundary cells* of the E^i.

From the elementary subdivision s_0 one can proceed to the *derived subdivisions*: These are obtained by further decomposition of S^n by n-planes, but this will not be done in utterly random fashion. We ask that each derived subdivision be obtained from a preceding subdivision by the operation of introducing a single convex i-cell, which separates an $(i + 1)$-cell into two convex $(i + 1)$-cells. This implies, in case the cell to be introduced is of dimension greater than 0, that the boundary cells of the new cell are already present. For example, if we wish to divide one of the n-cells of an elementary subdivision into two n-cells separated by an $(n - 1)$-cell, we may need to introduce some new $(n - 2)$-cells, etc., on the boundary first. This can be done through a series of steps $s_0, s_1, \cdots, s_k$ where s_0 is the elementary subdivision, each s_i

[3]In all proofs of the Jordan Curve Theorem the question arises as to what is assumed. Some proofs, during the early history of the theorem particularly, assume that the curve is made up of a finite number of analytic arcs, etc.; others treat the general case, but assume the theorem for the case where the curve is a polygon. The proof which is given below will be found to depend upon the statement just made above regarding the separation of the S^n by an n-plane in E^{n+1}—a fact evident immediately from the continuity of the real number system. *Domain* is a synonym for *connected open set.*

[4]We are not concerned with the fact that an i-cell is a topologically unique configuration, and do not employ this fact below.

for $j > 0$ is obtained from s_{j-1} by the introduction of a new cell whose boundary cells are already in s_{j-1}, and s_k is the final desired subdivision.

We shall now assume an infinite sequence of subdivisions $s_0, s_1, \cdots, s_j, \cdots$, where s_0 is the elementary subdivision and such that (1) for each natural number j, s_{j+1} is derived from s_j by the introduction of a single new i-cell $(0 \leqq i \leqq n - 1)$ which divides an $(i + 1)$-cell of s_j into two $(i + 1)$-cells, and (2) for arbitrary $\epsilon > 0$, there exists j such that for $h > j$, all cells of s_h are of diameter less than ϵ.[5]

5.2 Throughout the present section we assume that we are dealing with a fixed s_j. For each cell of s_j we adopt a symbol σ_k^r, where the superscript r denotes the dimension of the cell, and the subscript k is for enumerative purposes. For $r > 0$ we have a so-called "boundary relation"

$$\partial\sigma_k^r = \sum_m \eta_{kr}^m \sigma_m^{r-1} \tag{2.2a}$$

where m runs through all possible values, and η is 1 or 0 according as σ_m^{r-1} is a boundary cell of σ_k^r or not.

By an *r-chain* we mean any polynomial of the form $\sum c^k\sigma_k^r$ where the k runs through all possible values and the c's are all 0's and 1's. An r-chain must be distinguished from its associated complex. A *complex* is the geometric configuration, or more precisely the collection of cells, that is obtained by an arbitrary selection from s_j, with the single provision that if a cell is in the collection, so are all its boundary cells. (Thus a complex is a closed point set if we break it down into its individual points—which we do not do, however, as we wish to preserve the concept of a collection of cells.) The *associated complex* of an r-chain of the form $\sum c^k \sigma_k^r$ is obtained by selecting for our collection the sets of all cells σ_k^r for which $c^k = 1$, together with all cells necessary to satisfy the above condition regarding boundary cells. We denote this associated complex by the symbols $|\sum c^k\sigma_k^r|$. Thus an r-chain is a *polynomial,* while its associated complex is a *geometric entity.* The associated complex of a chain of the form[6] σ_k^r may be called a *cellular complex,* σ_k^r being a *cellular chain.* The cells in a cellular complex $|\sigma_k^r|$ are called the *faces* of the cell σ_k^r.[7] An r-dimensional complex, or simply r-complex, is a complex that is the associated complex of some r-chain; in other words, it consists only of r-cells and their faces. Except for the existence of other connotations that might lead to confusion, we might use the term "polyhedron", or "variety", instead of complex.

---

[5]In any metric space, the *diameter* $\delta(M)$ of a point set M is lub $\{\rho(x, y) \mid x, y \in M\}$.

[6]In analogy to ordinary algebra, we allow the coefficient 1 to be absorbed—σ_k^r is understood to have the coefficient 1.

[7]In particular, then, σ_k^r is a face of itself, and has faces of all dimensionalities from 0 to r. It may seem confusing, incidentally, to use the same symbol σ_k^r for both the cell (without its lower dimensional faces) and the cellular chain, and if we were to go on with the geometric concept of cell to any length, we would adopt another symbol, E_k^r, for the geometric cell, retaining σ_k^r only for the algebraic entity, the cellular chain. Most of our machinery is algebraic, however, so that no confusion should result in the long run.

For each r ($0 \leqq r \leqq n$) and complex K we define an abelian group $C^r(K)$ whose elements are the r-chains defined above, except that the σ_k^r's involved denote cells of K, and such that if $\sum c^k\sigma_k^r$ and $\sum d^k\sigma_k^r$ are two r-chains, then $\sum c^k\sigma_k^r + \sum d^k\sigma_k^r = \sum (c^k + d^k)\sigma_k^r$ where the coefficients in the right-hand member are reduced modulo 2. The identity of this group is obviously the chain all of whose coefficients are 0, and may itself be denoted by the symbol 0. Each chain is its own inverse.

If $C^r \in C^r(K)$, we call C^r an *r-chain* of K. And we call $C^r(K)$ the *group of r-chains* of K.

If $r > 0$, and $C^r = \sum c^k\sigma_k^r$, then the *boundary* of C^r, denoted by ∂C^r, is the chain

$$\sum c^k \partial\sigma_k^r = \sum_{m,k} c^k \eta_{kr}^m \sigma_m^{r-1} \tag{5.1a}$$

where the c's and η's are multiplied modulo 2. In other words, we deal with the polynomials which we call chains as algebraic polynomials with coefficients in the field of integers modulo 2. In particular, the distributive law holds, and as a consequence we have

5.2 Theorem. *The linear operator ∂ effects a homomorphism of the group $C^r(K)$, $r > 0$, into the group $C^{r-1}(K)$.*

The kernel of this homomorphism, which we denote by $Z^r(K)$, is the set of all $C^r \in C^r(K)$ such that $\partial C^r = 0$, and is called the *group of r-cycles* of K, each $Z^r \in Z^r(K)$ being called an *r-cycle* of K. The subgroup of $C^{r-1}(K)$ into which $C^r(K)$ is mapped by ∂ is called the *group of bounding $(r - 1)$-cycles*, and is denoted by $B^{r-1}(K)$. All chains ∂C^r are elements of $B^{r-1}(K)$, and since by our process of subdivision each $(r - 2)$-cell in $|\sigma_k^r|$ is itself a boundary cell of just two $(r - 1)$-cells of $|\sigma_k^r|$, it follows that

$$\partial(\partial\sigma_k^r) = \partial^2\sigma_k^r = 0. \tag{5.2a}$$

Hence by the linearity of ∂, application of ∂ to (5.1a) above gives

5.3 Theorem. *For every chain C^r, $r > 1$, $\partial^2 C^r = 0$. Hence, for $r > 0$, $B^r(K)$ is a subgroup of $Z^r(K)$.*

Before going further, let us consider the case $r = 0$. For present purposes, we make the convention that $\sum c^k\sigma_k^0$ is a *0-cycle* if and only if $\sum c^k = 0$. Evidently then a 0-chain is a 0-cycle if and only if an even number of its coefficients are $\neq 0$. Since the boundary of a 1-cell consists of exactly two 0-cells (because of the convexity condition), and since ∂ is linear, it follows that the boundary of a 1-chain is a 0-cycle, so that we have:

5.4 Theorem. *$B^0(K)$ is a subgroup of $Z^0(K)$.*

And we are now in the position to make the definition:

5.5 Definition. For each dimension r, the factor group $Z^r(K)/B^r(K)$ is called the *r-dimensional Betti group*, modulo 2, of K, and is denoted by $H^r(K)$.

Two elements of $Z^r(K)$ lie in the same coset of $H^r(K)$ if their difference ($=$ sum) modulo 2 is a bounding cycle. That a cycle $Z^r \in Z^r(K)$ is in the identity of $H^r(K)$, or what is equivalent, is the boundary of some $(r + 1)$-chain, is expressed by a relation

$$Z^r \sim 0 \qquad \text{on } K \tag{5.5a}$$

which is read "Z^r is homologous to zero on K," and called a *homology* or *homology relation*. (The "on K" may be omitted if no confusion results.) Thus two cycles Z^r, γ^r lie in the same coset of $H^r(K)$ if and only if

$$Z^r - \gamma^r \sim 0 \qquad \text{on } K. \tag{5.5b}$$

Relation (5.5b) may be written also

$$Z^r \sim \gamma^r \qquad \text{on } K, \tag{5.5c}$$

which is read "Z^r is homologous to γ^r on K." An alternative name for $H^r(K)$ is *r-dimensional homology group*, and it will usually be this name that we employ in later chapters, when we come to the so-called homology theory of general spaces.

A finite set of cycles $Z_1^r, \cdots, Z_m^r$ is called linearly independent relative to homology ($=$ lirh) on K if there exists no relation of the form

$$\sum c^i Z_i^r \sim 0 \qquad \text{on } K \tag{5.5d}$$

where the c^i's are 0's and 1's, but are not all 0. And correspondingly a set of elements of $H^r(K)$ are called lirh on K if no relation of the form (5.5d) exists where the Z_i^r are cycles in the respective elements. The number of linearly independent generators of $H^r(K)$, or what amounts to the same thing, the maximal number of r-cycles of K that are lirh on K, is called the *rth Betti number* (*modulo* 2) *of* K and is denoted by $p^r(K, 2)$.[8]

5.6 Definition. For any collection of cells K, the symbol $\|K\|$ will denote the point set consisting of all points that lie in cells of K. If C^r is a chain, we abbreviate $\| \, | C^r | \, \|$ to $\| C^r \|$. In particular, $\| \sigma^r \|$ is a closed r-cell.

5.7 Theorem. *If K is a complex, then a necessary and sufficient condition that $\| K \|$ be connected is that $p^0(K, 2) = 0$.*

Proof. Necessity. From the connectedness of $\overline{E}^1$ (I 12) and the fact that every two points of a cell may be joined by an arc of that cell, it follows (I 7.4) that a cell is connected. Hence if σ is any cell, then $\| \sigma \|$ is connected (I 7.2). Now if $\| K \|$ is connected, then $\| K^1 \|$, where K^1 is the complex obtained from K by deleting all but its 0- and 1-cells of K, is connected. For if $\| K^1 \| = A \cup B$ separate, and A_0, B_0 denote the collections of 0-cells in A, B respectively, then every 1-cell of K^1 has its boundary in A_0 alone or in B_0 alone. But this implies that $\| K \| = A' \cup B'$ separate, where $A' \supset A$, $B' \supset B$, and such that if σ is a cell of K, then $\| \sigma \| \subset A'$ or $\| \sigma \| \subset B'$.

[8] As we shall see later on, $H^r(K)$ is a vector space, and $p^r(K, 2)$ is its dimension.

Let σ_a^0, σ_b^0 be 0-cells of K^1. For each 0-cell σ^0 of K^1, let $St\sigma^0$ denote the point set consisting of σ^0 and all 1-cells σ^1 such that σ^0 is a boundary cell of σ^1. Then each $St\sigma^0$ is an open set and the collection $\{St\sigma^0\}$ covers $\| K^1 \|$. By Theorem I 12.3 there exists a simple chain $St\sigma_1^0, \cdots, St\sigma_i^0, \cdots, St\sigma_k^0$ from σ_a^0 to σ_b^0, where $\sigma_1^0 = \sigma_a^0$, $\sigma_k^0 = \sigma_b^0$. For each $i < k$, let σ_i^1 be a 1-cell in $St\sigma_i^0 \cap St\sigma_{i+1}^0$. Then $\partial \sum_{i=1}^{k-1} \sigma_i^1 = \sigma_1^0 + \sigma_k^0$. Since a 0-cycle Z^0 may always be put in the form $Z_1^0 + \cdots + Z_j^0 + \cdots + Z_m^0$, where each Z_j^0 is a binomial of the form $\sigma_c^0 + \sigma_d^0$, it follows that every Z^0 of K is homologous to zero on K. That is, $p^0(K, 2) = 0$.

The sufficiency follows from the fact that if every 0-cycle bounds, then in particular every 0-cycle formed from a pair of 0-cells σ_a^0, σ_b^0 bounds, which implies that every two 0-cells can be joined by a connected set consisting of a finite number of $\| \sigma^1 \|$'s (cf. Theorem I 7.4).

5.8 THEOREM. *The number $p^0(K, 2) + 1$ is the number of components of $\| K \|$, where K is any complex.*

PROOF. Denote the components of $\| K \|$ by $C_0, \cdots, C_k$; they are finite in number since each cell yields a connected point set and in particular k could not be greater than the number of cells in K. Let σ_i^0 be a 0-cell in C_i, $i = 0, \cdots, k$. Then the 0-cycles $Z_i^0 = \sigma_0^0 + \sigma_i^0$, $i = 1, \cdots, k$, are all nonbounding, and no linear combination of them bounds, so that $p^0(K, 2) \geqq k$. On the other hand, if Z^0 is an arbitrary 0-cycle of K, then $Z^0 = \gamma_0^0 + \cdots + \gamma_i^0 + \cdots + \gamma_k^0$, where $\| \gamma_i^0 \| \subset C_i$. If γ_i^0 is a 0-cycle, then $\gamma_i^0 \sim 0$ by Theorem 5.7. If not, then $\gamma_i^0 + \sigma_i^0$ is a 0-cycle, and $\gamma_i^0 + \sigma_i^0 \sim 0$. In either case, then,

$$\gamma_i^0 + c^i\sigma_i^0 \sim 0 \qquad \text{on } K \qquad (c^i = 0 \text{ or } 1). \tag{5.8a}$$

Adding relations (5.8a), we get

$$Z^0 + \sum c^i\sigma_i^0 \sim 0 \qquad \text{on } K. \tag{5.8b}$$

Since $Ki(Z^0) = 0$,[9] and since $Ki(Z^0 + \sum c^i\sigma_i^0)$ must be 0 inasmuch as $Z^0 + \sum c^i\sigma_i^0$ is a bounding chain, it follows that $\sum c^i = 0$. But $\sum c^i = 0$ implies that

$$\sum_{i=1}^{k} c^iZ_i^0 = \sum_{i=0}^{k} c^i\sigma_0^0 + \sum_{i=0}^{k} c^i\sigma_i^0 = \sum_{i=0}^{k} c^i\sigma_i^0. \tag{5.8c}$$

Therefore, from (5.8b) and (5.8c) follows that

$$Z^0 \sim \sum c^iZ_i^0 \qquad \text{on } K,$$

implying that the cycles Z_i^0 form a maximal set of 0-cycles that are lirh on K. Hence $p^0(K, 2) \leqq k$.

5.9 Returning now to the sequence of subdivisions $s_0, \cdots, s_j, \cdots$, suppose that K is a complex of some s_j. If in the passage to s_{j+1} a cell of K is the cell

[9] By $Ki(Z^0)$ we denote the sum (modulo 2 in the present case) of the coefficients of Z^0, where Z^0 is any 0-chain. See V 2.1.

which is subdivided in order to obtain s_{j+1}, then K no longer exists in s_{j+1}. However, there is a complex K' in s_{j+1} such that $\| K' \| = \| K \|$—it is a *subdivision of* K and contains the new cells introduced by the subdivision of s_j, as well as the other original cells of K.

5.10 THEOREM. *If K is a complex of s_j, and K' is its subdivision in s_{j+1}, then $p^r(K, 2) = p^r(K', 2)$, $r = 0, 1, \cdots, n$.*

PROOF. If $K = K'$, the result is trivial. We may suppose then, that in the passage from K to K' a cell σ^k of K is subdivided by a new cell σ^{k-1} into two new cells σ_1^k, σ_2^k. The only numbers $p^r(K, 2)$ that can be affected are $p^k(K, 2)$, $p^{k-1}(K, 2)$ and $p^{k-2}(K, 2)$. The first number can be affected by a change in the k-cycles. If Z^k is a k-cycle of K such that $\sigma^k \notin | Z^k |$, then Z^k is a cycle of K' also and bounds on K' according as it bounded or not on K. If $\sigma^k \in | Z^k |$, then Z^k may be written in the form $\sigma^k + Z_1^k$; and the chain $\sigma_1^k + \sigma_2^k + Z_1^k = \gamma^k$ is a cycle of K' which bounds or not according as Z^k bounds or not on K. And since no cycle of K' could contain one of the cells σ_1^k, σ_2^k without containing the other, every k-cycle of K' may be obtained from a cycle of K in the above manner. Consequently $p^k(K, 2) = p^k(K', 2)$.

A cycle Z^{k-1} of K is still a cycle of K', and bounds in K' according to whether it bounded in K or not. But of course K' may contain new cycles of the form $\gamma^{k-1} = \sigma^{k-1} + \gamma_1^{k-1}$. However, consider the cycle

$$Z^{k-1} = \gamma^{k-1} + \partial\sigma_1^k. \tag{5.10a}$$

Since $\partial\sigma_1^k = \sigma^{k-1} + \cdots$, the cell σ^{k-1} is not in $| Z^{k-1} |$. Now relation (5.10a) may be written in the form

$$Z^{k-1} - \gamma^{k-1} = \partial\sigma_1^k,$$

implying that

$$Z^{k-1} \sim \gamma^{k-1} \qquad \text{on } K'.$$

Hence every new $(k - 1)$-cycle is homologous to an old cycle of K, so that the number of $(k - 1)$-cycles that are lirh on K' is the same as on K. Hence $p^{k-1}(K, 2) = p^{k-1}(K', 2)$.

The only way $p^{k-2}(K, 2)$ could be affected is for a new bounding relation to be set up among the $(k - 2)$-cycles of K because of the new cell σ^{k-1}. This would be equivalent to a relation of the form

$$\partial(\sigma^{k-1} + C^{k-1}) = Z^{k-2}, \tag{5.10b}$$

where C^{k-1} and Z^{k-2} are chains of K. However, if we add to the relation (5.10b) the relation (cf. Theorem 5.3)

$$\partial(\partial\sigma_1^k) = 0,$$

we get

$$\partial[(\sigma^{k-1} + \partial\sigma_1^k) + C^{k-1}] = Z^{k-2},$$

where the chain in the brackets does not contain σ^{k-1}. Hence no new bounding relation is set up by the introduction of the cell σ^{k-1}.

As a corollary of Theorem 5.10 we have:

5.11 COROLLARY. *The Betti numbers $p^r(s_i, 2)$ are all 0 for $r < n$, and $p^n(s_i, 2) = 1$.*

(It is only necessary to observe that in the elementary subdivision s_0 every r-cycle, for $r < n$, is the boundary of a cellular chain, and that the only n-cycle is of the form $\sigma_1^n + \sigma_2^n$ and is necessarily nonbounding since there are no cells of dimension higher than n.)

5.12 *Open subsets of S^n.* Let U be an open subset of S^n, and let U_i denote the complex consisting of the set of all cells of s_i which with their faces lie in U. Hereafter, if C^r and L^r are chains of s_i and s_{i+k}, respectively, such that $\| C^r \| = \| L^r \|$, then we call L^r a *subdivision* of C^r. If C^r is a chain of some U_i, then we let it be an element of a set which we denote by $C^r(U)$, making the convention that the subdivisions of C^r in s_{i+k} $(k = 1, 2, \cdots)$ are the same element C^r of $C^r(U)$. From considerations such as those involved in the proof of Theorem 5.10 it follows that a chain which is a cycle of U_i will still be a cycle in every U_{i+k}; and if such a cycle bounds in U_i, it continues to bound in U_{i+k} for all k. Hence we may form sets $Z^r(U)$ consisting of all cycles of the U_i's, and $B^r(U)$ consisting of all cycles that bound in some U_{i+k}. (A cycle of U_i may fail to bound in U_i, yet bound in some U_{i+k} and therefore be an element of $B^r(U)$.)

The sets $C^r(U)$, $Z^r(U)$, $B^r(U)$ become groups if we let the sum of two elements be the sum determined by corresponding elements of the groups $C^r(U_j)$ for j large enough so that $C^r(U_j)$ contains such elements, of course. They are the group of r-chains, group of r-cycles and group of bounding r-cycles, respectively, of the open set U (all modulo 2). And the factor group $Z^r(U)/B^r(U)$ is the rth Betti, or homology, group, $H^r(U, 2)$, of the open set U. The rank of the latter group, which we denote by $p^r(U, 2)$—i.e., the maximum number of r-cycles of U that are lirh in U, we call the rth Betti number of U (modulo 2). It may of course be infinite, in which case we write $p^r(U, 2) = \infty$. In particular, we prove:

5.13 THEOREM. *The number $p^0(U, 2) + 1$ is the number of components of U.*

PROOF. If σ_a^0 and σ_b^0 are 0-cells of U_i that lie in the same component C of U, then there will exist U_{i+k} such that $\sigma_a^0 + \sigma_b^0$ lies in a single component of $\| U_{i+k} \|$, and hence $\sigma_a^0 + \sigma_b^0$ is homologous to zero thereon by Theorem 5.7. For by the following lemma there exists a broken line L joining σ_a^0 and σ_b^0, and just as soon as the value of j is so great that all cells of s_j are of diameter less than one-half the distance from L to $S^n - C$,[10] all points of L will lie in $\| U_{i+k} \|$. The remainder of the proof is based on considerations such as those used in proving Theorems 5.7 and 5.8 above.

[10] If A, B are subsets of a metric space, then by the distance from A to B—$\rho(A, B)$—we mean glb $\{\rho(x, y) \mid (x \in A) \;\&\; (y \in B)\}$.

5.14 LEMMA. *If x and y are points of the same component C of an open subset U of S^n, then x and y are the end points of an arc—as a matter of fact a broken line—of C.*

PROOF. Since S^n is lc, it follows from Theorem 3.1 above that C is open. Consequently $p \in C$ implies that $\rho(p, S^n - C)$ is positive. For each such point p let $U(p) = S(p, \rho(p, S^n - C)/2)$. By Theorem I 12.3 there exist $p_1, \cdots, p_i, \cdots, p_m$ such that the sets $U(p_i)$ form a simple chain of sets $U(p)$ from x to y. With $p_0 = x$ and $p_{m+1} = y$, let L_i denote the straight line interval whose end points are p_i and p_{i+1}. Then $\bigcup_{i=0}^m L_i$ is the required broken line.

5.15 THEOREM. *If Z^{n-1} ($\neq 0$) is a cycle of s_i, then there are exactly two chains C^n, K^n of s_i such that $\partial C^n = Z^{n-1} = \partial K^n$; furthermore, $\| C^n \| \cap \| K^n \| = \| Z^{n-1} \|$.*

PROOF. Since $p^{n-1}(s_i, 2) = 0$ by Corollary 5.11, there exists a chain C^n of s_i such that $\partial C^n = Z^{n-1}$. Denoting by Z^n the cycle of s_i which is the subdivision of the original cycle based on the two n-cells of s_0, we have $\partial Z^n = 0$, and hence $\partial(C^n + Z^n) = Z^{n-1}$. Let $K^n = C^n + Z^n$ (recall that all sums are reduced modulo 2). As $| C^n |$ and $| K^n |$ have no common n-cells and their union contains all the n-cells of s_i, the points common to $\| C^n \|$ and $\| K^n \|$ must all lie in lower-dimensional cells of s_i. By the mode of construction of s_i, each $(n - 1)$-cell is a face of exactly two n-cells. It follows that the points common to $\| C^n \|$ and $\| K^n \|$ must lie on the cells of $| Z^{n-1} |$.

Were there a third chain, say L^n, such that $\partial L^n = Z^{n-1}$, then $C^n + L^n$ and $K^n + L^n$ would be n-cycles of s_i that are lirh, contradicting the fact that $p^n(s_i, 2) = 1$.

5.16 COROLLARY. *If $x \in S^n$, and $Z^{n-1} \in Z^{n-1}(S^n - x, 2)$, then $Z^{n-1} \in B^{n-1}(S^n - x, 2)$.*

5.17 COROLLARY. *If a subset M of S^n consists of exactly two points, then $p^{n-1}(S^n - M, 2) = 1$.*

PROOF. Denote the two points which constitute M by x_1, x_2. Let $\rho(x_1, x_2) = \eta$, and choose j so that all cells of s_j are of diameter $< \eta/2$. Let s_{j1} be that subcomplex of s_j consisting of all closed n-cells that contain x_1. Let C^n be the chain such that $| C^n | = s_{j1}$. Then by Theorem 5.15 the cycle $Z^{n-1} = \partial C^n$ is nonbounding in $S^n - M$, and hence $p^{n-1}(S^n - M, 2) \geqq 1$.

Suppose Z_1^{n-1} and Z_2^{n-1} are cycles that are lirh in $S^n - M$. Denote the respective chains C^n which these bound, such that $\| C^n \|$ contains x_1, by C_1^n and C_2^n. For j great enough, the subcomplex s_{j1} defined above will be such that $\| s_{j1} \| \cap \| Z_i^{n-1} \| = 0$, $i = 1, 2$. Then if $Z^{n-1} = \partial C^n$ as in the preceding paragraph, we have, by addition to the relations $Z_i^{n-1} = \partial C_i^n$, that

$$Z^{n-1} + Z_i^{n-1} = \partial(C^n + C_i^n), \text{ mod } 2, \qquad i = 1, 2. \tag{5.17a}$$

Now $x_1 \notin \| C^n + C_i^n \|$, hence if we add relations (5.17a) we get

$$Z_1^{n-1} + Z_2^{n-1} = \partial(C_1^n + C_2^n), \text{ mod } 2,$$

where $x_i \notin \| C_1^n + C_2^n \|$. That is, $Z_1^{n-1} \sim Z_2^{n-1}$ in $S^n - (x_1 \cup x_2)$, contradicting the supposition that the cycles are lirh. Hence $p^{n-1}(S^n - M, 2) \leqq 1$.

5.18 Alexander Addition Theorem. *If A and B are closed subsets of S^n, $Z^r \in Z^r(S^n - A - B)$, $r < n - 1$, $C_1^{r+1} \in C^{r+1}(S^n - A)$, $C_2^{r+1} \in C^{r+1}(S^n - B)$, $L^{r+2} \in C^{r+2}(S^n - A \cap B)$ such that $\partial C_1^{r+1} = Z^r = \partial C_2^{r+1}$, $\partial L^{r+2} = C_1^{r+1} + C_2^{r+1}$, then $Z^r \sim 0$ in $S^n - A - B$. For $r = n - 1$, if $A \cap B \neq 0$ and C_1^{r+1}, C_2^{r+1} exist as before, then again $Z^r \sim 0$ in $S^n - A - B$.*

Proof. The sets $A' = \| L^{r+2} \| \cap A$, $B' = \| L^{r+2} \| \cap B$ are disjoint closed sets, so that $\eta = \rho(A', B') > 0$. Select j so that all cells of s_j are of diameter $<\eta/2$. Let L' be that chain such that L' consists of all closed $(r + 2)$-cells of $| L^{r+2} |$ meeting A'. Then $\| L^{r+2} + L' \| \subset S^n - A$. Let $\partial L' = \gamma^{r+1}$. Then we may write the relation

$$\partial(L^{r+2} + L') = (C_2^{r+1} + \gamma^{r+1}) + C_1^{r+1}. \tag{5.18a}$$

Now by Theorem 5.3, $\partial(C_2^{r+1} + \gamma^{r+1}) + \partial C_1^{r+1} = 0$, and by hypothesis, $\partial C_1^{r+1} = Z^r$. Adding these two relations we have that

$$\partial(C_2^{r+1} + \gamma^{r+1}) = Z^r. \tag{5.18b}$$

But $\| L^{r+2} + L' \| \subset S^n - A$, hence by (5.18a), $C_2^{r+1} + \gamma^{r+1} \in C^{r+1}(S^n - A, 2)$. By the choice of s_j, $\| \gamma^{r+1} \| \subset S^n - B$, and by hypothesis $C_2^{r+1} \subset S^n - B$. Thus $C_2^{r+1} + \gamma^{r+1} \in C^{r+1}(S^n - A - B, 2)$, so that by (5.18b), we may conclude that $Z^r \sim 0$ in $S^n - A - B$.

In case $r = n - 1$, then $C_1^n = C_2^n$—otherwise $| C_1^n + C_2^n | = s_i$ by Theorem 5.15, and either $\| C_1^n \|$ or $\| C_2^n \|$ meets $A \cap B$.

We may now prove, as a corollary of Theorem 5.18:

5.19 Theorem. *The n-sphere, S^n, $n > 1$, has all of the Properties* I, I′, II, III, IV, V, V′, of §4.

Proof. By Theorem 4.12 it will be sufficient to prove that S^n has Property I. Hence suppose A and B are disjoint closed subsets of S^n, and $x, y \in S^n - A - B$ such that neither A nor B separates x and y in S^n. Choose $\eta > 0$ so that $S(x, \eta) \cap (A \cup B) = 0$ and $S(y, \eta) \cap (A \cup B) = 0$, and j so that all cells of s_j are of diameter less than $\eta/2$. Then there exist 0-cells σ_x^0, σ_y^0 of s_j in $S(x, \eta)$, $S(y, \eta)$ respectively. As in the proof of Theorem 5.13, we may show that, since x and y—hence σ_x^0 and σ_y^0—lie in the same component of $S^n - A$, there exists $C_1^1 \in C^1(S^n - A, 2)$ such that $\partial C_1^1 = \sigma_x^0 + \sigma_y^0$. Similarly there exists $C_2^1 \in C^1(S^n - B, 2)$ such that $\partial C_2^1 = \sigma_x^0 + \sigma_y^0$. By Theorem 5.11, $C_1^1 + C_2^1 \sim 0$ in $S^n = S^n - A \cap B$. By Theorem 5.18, it follows that $\sigma_x^0 + \sigma_y^0 \sim 0$ in $S^n - A - B$. But this implies that σ_x^0 and σ_y^0 are in the same component of $S^n - A - B$—which in turn implies that x and y are in the same component of $S^n - A - B$.

Remark. In view of Theorem 5.19 it is evidently possible to state: If $x, y \in S^n$, $n > 1$, which are not separated by any one of a finite collection of

closed sets M_i, then $\bigcup M_i$ fails to separate x and y. Also, we have the following important lemma:

5.20 LEMMA. *If $x, y \in S^n$ are separated in S^n by a closed set K, then x and y are separated by a subcontinuum of K.*

For if $S^n - K = A \cup B$ separate, where $x \in A$, $y \in B$, then by Theorem 5.19 and Property V′, there exists a continuum $M \subset S^n - (A \cup B) = K$ such that M separates x and y in S^n.

5.21 THEOREM. *Let M be a homeomorph in S^n of $\| \sigma^r \|$, where σ^r is a cell obtained by subdivision of some S^k. Then $p^i(S^n - M, 2) = 0$ for all i.*

PROOF. The theorem is true for $r = 0$. For let Z^i be a cycle of some s_j in $S^n - M$. If $i = n - 1$, then $Z^i \sim 0$ in $S^n - M$ by Corollary 5.16. There are no cycles in s_j of dimension $> n - 1$ which lie in $S^n - M$. For $i < n - 1$, we may argue as follows: We may suppose that the point M always lies in an n-cell of the subdivisions of S^n. Let $\eta = \rho(M, \| Z^i \|)$, and let h be such that all cells of s_{j+h} are of diameter $<\eta/2$. The cell, σ^n, of s_{j+h} that contains M has an $(n - 1)$-sphere, K, as boundary. Let C^{i+1} be a chain of s_{j+h} such that $\partial C^{i+1} = Z^i$. If $\| C^{i+1} \| \cap \| \sigma^n \| = 0$, then $Z^i \sim 0$ in $S^n - M$. Otherwise, since $i + 1 < n$, the intersection $\| C^{i+1} \| \cap \| \sigma^n \| \subset K$, so that we still have $Z^i \sim 0$ in $S^n - M$.

Suppose the theorem proved for $r = m - 1$. Let Z^i be as above, and suppose $Z^i \not\sim 0$ in $S^n - M$. Now $M = A \cup B$, where A and B are the homeomorphs of two m-cells (and their faces) into which σ^m is subdivided in S^k by an $(m - 1)$-cell σ^{m-1}; let the homeomorph of the latter be denoted by C. Then $Z^i \not\sim 0$ in $S^n - A$, or $Z^i \not\sim 0$ in $S^n - B$—for otherwise by Theorem 5.18 (and the induction assumption) we would have $Z^i \sim 0$ in $S^n - M$. By repetition of this process, we would obtain a sequence of closed subsets $M_1, M_2, \cdots$, homeomorphs of m-cells (and their faces) in subdivisions of σ^m, such that each contains the following and the common part is a single point p (cf. Theorem I 12.8)—and where $Z^i \not\sim 0$ in $S^n - M_q$ for $q = 1, 2, \cdots$. But Z^i bounds in $S^n - p$ on some s_{j+h}, and hence in $S^n - M_q$ for q great enough. It follows that $Z^i \sim 0$ in $S^n - M$.

The Jordan-Brouwer separation theorem. We shall now prove the theorem on the separation of the n-sphere by an $(n - 1)$-sphere imbedded therein—a special case of which $(n = 2)$ is the classical Jordan Curve Theorem. It is just as easy, however, first to prove the following:

5.22 THEOREM. *Let K be an r-sphere imbedded in S^n. Then the Betti numbers of K and $S^n - K$ satisfy the following duality relations:*

(5.22a) $$p^r(K, 2) = p^{n-r-1}(S^n - K, 2) = 1,$$

(5.22b) $$p^s(K, 2) = p^{n-s-1}(S^n - K, 2) = 0, \qquad s \neq r.$$

PROOF. If $M \subset S^n$ and Z^i is a cycle of $S^n - M$ which fails to bound in $S^n - M$, let us express this fact by saying that Z^i *links* M in S^n. We are to prove that there is only one cycle lirh in $S^n - K$, and that this cycle is of dimension $n - r - 1$; in other words, that only one cycle, of dimension $n - r - 1$, links K, every other $(n - r - 1)$-cycle being homologous to some "multiple" (0 or 1) of this cycle in $S^n - M$. (The relations between the Betti numbers of K and $S^n - K$ then follow from Corollary 5.11.)

If $r = 0$, then K is a pair of points and the theorem is true for this case; cf. Corollary 5.17, and the proof of Theorem 5.21 above. We may then proceed by induction, with $r > 0$, supposing the theorem to be true for the case where K is of dimension less than r.

With K an r-sphere imbedded in S^n, $r > 0$, let $K = A \cup B$, where A and B are the homeomorphs of the closures of r-cells whose common part is an $(r - 1)$-sphere imbedded in S^n. For $s \neq n - r - 1$, let Z^s be a cycle of $S^n - K$. By Theorem 5.21 there exist chains $C_1^{s+1} \in C^{s+1}(S^n - A, 2)$, $C_2^{s+1} \in C^{s+1}(S^n - B, 2)$, such that $\partial C_1^{s+1} = Z^s = \partial C_2^{s+1}$. By the induction hypothesis, $C_1^{s+1} + C_2^{s+1}$ does not link $A \cap B$ since $s + 1 \neq n - (r - 1) - 1$. Hence by Theorem 5.18, $Z^s \sim 0$ in $S^n - K$. This proves relation (5.22b).

By the induction hypothesis there exists a cycle Z^{n-r} which links $A \cap B$. Then the set $A' = \| Z^{n-r} \| \cap A \neq 0$, else by Theorem 5.21, $Z^{n-r} \sim 0$ in $S^n - A$ and a fortiori in $S^n - A \cap B$. Similarly the set $B' = \| Z^{n-r} \| \cap B \neq 0$. Let $\eta = \rho(A', B')$, and let j be so great that all cells of s_j are of diameter $< \eta/2$. In s_j let L^{n-r} be that chain such that $| L^{n-r} |$ consists of all $(n - r)$-cells (and their faces) which have at least one face meeting B'. Then we have relations:

$$\text{(5.22c)} \qquad \begin{aligned} \partial L^{n-r} &= Z^{n-r-1} \qquad \text{in } S^n - A, \\ \partial(Z^{n-r} + L^{n-r}) &= Z^{n-r-1} \qquad \text{in } S^n - B. \end{aligned}$$

Now if there exists $M^{n-r} \in C^{n-r}(S^n - K, 2)$ such that $\partial M^{n-r} = Z^{n-r-1}$, then by Theorem 5.21 we would have relations

$$\text{(5.22d)} \qquad \begin{aligned} L^{n-r} + M^{n-r} &\sim 0 \qquad \text{in } S^n - A, \text{ hence in } S^n - A \cap B, \\ (Z^{n-r} + L^{n-r}) + M^{n-r} &\sim 0 \qquad \text{in } S^n - B, \text{ hence in } S^n - A \cap B. \end{aligned}$$

But adding relations (5.22d) we get $Z^{n-r} \sim 0$ in $S^n - A \cap B$, contradicting the fact that Z^{n-r} links $A \cap B$. Hence we conclude that Z^{n-r-1} links K.

Suppose some other cycle γ^{n-r-1} also links K. Then $Z^{n-r-1} + \gamma^{n-r-1} \sim 0$ in $S^n - K$. For by Theorem 5.21 there exist relations $\partial N_1^{n-r} = \gamma^{n-r-1}$ in $S^n - A$, $\partial N_2^{n-r} = \gamma^{n-r-1}$ in $S^n - B$, and hence, applying Theorem 5.18 as before, the cycle $N_1^{n-r} + N_2^{n-r}$ links $A \cap B$. But by the induction hypothesis

$$\text{(5.22e)} \qquad N_1^{n-r} + N_2^{n-r} \sim Z^{n-r} \qquad \text{in } S^n - A \cap B.$$

Note, then, that we have the following relations [cf. relations (5.22c)]:

$$\text{(5.22f)}\qquad \begin{aligned} \partial(L^{n-r} + N_1^{n-r}) &= Z^{n-r-1} + \gamma^{n-r-1} \qquad \text{in } S^n - A, \\ \partial[(Z^{n-r} + L^{n-r}) + N_2^{n-r}] &= Z^{n-r-1} + \gamma^{n-r-1} \qquad \text{in } S^n - B; \end{aligned}$$

and, as a consequence of relation (5.22e), a relation

$$\text{(5.22g)}\qquad \partial C^{n-r+1} = Z^{n-r} + N_1^{n-r} + N_2^{n-r} \qquad \text{in } S^n - A \cap B.$$

But relations (5.22f), (5.22g) imply that $Z^{n-r-1} + \gamma^{n-r-1} \sim 0$ in $S^n - K$, by Theorem 5.18. That is, every other $(n - r - 1)$-cycle of $S^n - K$ is linearly dependent on Z^{n-r-1} in the sense of homology and consequently relation (5.22a) holds.

5.23 JORDAN-BROUWER SEPARATION THEOREM. *If K is an $(n - 1)$-sphere imbedded in S^n, then $S^n - K$ consists of just two disjoint domains of which K is the common boundary.*

PROOF. That $S^n - K$ has just two components A and B follows from relation (5.22a) and Theorem 5.13. To show that K is the common boundary of A and B, let $x \in K$ and $\epsilon > 0$. Now $K = f(S^{n-1})$, where f is a homeomorphism and S^{n-1} is the sphere of which K is the homeomorph. In the subdivisions of S^{n-1}, let σ^{n-1} be a cell containing $f^{-1}(x)$, small enough so that the set $D = f(\sigma^{n-1})$ is of diameter $< \epsilon$—possible because of the continuity of f (cf. I 5.1). Then $D' = K - D = f(S^{n-1} - \sigma^{n-1})$ is an $(n - 1)$-cell and its boundary imbedded in S^n, and consequently $p^0(S^n - D', 2) = 0$ by Theorem 5.21. If $a \in A$, $b \in B$, then there exists an arc L from a to b in $S^n - D'$ by Lemma 5.14. If a' and b' are respectively the first and last points of $K \cap L$ on L in the order from a to b, then a' is a limit point of A in $S(x, \epsilon)$ and b' is a limit point of B in $S(x, \epsilon)$. Since ϵ was arbitrary, it follows that x is itself a limit point of both A and B.

5.24 COROLLARY (JORDAN CURVE THEOREM). *If K is the homeomorph of a circle in S^2, then $S^2 - K$ consists of exactly two disjoint domains of which K is the common boundary.*

REMARK. Since the euclidean n-space, E^n, is related to S^n by the fact that $E^n = f(S^n - x)$ where x is an arbitrary point of S^n and f is a homeomorphism, it follows at once that in Theorem 5.23, "E^n" may be substituted for "S^n", and similarly in Theorem 5.24 "E^2" may be substituted for "S^2".

ADDITIONAL REMARKS. Theorem 5.22 is a special case of the Alexander Duality Theorem, mentioned in the historical remarks, chap. I, §6. Since we shall later prove this duality theorem for the very general case of the spherelike generalized manifolds, we need not go any further in this direction at present. As we stated above, we reverted to the matter at present chiefly for the proofs of such theorems as 5.19 and 5.24. Moreover, the reader unfamiliar with algebraic topology should be greatly aided, when we come to the homology

theory of general spaces, by the previous reading of the special homology theory which we have just described in this section.

Before proceeding with the discussion of Peano spaces, we shall prove other theorems which follow quickly from the above theorems and are needed in the immediate sequel.

5.25 THEOREM. *In order that a subset M of S^n should be connected, it is necessary that $p^{n-1}(S^n - M, 2) = 0$. And if M is closed, this condition is also sufficient.*

PROOF. Let M be a connected subset of S^n, and $Z^{n-1} \in Z^{n-1}(S^n - M, 2)$. By Theorem 5.15, Z^{n-1} bounds exactly two chains C^n, K^n, and these are such that $\| C^n \| \cup \| K^n \| = S^n$ and $\| C^n \| \cap \| K^n \| = \| Z^{n-1} \|$. Since $M \cap \| Z^{n-1} \| = 0$, either $M \cap \| C^n \| = 0$ or $M \cap \| K^n \| = 0$, since M is connected. In either case $Z^{n-1} \sim 0$ in $S^n - M$.

Conversely, suppose M is a closed subset of S^n and that $M = A \cup B$ separate. Let s_i be a subdivision of S^n whose cells are all of diameter $< \rho(A, B)/2$. Let C^n be the n-chain such that $\| C^n \|$ is the union of those closed n-cells that meet A. Then $\partial C^n = Z^{n-1} \in Z^{n-1}(S^n - M, 2)$, and by Theorem 5.15 there is only one other n-chain, K^n, such that $\partial K^n = Z^{n-1}$, etc., and of necessity $B \subset \| K^n \|$. Hence $p^{n-1}(S^n - M, 2) \neq 0$, since $Z^{n-1} \nsim 0$ in $S^n - M$.

5.26 THEOREM. *If A and B are closed subsets of S^n and $Z^r \in Z^r(S^n - A \cup B, 2)$ links neither A nor B, then if no $(r + 1)$-cycle links $A \cap B$, the cycle Z^r bounds in $S^n - A \cup B$.*

(Theorem 5.26 is an immediate consequence of Theorem 5.18.)

5.27 THEOREM. *If A and B are closed subsets of S^n neither of which is linked by an $(r + 1)$-cycle, but such that $A \cap B$ is linked by an $(r + 1)$-cycle, then $A \cup B$ is linked by an r-cycle.*

PROOF. Let Z^{r+1} link $A \cap B$, and let s_i be such that the subdivision of Z^{r+1} therein has the property that chains C_1^{r+1}, C_2^{r+1} exist such that $C_1^{r+1} + C_2^{r+1} = Z^{r+1}$, and $\| C_1^{r+1} \| \cap A = 0 = \| C_2^{r+1} \| \cap B$. Let $\partial C_1^{r+1} = Z^r = \partial C_2^{r+1}$, and suppose that there exists $L^{r+1} \in C^{r+1}(S^n - A \cup B, 2)$ such that $\partial L^{r+1} = Z^r$. Then by hypothesis,

$$\begin{aligned} &C_1^{r+1} + L^{r+1} \sim 0 \text{ in } S^n - A, \text{ hence in } S^n - A \cap B, \\ (5.27a)\qquad & \\ &C_2^{r+1} + L^{r+1} \sim 0 \text{ in } S^n - B, \text{ hence in } S^n - A \cap B. \end{aligned}$$

Adding relations (5.27a), we get that $Z^{r+1} \sim 0$ in $S^n - A \cap B$, contradicting the fact that Z^{r+1} links $A \cap B$. We conclude, then, that Z^r must link $A \cup B$.

As a consequence of Theorems 5.26 and 5.27 we have:

5.28 THEOREM. *If A and B are closed subsets of S^n, neither of which is linked by an r- or $(r + 1)$-cycle, then a necessary and sufficient condition that $A \cup B$ be linked by an r-cycle is that $A \cap B$ be linked by an $(r + 1)$-cycle.*

An interesting special case of Theorem 5.28 is the following:

5.28a THEOREM. *If A and B are subcontinua of S^2 neither of which separates S^2, then a necessary and sufficient condition that $A \cup B$ separate S^2 is that $A \cap B$ be not connected.*

[Cf. Theorems 5.13 and 5.25.]

Now whereas Theorem 5.28a deals with separation of an arbitrary pair of points of S^2, the following theorem, on the other hand, treats of a specified pair:

5.29 THEOREM. *If x and y are points of S^2 which are not separated by either of the closed sets A and B, and $A \cap B$ is connected, then x and y are not separated by $A \cup B$.*

Theorem 5.29 is a corollary of Theorem 5.25 and 5.26.

It will also be noted that the fact that S^n has Property IV is a corollary of Theorem 5.26. However, as a corollary of Theorem 5.29 we have the stronger result for S^2:

5.30 COROLLARY. *If M is the common boundary of two disjoint domains in S^2, then no closed and connected subset of M disconnects M.*

5.31 DEFINITION. An open subset U of S^n is called *uniformly locally connected in dimension r*—abbreviated *r-ulc*—if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $Z^r \in Z^r(U, 2)$ and the diameter of Z^r is $< \delta$, then there exists $C^{r+1} \in C^{r+1}(U, 2)$ such that $\partial C^{r+1} = Z^r$ and the diameter of $\| C^{r+1} \|$ is $< \epsilon$.

In connection with the case $r = 0$ of Definition 5.31 we also give the following definition:

5.32 DEFINITION. A metric space M is called *uniformly locally connected* (*ulc*) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in M$ and $\rho(x, y) < \delta$, then M contains a connected set N such that $x, y \in N$ and the diameter of N is $< \epsilon$.

Since the method of proof has been well demonstrated above in the proofs of such theorems as Theorem 5.7, we leave to the reader the proof of the following theorem:

5.33 THEOREM. *In order that an open subset U of S^n should be ulc it is necessary and sufficient that U be 0-ulc.*

5.34 LEMMA. *In S^n let U be an open set and $Z^r \in Z^r(U, 2)$ such that $Z^r \sim 0$ in $S^n - B$, where B is the boundary of U. Then $Z^r \sim 0$ in U.*

PROOF. Since we have $\partial C^{r+1} = Z^r$ in $S^n - B$, we may write

$$\partial(C_1^{r+1} + C_2^{r+1}) = Z^r, \tag{5.34a}$$

where $\| C_1^{r+1} \| \subset U$ and $\| C_2^{r+1} \| \subset S^n - \overline{U}$. But $\| Z^r \| \subset U$, $\partial(C_1^{r+1} + C_2^{r+1}) = \partial C_1^{r+1} + \partial C_2^{r+1}$, and Z^r is a cycle of U; it follows from (5.34a) that $\partial C_1^{r+1} = Z^r$.

5.35 THEOREM. *Let K be a k-sphere imbedded in S^n. Then the open set $S^n - K$ is r-ulc for all dimensions $r \neq n - k - 1$, and also in case K is an $(n - 1)$-sphere, the individual complementary domains of K are 0-ulc.*

PROOF. Suppose that $U = S^n - K$ is not r-ulc. Then there exists $\epsilon > 0$ such that for every integer m there exists a $Z^r_m \in Z^r(U, 2)$ such that $\delta(\|Z^r_m\|) < 1/m$ but such that there exists no $C^{r+1} \in C^{r+1}(U, 2)$ for which $\partial C^{r+1} = Z^r_m$ and $\delta(\|C^{r+1}\|) < 2\epsilon$. Let $x_m \in \|Z^r_m\|$. Then since S^n is compact. the set $\{x_m\}$ has a limit point x, which evidently must be a point of K.

Now $K = f(S^k)$, where f is a homeomorphism. Let σ^k be a cell of S^k containing $f^{-1}(x)$ and such that the set $D = f(\sigma^k) \subset S(x, \epsilon)$. There exists $\delta > 0$ such that $K \cap S(x, \delta) \subset D$, and there exists an m such that $\|Z^r_m\| \subset S(x, \delta)$. Denote $K - D$ by D', and $F(x, \epsilon)$ by F.

Since $r \neq n - k - 1$, we have from Theorem 5.22:

$$\partial C_1^{r+1} = Z^r_m \quad \text{in } S^n - K; \text{ a fortiori in } S^n - \overline{D}. \tag{5.35a}$$

Also, we have

$$\partial C_2^{r+1} = Z^r_m \quad \text{in } S(x, \delta); \text{ a fortiori in } S^n - (D' \cup F). \tag{5.35b}$$

Evidently $\overline{D} \cap (D' \cup F) = \overline{D} \cap D'$, and since the latter is a subset of D', which is the homeomorph in S^n of a closed cell, we have by Theorem 5.21 that

$$C_1^{r+1} + C_2^{r+1} \sim 0 \quad \text{in } S^n - (\overline{D} \cap D'). \tag{5.35c}$$

Relations (5.35a)—(5.35c) imply, according to Theorem 5.18, that $Z^r_m \sim 0$ in $S^n - [\overline{D} \cup (D' \cup F)] = S^n - (K \cup F)$, and from Lemma 5.34 we conclude that $Z^r_m \sim 0$ in $U \cap S(x, \epsilon)$. Thus the supposition that U is not r-ulc leads to a contradiction.

Now let $k = n - 1$ and $r = 0$, and let U be one of the domains complementary to K. We may then proceed as above to obtain x, $\{Z^0_m\}$, $\epsilon > 0$, etc. Relation (5.35a) is obtainable since $Z^0_m \sim 0$ in U (cf. Lemma 5.14 and the proof of Theorem 5.13), and relations (5.35b), (5.35c) are obtainable as before.

5.36 DEFINITION. A point a is called *arcwise accessible* from a point set B if $b \in B$ implies the existence of an arc T with end points a and b such that $T - a \subset B$. If A is a point set every point of which is arcwise accessible from some point set B, then we call A arcwise accessible from B.

In Example I 10.13, if $D = \{(\rho, \theta) \mid (\rho < 1)\&(\rho, \theta) \notin S\}$ (that is, the bounded domain in the (ρ, θ)-plane complementary to S), then no point of A except $(0, 0)$ is arcwise accessible from D. However, all points of $S - A$ are arcwise accessible from D.

5.37 THEOREM. *Let K be a k-sphere imbedded in S^n. Then K is arcwise accessible from $S - K$.*

PROOF. By Theorem 5.35, $S - K$ is 0-ulc if $k \neq n - 1$, and if $k = n - 1$, then the domains complementary to K are 0-ulc. Consequently if U is a com-

ponent of $S - K$ and $x \in K$, there exists by Theorem 5.33 a sequence $\eta_1, \eta_2, \cdots, \eta_i, \cdots$, of positive numbers such that (1) $\lim \eta_i = 0$, and (2) the set $U \cap S(x, \eta_{i+1})$ lies in a single component of $U \cap S(x, \eta_i)$.

Now by methods similar to those used in proving Theorem 5.23, it may be shown that every point of K is a limit point of U. Let p_1 be any point of U, and for each $i > 1$ let $p_i \in U \cap S(x, \eta_i)$. By Lemma 5.14, there exist arcs T_1 with end points p_1 and p_2 in U, and T_i with end points p_i and p_{i+1} in $U \cap S(x, \eta_{i-1})$ for $i > 1$. The point x, together with obvious portions of the arcs T_i, $i = 1, 2, \cdots$, forms an arc from p_1 to x that meets K only in the point x.

In the following theorem we give a characterization of the S^1 in S^2 analogous to that given by Schoenflies, except that the conditions employed are weaker:

5.38 Theorem. *A necessary and sufficient condition that a subset M of S^2 should be an S^1 is that it be a common boundary of two disjoint domains D_1 and D_2, from each of which M is arcwise accessible.*

Proof of sufficiency. Since, by Theorem 4.12, S^2 has Property IV, the set M is a continuum. By Corollary 5.30, M has no cut points.

Let $x, y \in M$, $x \neq y$. Then from the accessibility property of M it follows that there exists an S^1, which we denote by J, consisting of two arcs A_i, $i = 1, 2$, having x and y as end points and such that $A_i - x - y \subset D_i$. By the Jordan Curve Theorem, $S - J$ is the union of two domains A and B having J as common boundary.

Each of the sets A, B contains points of M. For consider the point set $A \cup a_1 \cup a_2$, where $a_i \in A_i - x - y$; since A is connected, it must be a connected set. Hence by Theorem I 7.8, $A \cap M \neq 0$. Thus $M - x - y = (A \cap M) \cup (B \cap M)$ separate.

It follows from Theorem I 11.21 that M is an S^1.

Bibliographical comment

§1. Regarding the early history of local connectedness, see I 6; also Moore [e]. Lcw corresponds to Hausdorff's [H_2, 156] "im Punkte x lokal zusammenhängend;" and for spaces employing open sets as neighborhoods, lcw is the same as what Moore calls [Mo, 94] "connected im kleinen."

§2. Wilder [m]. In euclidean spaces, Theorem 2.4 was proved in Wilder [g, Th. 7]. A much more general theorem was given in Wilder [m, Th. 13]. Regarding Corollary 2.6, see Whyburn [d, Th. 2] and Wilder [g, 655] and [j, 52]. And regarding the characterizations of the 1-sphere, see Wilder [j, §3].

§3. Regarding Theorem 3.1, see Kuratowski [a], Hahn [b] and Hausdorff [H_2, 156]; and Theorem 3.6, see Wilder [h]; Theorem 3.8, Wilder [e].

§4. Relations between Properties I–V in metric, lc continua (Peano spaces) were given by Kuratowski [b] and van Kampen [a]. See also Wilder [A_1].

§5. Theorem 5.18 is Corollary W^i of Alexander [a]. Theorem 5.19 embodies a number of classical theorems on the n-sphere; for example, that S^n has property

I′ is the Phragmen-Brouwer Theorem (the connection with Theorem 5.18 was first noticed by Alexandroff [h] who derived Theorem 5.26 from it); that S^2 has properties II and V was shown by Brouwer in [e]. Theorem 5.28a was proved by A. Mullikin [a, Th. 4] (a comparison between her proof, using methods in common use at the time, with that given here is revealing). Theorem 5.35 was first proved by Wilder for the case $k = n - 1$ in [c], and for the general k in [A_7]. The accessibility theorem, 5.37, was proved for $k = n - 1$ by Brouwer [d] and by Mazurkiewicz [a] for $k < n - 1$. Theorem 5.38 embodies the Schoenflies converse of the Jordan Curve Theorem, although Schoenflies assumed [S, V] that the complement of the set in question has *exactly* two domains; that the latter is unnecessary was first noticed by P. M. Swingle [a].

CHAPTER III

PEANO SPACES; CHARACTERIZATIONS OF S^2 AND THE 2-MANIFOLDS

1. **Peano continua.** As stated in the historical remarks (I 6), the notion of continuous curve, originally due to C. Jordan, has proved most fruitful in the set-theoretic topology. As defined by Jordan, a plane continuous curve is a set of points (x, y) which may be obtained by functions $x = f(t)$, $y = g(t)$ which are continuous in the real variable t as t varies from 0 to 1. The generalized notion of continuous curve, usually called the Peano continuum (I 6), encompasses any subset M of a metric space such that $M = f(\bar{E}^1)$ where f is a continuous mapping (I 5) and $\bar{E}^1$ is the real number interval $0 \leqq x \leqq 1$. As remarked previously (loc. cit.), the fulfillment of the order of ideas begun by Peano's solution of the space-filling curve problem may be considered to have been achieved with the Hahn-Mazurkiewicz discovery of the topological characterization of continuous curves by the lc property. For the sake of completeness, and because of the useful by-products of the proof, we give in the present section the Hahn-Mazurkiewicz theorem.

We begin with some general lemmas concerning mappings. We recall that a *Hausdorff space* is a space which satisfies the axioms I 4.5 and I 4.7, as well as the separation axiom which states that if x and y are distinct points, then there exist disjoint open sets which contain x and y respectively. Whenever an explicit statement is lacking in what follows, regarding the form of space under consideration, we shall understand that a Hausdorff space is intended.

1.1 LEMMA. *A compact subspace of a Hausdorff space S is closed in S.*

PROOF. Let M be a compact subspace of a Hausdorff space S, and let $p \in S - M$. For each $x \in M$, let $U(x)$, $V(x)$ be disjoint open subsets of S containing x and p respectively. As M is compact, and each set $U(x) \cap M$ is open in M, a finite number, $U(x_1), \cdots, U(x_k)$, of the sets $U(x)$ cover M. Then (I 4.6) $\bigcap_{i=1}^{k} V(x_i)$ is an open subset of $S - M$ which contains p. Hence $S - M$ is open (I 4.2) and M is closed in S.

1.2 LEMMA. *The result of a continuous mapping of a compact space is compact.*

PROOF. Suppose f maps a compact space S onto a space S', and that $\{U_\nu\}$ is a collection of open sets U_ν covering S'. Each $f^{-1}(U_\nu)$ is open (I 5.1a), and since S is compact, there exists a finite number of the sets U_ν, say $U_\nu^{(1)}, \cdots, U_\nu^{(k)}$ such that $f^{-1}(U_\nu^{(1)}), \cdots, f^{-1}(U_\nu^{(k)})$ cover S. Since $ff^{-1}(U_\nu^{(i)}) = U_\nu^{(i)}$, it follows that $S' = \bigcup_{i=1}^{k} U_\nu^{(i)}$.

Definition. A mapping is called *closed* if it maps closed sets into closed sets.

1.3 Lemma. *A continuous mapping of a compact space into a Hausdorff space is a closed mapping.*

Proof. Suppose $f : S \to S'$, and let F be a closed subset of S. By Theorem I 12.11, F is compact. Then by Lemma 1.2 the set $f(F)$ is compact, and by Lemma 1.1 $f(F)$ is closed.

1.4 Lemma. *If f is a continuous mapping of S onto S' and C' is a component of S', then $f^{-1}(C')$ is the union of components of S.*

Proof. Consider a component A of S. The set $f(A)$ is connected (I 5.2a), and therefore if $f(A) \cap C' \neq 0, f(A) \subset C'$.

1.5 Lemma. *If S is lc and f is a continuous and closed mapping of S, then $f(S)$ is lc.*

Proof. Let $f(S) = S'$, and suppose C' is a component of an open subset U' of S'. As f is continuous, $f^{-1}(U')$ is open (I 5.1a), and by Lemma 1.4, $f^{-1}(C')$ is the union of components of $f^{-1}(U')$. But components of $f^{-1}(U')$ are open by Theorem II 3.1, and therefore $f^{-1}(C')$ is open (I 4.4). As $ff^{-1}(C') = C'$ and f is a closed mapping, the set $ff^{-1}(S' - C')$ is closed and hence C' is open. That S' is lc then follows from Theorem II 3.1.

We can now state the following theorem, whose proof follows from the above lemmas:

1.6 Theorem. *The result of a continuous mapping of a compact lc space is a compact lc space.*

1.7 Definition. A collection $\mathfrak{U}$ of sets is called *finitely additive* if every union of a finite number of its elements is also an element of $\mathfrak{U}$.

1.8 Definition. A space S is called *perfectly separable* if it has a countable system of neighborhoods equivalent to its defining system. Evidently *every subspace of a perfectly separable space is itself perfectly separable.*

In particular, a Hausdorff space is perfectly separable if and only if it has a countable set of open sets which is equivalent to the set of all open sets; and such a countable set we shall call a *countable basis* for the space. Every such countable basis may be considered as finitely additive (cf. I 4).

1.9 Lemma. *If S is compact, has a finitely additive basis $\mathfrak{U}$, and F is a closed set contained in the open set U' of S, then there exists a $U \in \mathfrak{U}$ such that $F \subset U \subset U'$.*

Proof. As S is compact, F is compact (I 12.11). And as U' is open, $x \in F$ implies that there exists $U(x) \in \mathfrak{U}$ such that $x \in U(x) \subset U'$. Then the union of a finite number of the $U(x)$'s covering F is a U of the type asserted by the lemma.

1.10 THEOREM. *If f is a continuous mapping of a compact Hausdorff space S onto a compact Hausdorff space T, and S has a countable basis, then T has a countable basis.*

PROOF. Let $\{U_i\}$ be a finitely additive countable basis for S. Then, denoting the complement of a set by "C", each set CU_i is closed, fCU_i is closed by Lemma 1.3, and hence $CfCU_i$ is open in T. We shall show that the sets $CfCU_i$ form a countable basis for T.

Let $x \in V \subset T$, where V is open. Then $f^{-1}(x)$ is a closed subset of the open set $f^{-1}(V)$ in S (I 5.1a), and therefore by Lemma 1.9 there exists U_i such that $f^{-1}(x) \subset U_i \subset f^{-1}(V)$. Taking complements, this implies $Cf^{-1}(x) \supset CU_i \supset Cf^{-1}(V)$. In view of the fact that $Cf^{-1} = f^{-1}C$, we then have

$$f^{-1}Cx \supset CU_i \supset f^{-1}CV. \tag{1.10a}$$

Applying f to all members of (1.10a) we get

$$ff^{-1}Cx \supset fCU_i \supset ff^{-1}CV. \tag{1.10b}$$

But (1.10b) may be written $Cx \supset fCU_i \supset CV$, and therefore, since $CCM = M$, $x \in CfCU_i \subset V$.

1.11 LEMMA. *If A and B are disjoint closed subsets of a normal space S, then there exists a real single-valued continuous function $f(x)$, defined for all $x \in S$, such that $0 \leqq f(x) \leqq 1$, $f(x) = 0$ for $x \in A$, and $f(x) = 1$ for $x \in B$.*

PROOF. If there exists an open set U such that $A \subset U \Subset S - B$ and $\overline{U} - U = 0$, then we may let $f(x) = 0$ for $x \in U$ and $f(x) = 1$ for $x \in S - U$. Otherwise, we proceed as follows: Denote by $U(1/2)$ an open set which contains A and whose closure does not meet B; this is possible because of the normality of S. Again, applying the normality condition to the disjoint pair A, $S - U(1/2)$, we obtain an open set $U(1/4) \supset A$ whose closure fails to meet $S - U(1/2)$; and relative to the pair $\overline{U}(1/2)$, B, an open set $U(3/4) \supset \overline{U}(1/2)$ whose closure fails to meet B. Proceeding in this fashion we obtain for every dyadic proper fraction $r = k/2^n$ an open set $U(r)$ such that if $r < r'$, then $\overline{U}(r) \subset U(r')$.

To define $f(x)$, if there exists r such that $x \notin U(r)$, let $f(x) = \text{lub}\{r \mid x \notin U(r)\}$; otherwise $f(x) = 0$. Then $0 \leqq f(x) \leqq 1$, $f(x) = 0$ for $x \in A$ and $f(x) = 1$ if $x \in B$. And if I is an open interval ab of the real number interval $[0, 1]$, we have $f^{-1}(I) = \bigcup \{U(r) \mid r < b\} - \bigcap \{\overline{U}(r) \mid r > a\}$, which is an open subset of S, and hence f is continuous (I 5.1a).

1.12 DEFINITION. By the *fundamental parallelopiped of Hilbert space* we mean a metric space P whose points are the (type ω) sequences $\{x_n\}$, $0 \leqq x_n \leqq 1$, and such that if $x = \{x_n\}$ and $y = \{y_n\}$, then $\rho(x, y) = \sum_{n=1}^{\infty} | x_n - y_n |/2^n$. Note that this space is compact.

1.13 DEFINITION. A space S is called *metrizable* if there exists a distance function (I 3) $\rho(x, y)$ such that the system of spherical neighborhoods defined

by $\rho(x, y)$ is equivalent to the defining system of S. No space is metrizable unless it has the topological properties of a metric space—as for example the property of being completely normal (cf. Theorem II 4.11).

1.14 THEOREM. *Every perfectly separable normal space is metrizable; indeed, every such space may be imbedded in the fundamental parallelopiped of Hilbert space.*

PROOF. Let S be a perfectly separable normal space. Denoting the countable defining system of neighborhoods of S by $\{U_n\}$, we arrange in a sequence the set of all pairs $(U_{k(n)}, U_{m(n)})$ such that $\overline{U}_{k(n)} \subset U_{m(n)}$; by Theorem 1.11 there exists for each n a continuous function $f_n(x)$ defined over S such that $0 \leqq f_n(x) \leqq 1$ and whose values are 0 and 1 on the sets $\overline{U}_{k(n)}$ and $S - U_{m(n)}$ respectively. Let $f(x) = \{f_n(x)\}$. Then $f(x)$ is a continuous mapping of S into P.[1] Furthermore, $f(x)$ is 1-1.

To show that f is a homeomorphism, we must prove f^{-1} continuous. Suppose $x \notin \overline{M} \subset S$; then $f(x) \notin f(\overline{M})$. For there exists a natural number n such that $x \in U_{m(n)} \subset S - \overline{M}$, and an index $k(n)$ such that $x \in \overline{U}_{k(n)} \subset U_{m(n)}$; hence for $y \in \overline{M}$, $f_n(x) = 0$, $f_n(y) = 1$ and consequently $\rho(f(x), f(y)) \geqq 1/2^n$. Thus a sphere $S(f(x), 1/2^n)$ in P contains no point of $f(\overline{M})$.

1.15 LINDELÖF THEOREM. *In a perfectly separable space S, if $\mathfrak{U}$ is a covering of a point set M by open sets, then a countable subset of $\mathfrak{U}$ covers M.*

PROOF. Let $\{U_n\}$ be a countable basis for S, and for each U_n let $\mathcal{G}_n = \{U \mid (U \in \mathfrak{U})\&(U \supset U_n)\}$. For each n, let $V_n \in \mathcal{G}_n$. Then $\{V_n\}$ is a countable subset of $\mathfrak{U}$ covering M. For suppose $x \in M$. Then there exists $U \in \mathfrak{U}$ such that $x \in U$. Also, there exists U_n such that $x \in U_n \subset U$ and consequently $U \in \mathcal{G}_n$, implying $\mathcal{G}_n \neq 0$. Then $x \in V_n$.

A natural generalization of the notion of equivalent sets of neighborhoods is contained in the following definition:

1.16 DEFINITION. A set $\mathfrak{U}$ of neighborhoods is called *equivalent* to the set of all neighborhoods (or to the defining system) *relative to $M \subset S$* if given a neighborhood U of $x \in M$ there always exists $V \in \mathfrak{U}$ such that V is a neighborhood of x and $V \subset U$. The extension to a definition of equivalence of two arbitrary collections of neighborhoods relative to a set M should be obvious.

1.17 LEMMA. *If S is either* (1) *metric or* (2) *perfectly separable, and $x \in S$, then there exists a countable collection of open sets equivalent to the set of all open sets relative to x.*

---

[1]Given $p = \{f_n(x)\} \in P$, and $\epsilon > 0$, choose $\delta_n > 0$ so that $\rho(x, y) < \delta_n$ implies that $| f_n(x) - f_n(y) | < \epsilon/2^{n+1}$—which is possible because of the continuity of each function $f_n(x)$. Then choose m so that $1/2^m < \epsilon/2$, and let $\delta = \min(\delta_1, \cdots, \delta_m)$. Then if $y \in S(x, \delta)$ in S, we have the relations $\rho[f(x), f(y)] = \Sigma^\infty_1 | f_n(x) - f_n(y) |/2^n = \Sigma^m_1 | f_n(x) - f_n(y) |/2^n + \Sigma^\infty_{m+1} | f_n(x) - f_n(y) |/2^n < (\epsilon/2^2 + \cdots + \epsilon/2^{m-1}) + 1/2^m < \epsilon/2 + \epsilon/2 = \epsilon$.

Proof. If S is metric, the desired collection may be taken as $\{S(x, 1/n) \mid n$ a natural number$\}$. If S has a countable basis $\{U_n\}$, the desired collection may consist of all U_n such that $x \in U_n$.

1.18 Remark. A space having the property that for some point x there exists a countable collection of open sets equivalent to the set of all open sets relative to x is said to *satisfy the first Hausdorff countability axiom* at x [H, 263(E)] or to be *of countable character* at x. A space that is of countable character at all its points will be called a space of countable character.

1.19 Definition. A point x is said to be a *sequential limit point* (slp) of a sequence of (not necessarily distinct) points $\{p_n\}$ if every neighborhood of x contains all but a finite number of the points p_n. The relationship may be expressed by the symbols x slp $\{p_n\}$. The fact that x slp $\{p_n\}$ is frequently expressed by the statement $\{p_n\}$ *converges to* x. And if for given sequence $\{p_n\}$ there exists x such that x slp $\{p_n\}$, the sequence $\{p_n\}$ may be called *convergent*.

1.20 Lemma. *In a Hausdorff space, no sequence has two distinct sequential limit points.*

Remark. The notion of sequential limit point in topology is analogous to that of limit of a sequence in real number theory. Thus, just as a sequence of real numbers may converge to a limit because of endless repetition of the same number, so a sequence of points consisting of the same point endlessly repeated has that same point as sequential limit point.

1.21 Theorem. *If S is either* (1) *metric, or* (2) *perfectly separable, and $M \subset S$ has a limit point p, then p is a sequential limit point of a sequence of distinct points of M.*

[Note that if $\{U_n\}$ is a countable collection of open sets equivalent to the set of all open sets relative to p, then each set $M \cap U_n$ is infinite and has p as a limit point (I 9.1), (I 7.14).]

1.22 Lemma. *Every countably compact subset of a perfectly separable Hausdorff space is compact.*

Proof. Since a subset of a perfectly separable space is itself perfectly separable, the lemma follows from Theorems 1.15 and I 12.8.

1.23 Corollary. *For subsets of a perfectly separable space, "compact" and "countably compact" are equivalent.*

Remark. In view of Corollary 1.23, when dealing with perfectly separable spaces we usually drop the adjective "countably" even though the property desired may be only that of countable compactness.

1.24 Corollary. *Every countably compact subset of a perfectly separable Hausdorff space is closed.*

(Cf. Lemma 1.1.)

1.25 LEMMA. *In a metric space, separability and perfect separability are equivalent.*

If S is metric and separable, let $K \subset S$ be a countable set such that $\overline{K} = S$. Then the collection $\mathfrak{U} = \{S(x, r) \mid (x \in K)\&(r \text{ is rational})\}$ is a countable basis for S.

REMARK. In the proof of Lemma 1.25, evidently the number r can be restricted to be always less than an arbitrarily given fixed number.

1.26 THEOREM. *Every compact metric space is perfectly separable.*

PROOF. For each natural number n, let $\mathfrak{U}_n = \{S(x, 1/n) \mid x \in S\}$, and let $\mathfrak{B}_n$ be a finite subset of $\mathfrak{U}_n$ covering S. Then the collection $\mathfrak{B} = \bigcup \mathfrak{B}_n$ is a countable basis for S.

1.27 THEOREM. *Every compact Hausdorff space is normal.*

PROOF. Let A, B be disjoint closed subsets of the compact Hausdorff space S. The sets A, B are compact by Theorem I 12.11. For each $x \in A$ and $y \in B$ let $U(x, y)$ and $V(x, y)$ be disjoint open sets containing x and y, respectively. For each x, let $V(x)$ be the union of a finite collection $V(x, y_i)$ of the sets $V(x, y)$ that cover B, and let $U(x) = \bigcap U(x, y_i)$. Let U be the union of a finite collection $\{U(x_j)\}$ of the sets $U(x)$ that cover A, and $V = \bigcap V(x_j)$. Then U and V are disjoint open sets containing A and B, respectively.

1.28 THEOREM. *The result of a continuous mapping of a compact metric space into a Hausdorff space is a compact metrizable space.*

PROOF. By Lemma 1.2, if S is compact metric and f is a continuous mapping of S into a Hausdorff space, then $S' = f(S)$ is compact. By Theorem 1.26, S is perfectly separable and therefore S' is perfectly separable by Theorem 1.10. As every compact space is normal by Theorem 1.27, the theorem now follows from Theorem 1.14.

From Theorems 1.6 and 1.28 we now have:

1.29 THEOREM. *The result of a continuous mapping of the real number interval* [0, 1] *into a Hausdorff space is a compact, metric, lc space.*

2. Topological characterization of Peano continua. We now turn to the converse of Theorem 1.29.

DEFINITION. By the *Cantor ternary set* we mean the set T of real numbers x, $0 \leqq x \leqq 1$, such that x is expressible in the ternary number scale without use of the digit 1. In geometric terms, T is often described as $\bigcap M_n$, where M_1 is a straight line interval, M_2 is the result of deleting the open middle third interval from M_1, M_3 is the result of deleting the open middle third intervals from the components of M_2, and so on.

2.1 THEOREM. *In order that a space S should be compact and metric, it is*

necessary and sufficient that it be the result of a continuous mapping of the Cantor ternary set T into a Hausdorff space.

PROOF. The sufficiency is of course a consequence of Theorem 1.28 above.

Suppose S is a compact metric space. Then $S = \bigcup_1^{n_0} S_i$, where n_0 is finite, S_i is closed, and $\delta(S_i) < 1$. Let $T = \bigcup_1^{n_0} T_i$, where the T_i's are disjoint closed sets, and let T_i correspond to S_i.

Next, express each S_i as $\bigcup_{j=1}^{n_i} S_{ij}$ where S_{ij} is closed and $\delta(S_{ij}) < 1/2$; and then express the corresponding T_i as $\bigcup_{j=1}^{n_i} T_{ij}$, where again the T_{ij} are disjoint closed sets, and let T_{ij} correspond to S_{ij}.

In continuing this process, the decomposition of T is made in such a manner that the sets involved at each step decrease uniformly to zero in diameter as the number of steps in the process increases indefinitely. The sets in the decomposition of S at the nth stage are all to be of diameter $<1/n$, and every set obtained from S at the nth step is decomposed into at least two closed subsets in the $(n + 1)$th step.

Now if $x' \in S$, then x' is the common point of at least one sequence $S_i \supset S_{ij} \supset \cdots$. And for each such sequence, the corresponding sequence in T, $T_i \supset T_{ij} \supset \cdots$ has a unique point x which is common to all elements of the sequence. We let $f(x) = x'$. (Evidently there may be many x's such that $f(x) = x'$, but we only ask that $f(x)$ be single-valued, since generally a single-valued inverse would be impossible.)

To show $f(x)$ continuous, consider a fixed pair x, x' such that $x' = f(x)$, and arbitrary $\epsilon > 0$. Determine an integer k such that $1/k < \epsilon$; then the subsets $S_{ij\ldots}$ of S with k indices are all of diameter $<\epsilon$. In the decomposition of T at the kth step of the above process, let the set that contains x be denoted by $T_k(x)$. There exists $\delta > 0$ such that $S(x, \delta) \cap [T - T_k(x)] = 0$. Hence if $\bar{x} \in S(x, \delta)$, then $\bar{x} \in T_k(x)$ and $f(\bar{x}) \in S(x', \epsilon)$.

2.2 LEMMA. *If a compact metric space is lc, then it is ulc* (II 5.32).

2.3 LEMMA. *If S is a compact, metric, connected, lc space, and $a, b \in S$, then there exists a continuous mapping f of the real number interval $\bar{E}^1 = [0, 1]$, such that $f(0) = a$, $f(1) = b$, and in general $f(x) \in S$.*

PROOF. Define the function $f(x)$ as follows: Let $f(0) = a$ and $f(1) = b$. By Theorem I 12.3, there exists a simple chain $\mathfrak{C}_0 = [U_1, \cdots, U_{n(0)}]$ of open connected subsets—i.e., domains—of S from a to b such that each U_i is of diameter <1. For each i, $1 \leqq i < n(0)$, let $x_i \in U_i \cap U_{i+1}$. Then let $f(i/n(0)) = x_i$.

Denoting a, b respectively by x_0, $x_{n(0)}$, for each i, $1 \leqq i \leqq n(0)$, let $\mathfrak{C}_i = [U_{i1}, \cdots, U_{in(i)}]$ be a simple chain of domains from x_{i-1} to x_i such that each $\bar{U}_{ij} \subset U_i$ and $\delta(U_{ij}) < 1/2$. Let $x_{ij} \in U_{ij} \cap U_{i,j+1}$, $1 \leqq j \leqq n(i) - 1$, and let $f(i/n(0) + j/n(0)n(j)) = x_{ij}$.

At the next step we introduce simple chains $\mathfrak{C}_{ij}$ in the sets U_{ij} made up of domains of diameter $<1/3$, etc. This process defined inductively generates

functional values for a countable subset M dense in $\bar{E}^1$, which can then be extended to the points of $\bar{E}^1 - M$ so as to provide for the continuity of $f(x)$.

2.4 LEMMA. *Let S be a compact, metric, lc space. Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $a, b \in S$ for which $\rho(a, b) < \delta$, then there exists a continuous mapping f of the real number interval $\bar{E}^1 = [0, 1]$ into S such that $f(0) = a$, $f(1) = b$ and $\delta[f(\bar{E}^1)] < \epsilon$.*

INDICATION OF PROOF. By Lemma 2.2, there exists $\delta > 0$ such that if $\rho(a, b) < \delta$, then there exists a connected subset M of S containing a and b such that $\delta(M) < \epsilon$. Hence one can commence the process used in proving Lemma 2.3 above with the elements of the chain $\mathfrak{C}_0$ forming a point set of diameter $< \epsilon$.

2.5 THEOREM. *In order that a space S should be compact, metric, connected and lc, it is necessary and sufficient that it be the result of a continuous mapping of the real number interval* $[0, 1]$ *into a Hausdorff space.*

PROOF. The sufficiency is a consequence of Theorems 1.29 and I 5.2a.

To prove the necessity, let S be compact, metric, connected and lc. Since S is compact and metric, there exists, by Theorem 2.1, a continuous function $f(x)$, $x \in T$, such that $f(T) = S$. Denote the intervals complementary to T in the real number interval $[0, 1]$ by $I_1, I_2, \cdots, I_n, \cdots$, and denote the end points of I_n by a_n, b_n. If $f(a_n) = f(b_n)$, then we let $f_n(x) = f(a_n)$ for all numbers x between a_n and b_n. Otherwise, there exist, by application of Lemma 2.4, mappings $f_n(I_n)$ into S such that (1) $f_n(a_n) = f(a_n)$ and $f_n(b_n) = f(b_n)$; (2) $\delta[f_n(I_n)] < \epsilon_n$, $\lim \epsilon_n = 0$. Finally, if we let $f(x) = f_n(x)$ for $x \in I_n$, $f(x) = f(x)$ if $x \in T$, then the new function $f(x)$ is the continuous mapping desired.

3. **Peano spaces.** Throughout the present section we shall deal with a space C having the following properties: (1) nondegenerate, perfectly separable and normal; (2) locally compact; (3) connected; (4) lc. Such a space we shall call a *Peano space*. In case C is compact, it constitutes a *Peano continuum*.

Since C is perfectly separable and normal, we may assume that it has been assigned a metric $\rho(x, y)$, in view of Theorem 1.14.

Let $p \in C$, $\epsilon > 0$, and let $R(p, \epsilon)$ be the component of $S(p, \epsilon)$ determined by p. Then $R(p, \epsilon)$ is a domain (Theorem II 3.1) of C which we shall call an ϵ-region; we frequently drop the ϵ, however, using merely the term *region* when not interested in the diameter. And since C is locally compact, we shall assume without explicit mention hereafter that the closures of all regions employed are compact. Clearly the set of all regions is equivalent to the set of all open subsets of C.

Given $p \in C$, $\epsilon > \epsilon' > 0$, we call $R(p, \epsilon')$ a *shrinkage* of $R(p, \epsilon)$. If $\eta = \epsilon - \epsilon'$, we may call $R(p, \epsilon')$ an *η-shrinkage* of $R(p, \epsilon)$.

If $a, b \in C$, then by $C(a, b)$ we shall denote a simple chain of regions from a to b.

3.1 Lemma. *Let M be a compact subset of a region $R = R(p, \epsilon)$. Then some shrinkage of R covers M.*

Proof. For each $x \in R$ let $R(x) = R(x, \eta_x)$, where $\eta_x = \rho(x, \bar{R} - R)/2$. The set $M \cup p$ is compact and is therefore covered by a finite set of the $R(x)$'s, say $R(x_1), \cdots, R(x_m)$. By Theorem I 12.3 there exists for each i, $i = 1, \cdots, m$, a $C(p, x_i)$ consisting of elements of the collection $\{R(x)\}$. Let $A = \bigcup [R(x_i) \cup C(p, x_i)]$. Then $\bar{A}$ is a subcontinuum of R containing $M \cup p$.

Let ϵ' be a positive number $<\epsilon$ such that $S(p, \epsilon') \supset \bar{A}$. Since $\bar{A}$ is connected, $R(p, \epsilon')$ contains $\bar{A}$, and as $\epsilon' < \epsilon$, $R(p, \epsilon')$ is a shrinkage of R containing M.

3.2 Lemma. *Let M be a compact point set and $\mathfrak{R}$ a collection of regions covering M. Then there exists an $\eta > 0$ such that the set $G(\eta)$ consisting of η-shrinkages of the regions of $\mathfrak{R}$ also covers M.*

Proof. If $\mathfrak{R}$ is infinite, we select a finite set $R_1, R_2, \cdots, R_k$ of regions of $\mathfrak{R}$ which covers M, and continue to call this set $\mathfrak{R}$. Evidently an η which satisfies the conclusion of the lemma for the new $\mathfrak{R}$ will also do so for the old $\mathfrak{R}$, and we therefore carry out the proof for the finite case. Also, if $k = 1$, the lemma follows from Lemma 3.1.

Using a mathematical induction argument, suppose the lemma holds for every covering of $k - 1$ regions, $k > 1$, and let $\mathfrak{R}$, k be as above. Let $A = M - M \cap \bigcup_{i=1}^{k-1} R_i$. If $A = 0$, the lemma follows from the induction assumption. Otherwise, let $\eta' > 0$ be such that an η'-shrinkage R_k' of R_k covers A.

Now $M \cap R_k' \supset M \cap A = A$. Hence $M - M \cap R_k' \subset M - A = M \cap \bigcup_{i=1}^{k-1} R_i$. That is, $M - M \cap R_k'$ is a closed set covered by the $k - 1$ regions $R_1, \cdots, R_{k-1}$. Hence by the induction assumption there exists $\eta'' > 0$ such that η''-shrinkages of $R_1, \cdots, R_{k-1}$ cover $M - M \cap R_k'$. Evidently η-shrinkages, where $\eta = \min(\eta', \eta'')$, of $R_1, \cdots, R_k$ cover M.

Remark. Any positive number $<\eta$ will obviously still satisfy the requirements of Lemma 3.2.

3.3 Theorem. *If $p \in C$ and $\epsilon > 0$, then there exists a domain U which is ulc and such that $p \in U \subset S(p, \epsilon)$.*

Proof. We may assume $\epsilon < 1$ as well as small enough so that $\bar{S}(p, \epsilon)$ is compact. For each natural number n, let $\epsilon_n = 1/2^n$. Our general process will be to commence with an $\epsilon \cdot \epsilon_2$-region U_{00} containing p, which we augment with $\epsilon \cdot \epsilon_{n+2}$-regions U_{ni} where for each n, U_{ni} meets some $U_{n-1,j}$, so that the union $U_{00} \cup \bigcup_{n,i} U_{ni}$ will be of diameter $<\epsilon$. It will be assumed throughout the discussion that this condition relative to the diameters of the sets U_{ni} is complied with, without necessarily making explicit mention of the fact. We shall denote the boundary of a U_{ni} by B_{ni}.

As B_{00} is compact, it may be covered by a finite set of $\epsilon \cdot \epsilon_3$-regions $R_1, \cdots, R_{n(1)}$. Let η be such that an η-shrinkage of these regions still covers B_{00} (Lemma 3.2). Suppose it is true that if R_1 has a boundary point in common with any R_j, $j > 1$, then it overlaps R_j; then in this case we let $R_1 = U_{1,1}$.

If this is not the case, then $U_{1,1}$ is an η'-shrinkage of R_1, $\eta' < \eta$, which does have this property, while at the same time overlapping the same regions R_j that R_1 overlapped—this may be accomplished by use of Lemma 3.2, by selecting a point x_j in each nonempty $R_1 \cap R_j$ and applying the lemma to the closed set $M = \bigcup x_j$. We next shrink R_2 if necessary, so that it has similar properties relative to $U_{1,1}$, R_3, $\cdots$, $R_{n(1)}$, and call the resulting set $U_{1,2}$; and so on until from $R_{n(1)}$ we get a region $U_{1,n(1)}$ (actually, $R_{n(1)} = U_{1,n(1)}$).

Let $U_{1,0} = U_{00} \cup \bigcup U_{1i}$. Now $U_{1,0}$ is a connected open set, of which we assert that there exists a $\delta_1' > 0$ such that if a and b are points of $U_{1,0}$ such that $\rho(a, b) < \delta_1'$, then there exists $C(a, b) \subset U_{1,0}$ such that $\delta[C(a, b)] < \epsilon_1$ (cf. the last sentence preceding Lemma 3.1). For suppose this is not the case. Then, since $\overline{U}_{1,0}$ is compact, there must exist $c \in B_{1,0}$ such that in each $S(c, \epsilon_n)$ there is a pair a_n, $b_n \in U_{1,0}$ not joined by a $C(a_n, b_n) \subset U_{1,0}$ of diameter $<\epsilon_1$. Now $B_{1,0} \subset \bigcup B_{1i}$ and c is therefore in some B_{1i}, say $B_{1,1}$. And as $c \notin B_{00}$, we may select n so great that $S(c, \epsilon_n)$ contains no point of U_{00}, as well as no point of a U_{1i} unless $U_{1,1}$ and that U_{1i} overlap. But then a_n, b_n either lie in one region U_{1i} or in overlapping regions U_{1i}, U_{1j}, and in either case we have a $C(a_n, b_n)$ of diameter $<\epsilon_1$, since the U_{1i} are of diameter $<\epsilon_3$. Thus a δ_1' of the sort described above must exist. For use later on we record the (trivial) fact that if $\delta_1 = \delta_1'/3$, $a, b \in U_{1,0}$ such that $\rho(a, b) < \delta_1$, then there exists a $C(a, b)$ such that the diameter of $C(a, b) < 3\epsilon_1$.

We now cover $B_{1,0}$ by a finite set of regions K_1, K_2, $\cdots$, $K_{n(2)}$ of diameters $<\epsilon_1 \cdot \delta_1$. By shrinking the K's we obtain a set of regions U_{2i}, $i = 1, 2, \cdots$, $n(2)$, still covering $B_{1,0}$, such that if two of them have a common boundary point then they overlap. Then the set $U_{2,0} = U_{1,0} \cup \bigcup U_{2i}$ is such that if $a, b \in U_{2,0}$, $\rho(a, b) < \delta_1$, then there exists a $C(a, b) \subset U_{2,0}$ such that the diameter of this set $C(a, b)$ is $< 3\epsilon_1$. For if a, for instance, is not in $U_{1,0}$, it is in a set U_{2i} which contains a point a' of $U_{1,0}$, and similarly if b is not in $U_{1,0}$ it is in a set U_{2j} which contains a point b' of $U_{1,0}$; and as $\rho(a', b') < \delta_1'$, there exists a $C(a', b') \subset U_{1,0}$ of diameter $<\epsilon_1$. From this chain and the regions U_{2i}, U_{2j} we may get a $C(a, b) \subset U_{2,0}$ of diameter $<3\epsilon_1$. Finally, by an argument identical with that used in the paragraph above we prove that there exists a δ_2' such that if $a, b \in U_{2,0}$ and $\rho(a, b) < \delta_2'$, then there exists a $C(a, b) \subset U_{2,0}$ such that $\delta[C(a, b)] <\epsilon_2$. And we record the trivial fact that if $\delta_2 = \delta_2'/3$ and $\rho(a, b) < \delta_2$, then there exists a $C(a, b)$ of diameter $< 3\epsilon_2$.

Instead of describing the general nth stage of the process at this point, we shall describe the next, third stage—the general inductive definition should be clear thereafter without explicit formulation. Proceeding as in the first two stages, we cover $B_{2,0}$ by regions U_{3i}, $i = 1, 2, \cdots, n(3)$, such that $\delta(U_{3i}) < \min(\epsilon_2\delta_1, \epsilon_1\delta_2)$. Let $U_{3,0} = U_{2,0} \cup \bigcup U_{3i}$. We assert first that if $a, b \in U_{3,0}$ and $\rho(a, b) < \delta_1$, then there exists a $C(a, b) \subset U_{3,0}$ of diameter $< 3\epsilon_1$. We have already shown this if $a, b \in U_{2,0}$. If $a \notin U_{2,0}$, then a is in a U_{3i} of diameter $<\epsilon_2\delta_1$ which contains a point a' of a U_{2j} of diameter $< \epsilon_1\delta_1$, which in turn contains a point a'' of $U_{1,0}$; thus a and a'' are joined by the connected set

$U_{3i} \cup U_{2i}$ whose diameter is $<(\epsilon_2 + \epsilon_1)\delta_1 < \delta_1$. Similarly b is joined to a point b'' of $U_{1,0}$ by a connected set of diameter $<\delta_1$. As $\rho(a'', b'') < 3\delta_1 = \delta_1'$, there exists a $C(a'', b'')$ in $U_{1,0}$ of diameter $<\epsilon_1$. The remainder of this part of the argument should be clear. We next assert that if $\rho(a, b) < \delta_2$, then there exists a $C(a, b)$ in $U_{3,0}$ of diameter $<3\epsilon_2$—the proof is similar to that already employed above. And finally we prove the existence of a $\delta_3' > 0$ such that if $\rho(a, b) < \delta_3'$, then there exists a $C(a, b)$ of diameter $<\epsilon_3$. We define $\delta_3 = \delta_3'/3$.

In general, we set up a $U_{n0} = U_{n-1,0} \cup \bigcup U_{ni}$, and a number $\delta_n > 0$ such that for $1 \leqq k \leqq n$, $a, b \in U_{n0}$, and $\rho(a, b) < \delta_k$, there will exist a $C(a, b)$ of diameter $< 3\epsilon_k$.

Let $U = \bigcup U_{n0}$. Then U is a connected open subset of $S(p, \epsilon)$. We assert that for arbitrary $\eta > 0$, there exists a $\delta > 0$ such that if $a, b \in U$ and $\rho(a, b) < \delta$, then there exists a $C(a, b)$ of diameter $<\eta$. For take n such that $3\epsilon_n < \eta$ and let $\delta = \delta_n$. There exists $k > n$ such that $a, b \in U_{k0}$. By the method of construction we have already provided that if $a, b \in U_{k0}$ and $\rho(a, b) < \delta_n$, then there exists a $C(a, b)$ in U_{k0} of diameter less than $3\epsilon_n < \eta$. Hence U is ulc.

It will be noticed that we have also proved:

3.4 THEOREM. *If M is a compact point set, then for an arbitrary $\epsilon > 0$ there exists a ulc, open set U such that $M \subset U \subset S(M, \epsilon)$; and if M is connected then U is connected, and in any case U has a finite number of components whose closures are disjoint.*

For the single point case we have the following important corollary:

3.5 THEOREM. *If $p \in C$, $\epsilon > 0$, $\eta > 0$, and $\overline{R}(p, \epsilon)$ is compact, then C contains a ulc, connected open set U such that $\overline{R}(p, \epsilon) \subset U \subset R(p, \epsilon + \eta)$.*

3.6 THEOREM. *If a point set M is ulc, then $\overline{M}$ is lc.*[2]

PROOF. By Theorem II 1.8 it will be sufficient to show that $\overline{M}$ is lcq. Let $p \in \overline{M}$, $\epsilon > 0$. Since M is ulc, there exists $\delta > 0$ such that if $x, y \in M$ and $\rho(x, y) < 2\delta$, then some connected subset of M of diameter $<\epsilon/2$ contains x and y. We may suppose $\delta < \epsilon/2$.

We assert that all points of $\overline{M} \cap S(p, \delta)$ are in the same quasi-component of $\overline{M} \cap S(p, \epsilon)$. For suppose $x, y \in \overline{M} \cap S(p, \delta)$ and that $\overline{M} \cap S(p, \epsilon) = A \cup B$ separate, where $x \in A$, $y \in B$. Let $\eta > 0$ be such that $S(x, \eta) \subset S(p, \delta) - B$, and let $x' \in M \cap S(x, \eta)$. Then $x' \in M \cap A \cap S(p, \delta)$. Similarly there exists $y' \in M \cap B \cap S(p, \delta)$. By our selection of δ, there exists a connected set K in M such that $x', y' \in K \subset S(p, \epsilon)$. But then K is a connected subset of $A \cup B$ which has points in both A and B, which is impossible (I 7.1).

As a corollary of Theorems 3.4 and 3.6 we now have:

3.7 THEOREM. *If M is a compact point set, then for arbitrary $\epsilon > 0$ there*

[2]See Theorem IV 4.12.

exists a closed set K and positive number η *such that* (1) $S(M, \eta) \subset K \subset S(M, \epsilon)$; (2) *the components of K form a finite set of Peano continua*; (3) *if M is connected, K is connected.*

In particular, then, *arbitrarily small neighborhoods of each point of C lie in arbitrarily small Peano continua.*

For the next theorem we need the following lemma:

3.8 LEMMA. *If* $M_1, M_2, \cdots, M_n, \cdots$ *is a sequence of compact, connected, nonempty subsets of a Hausdorff space S such that for each n,* $M_n \supset M_{n+1}$, *then* $\bigcap M_n$ *is nonempty, closed and connected.*

PROOF. That the set $K = \bigcap M_n$ is nonempty and closed is a consequence of I 4.3 and I 12.8. Suppose $K = A \cup B$ separate. Since M_1 is a compact space, there exist by Theorem 1.27 disjoint open subsets U, V of M_1 such that $U \supset A$, $V \supset B$. Denoting the boundary of U in M_1 by F, let $F_n = F \cap M_n$. Each of the sets F_n is nonempty by Theorem I 7.8, and since each M_n is closed by Lemma 1.1, F_n is closed. Hence $\bigcap F_n \neq 0$ by I 12.8. But $\bigcap F_n \subset \bigcap M_n$, contradicting the supposition that $\bigcap M_n$ has no points on F.

3.9 THEOREM. *Every two points of a region R are the end points of an arc of that region.*

PROOF. If a and b are points, then by a *regular* ϵ-chain (of R) from a to b we mean a simple chain (I 12.2) of Peano continua C_i, $i = 1, \cdots, n$, from a to b such that $C_i \cap C_{i+1}$ is a point and $\delta(C_i) < \epsilon$. Then, R being a region and $a, b \in R$, we assert that for every $\epsilon > 0$ there is a regular ϵ-chain in R from a to b. For let A denote the set of all $x \in R$ such that there is such a chain from a to x, the number ϵ being now fixed. Then A is open in R. To see this, let $x \in A$, $\mathfrak{C}$ a regular ϵ-chain in R from a to x, and L the link (I 12.2) of $\mathfrak{C}$ containing x. Let $\eta > 0$ be such that

$$\eta < \min \{\epsilon - \delta(L), \rho[x, F(R) \cup (\mathfrak{C} - L)]\}.$$

By Theorem 3.7, there exists a Peano continuum C' such that $x \in C'$, $\delta(C') < \eta$, and all points of C in some neighborhood of x lie in C'. Then the chain obtained from $\mathfrak{C}$ by replacing, in $\mathfrak{C}$, the set L by $L \cup C'$ is a regular ϵ-chain from a to x in R having x as an interior point.

The set A is also closed in R. For suppose $y \in R$ is a limit point of A. Let $C_1(y)$ be a Peano continuum in R such that for some neighborhood $U(y)$, $U(y) \subset C_1(y)$ and $\delta[C_1(y)] < \epsilon$. Since $A \cap U(y) \neq 0$, $A \cap C_1(y) \neq 0$, and therefore for some $x \in C_1(y)$ there exists a regular ϵ-chain $\mathfrak{L}$ from a to x consisting of continua $L_1, L_2, \cdots, L_n$. Evidently we may suppose $L_i \cap C_1(y) = 0$ for $i < n$, and if $y \in L_n$, then $\mathfrak{L}$ is a regular ϵ-chain from a to y. Suppose, however, that $y \notin L_n$. Let $0 < \epsilon_1 < \min [1/2, \rho(y, L_n)]$; then, by Theorem 3.7, $C_1(y)$ is the union of a finite collection $\mathfrak{C}_1$ of Peano continua of diameter $< \epsilon_1$. Let H_1 be the union of those elements of $\mathfrak{C}_1$ that fail to meet L_n, and let Y_1 be the component of H_1 containing y. Then at least one element, K_2, of $\mathfrak{C}_1$

contains both a point of Y_1 and a point of L_n. In general, for $k > 1$, let $0 < \epsilon_k < \min [1/2k, \rho(Y_{k-1}, L_n)]$, and express K_k as the union of a finite collection $\mathfrak{C}_k$ of Peano continua of diameter $<\epsilon_k$. Let H_k be the union of those elements of $\mathfrak{C}_k$ that fail to meet L_n, and let Y_k be the component of $Y_{k-1} \cup H_k$ that contains y. Then at least one element K_{k+1} of $\mathfrak{C}_k$ contains both a point of Y_k and a point of L_n. Let $Y = \bigcup_{k=1}^{\infty} Y_k$.

Now $\bigcap K_k = L_n \cap \bigcap K_k$ is a single point p because of the restriction on the diameters of the sets K_k, and since each Y_k is closed, $Y \cap L_n = 0$ and $Y - Y_k \subset K_k$, it follows that the set $\overline{Y} - Y = \bigcap K_k = p$. It is readily shown that $\overline{Y}$ is a Peano continuum of diameter $<\epsilon$ having just p in common with L_n, and the collection whose elements are $L_1, L_2, \cdots, L_n, \overline{Y}$, is a regular ϵ-chain of Peano continua from a to y. Hence $y \in A$ and we conclude that A is a closed set. And now, since A is both open and closed in the connected set R, $A = R$.

To show the existence of an arc from a to b in R we now proceed as follows: Let C_1 be the set of all points in some regular 1-chain of R from a to b. For each $n > 1$, let C_n be the set of all points in some regular $1/n$-chain of C_{n-1} from a to b. Let $K = \bigcap_{n=1}^{\infty} C_n$. We shall show that K is an arc of R from a to b.

Since the sets C_n are compact and connected, K must be a continuum by Lemma 3.8. Every $x \in K - (a \cup b)$ separates a and b in K. For let P_n be the set of all points of C_n that lie on two links of C_n, and $P = \bigcup P_n$. Then $P \subset K$ and every point of P separates a and b in K. Let $x \in K - P - a - b$. Then for n large enough, x lies in exactly one link L_n of C_n, and $L_n \cap (a \cup b) = 0$. If we let C_{na} denote the union of those links of C_n that precede L_n and C_{nb} denote the union of those that are preceded by L_n, in the order from a to b, then $K - x = (\bigcup K \cap C_{na}) \cup (\bigcup K \cap C_{nb})$, and this is easily seen to be a separation. That K is an arc now follows from Theorem I 11.15.

3.10 Theorem. *Every two points of a domain are the end points of an arc of that domain.*

The proof follows easily from the simple chain theorem I 12.3 and Theorem 3.9.

3.11 Corollary. *Every two points of C are the end points of an arc of C.*

3.12 Definition. A space S is called *locally arcwise connected* at $x \in S$ if every neighborhood U of x contains a neighborhood V of x such that if a, $b \in V$, then U contains an arc from a to b. A space S is called locally arcwise connected if it is locally arcwise connected at every $x \in S$. Note that as a consequence of 3.9:

3.13 *The space C is locally arcwise connected.*

3.14 If $M \subset S$, and every two points of M are joined by an arc of S, then M is said to be *arcwise connected through* S. If $M = S$, we say simply that S is *arcwise connected*. For example, by virtue of Theorem 3.10, every domain of C is arcwise connected.

3.15 THEOREM. *If x is a non-cut point of C and $\epsilon > 0$ such that $F(x, \epsilon)$ is compact, then there exists $\delta > 0$ such that $F(x, \epsilon)$ is arcwise connected through $C - S(x, \delta)$.*

PROOF. By Theorem 3.7, $F(x, \epsilon)$ is contained in the union of a finite number of Peano continua $C_1, \cdots, C_k$ which lie in $C - x$. And, since $C - x$ is a domain, by Theorem 3.10 these continua are joined by a finite number of arcs in $C - x$ which, together with the continua $C_1, \cdots, C_k$, form a Peano continuum $K \subset C - x$. Let $\delta = \rho(x, K)$.

3.16 COROLLARY. *If x is a non-cut point of C and $\epsilon > 0$, then there exists a $\delta > 0$ such that all points of $C - S(x, \epsilon)$ lie in one component of $C - S(x, \delta)$.*

3.17 DEFINITION. A point $p \in C$ is called an *end point* of C if for every $\epsilon > 0$ there exists a point x such that $C - x = C_1 \cup C_2$ separate and $p \in C_1 \subset S(p, \epsilon)$.

3.18 DEFINITION. With every non-cut point p of C, we associate a set C_p which consists of all points x of C such that for no $y \in C$ is $C - y = C_1 \cup C_2$ separate such that $p \in C_1$, $x \in C_2$. In other words (see I 5.11), C_p is the set of all points x of C such that no point of C separates p and x in C. Evidently if p is an end point of C, then C_p consists only of p itself. And *if C has no cut points, C is itself a C_p.*

3.19 DEFINITION. By a *cyclic element* of C will be meant either a cut point[3] of C or a set C_p.

For example, it follows easily that what we have hitherto been calling the end points of an arc (I 11.1) are also end points in the sense defined in 3.17 (*if the arc is the space C*); and the other points of the arc, being cut points, are themselves cyclic elements of the arc. Every point of an arc, then, constitutes a cyclic element of the arc.

The space consisting of two tangent circles in a plane, as a space C, has just three cyclic elements, consisting respectively of the point of tangency and the two sets of points lying on the respective circles. Incidentally, this example shows that cyclic elements are generally not disjoint.

3.20 THEOREM. *A point $p \in C$ is a C_p if and only if it is an end point.*

PROOF. If p is a set C_p, then it is by definition a non-cut point. Hence, if $\epsilon > 0$, there exists by Corollary 3.16 a $\delta > 0$ such that all points of $C - S(p, \epsilon)$ lie in one component K of $C - S(p, \delta)$. Let px be an arc such that $px \cap K = x$. Since p is a C_p, there exists a point y such that $C - y = C_1 \cup C_2$ separate and such that $p \in C_1$, $x \in C_2$. Evidently $y \in px$, and consequently $y \notin K$. Hence K must lie in C_2 and $C_1 \subset S(x, \epsilon)$.

We have already observed in 3.18 that every end point p constitutes its own C_p.

[3] More precisely, "the set consisting of a cut point of C", but we say simply "cut point of C" in the interest of brevity.

3.21 THEOREM. *Every set C_p is closed.*

PROOF. If $x \notin C_p$, then there exists y such that $C - y = C_1 \cup C_2$ separate where $p \in C_1$, $x \in C_2$. Since C_p must lie in C_1, x cannot be a limit point of C_p.

We now recall Definition II 5.36 of arcwise accessibility. Throughout the present chapter we shall use the term "accessible" to mean "arcwise accessible".

3.22 DEFINITION. If K and M are point sets such that $K \subset M$ and $\overline{K} \supset M$, then K is said to be *dense in M.*

For example, a space is separable (I 10.22) if it has a countable subset which is dense in it. It will be noted in the example referred to in II 5.36 (this was Example I 10.13) that although the indicated portion of the boundary contains no accessible points, every point of it is a limit point of accessible points. This is an important property of the boundaries of domains, not only in the plane but in any space of type C:

3.23 THEOREM. *If D is a domain, then the set of all boundary points of D which are accessible from D is dense in the boundary.*

PROOF. Denoting the boundary of D by B, let $x \in B$ and $\epsilon > 0$. Let R be an ϵ-region of x, and $y \in D \cap R$. Then there exists an arc xy in R by Theorem 3.9, and if p is the first point of B on xy in the order from y to x, then p is accessible from D.

3.24 THEOREM. *If M is a set C_p, then every component of $C - M$ has exactly one limit point in M.*

PROOF. Let K be a component of $C - M$. By Corollary II 3.4, K has at least one limit point in M. Suppose K has two limit points x, y in M. With $\epsilon < \rho(x, y)/2$, let u, v be accessible boundary points of K in $S(x, \epsilon)$, $S(y, \epsilon)$ respectively—such points exist by Theorem 3.23. Being a domain (Theorem II 3.1), K is arcwise connected by Theorem 3.10. It follows that $K \cup u \cup v$ contains an arc uv.

Let w be a point of uv in K, and (since M is a C_p) q a point such that $C - q = C_1 \cup C_2$ separate, where $p \in C_1$, $w \in C_2$. Then necessarily $M - q \subset C_1$ and therefore either $u \in C_1$ or $v \in C_1$. But then either the portion of uv from u to w, or the portion from v to w, is in C_1, which is impossible.

3.25 THEOREM. *If M is a connected point set and K is a C_p, then $M \cap K$ is connected.*

PROOF. Suppose $M \cap K = A \cup B$ separate. Then $M = M(A) \cup M(B)$ disjoint, where $M(A)$ consists of A together with points of M lying in components of $C - K$ whose boundary points lie in A (cf. Theorems I 7.8 and 3.24); $M(B)$ is defined similarly relative to B. As M is connected, $M(B)$, say, contains a limit point, x, of $M(A)$. Then $x \in B$, since each point of $M(B) - B$ is in a domain containing no point of $M(A)$. However, if $\epsilon < \rho(x, A)$, then an ϵ-region R of x containing points of $M(A)$ must contain points of $M(A) - A$,

hence points of some component L of $C - K$ whose boundary point is in A. A violation of Theorem 3.24 is now easily established by means of the set $L \cup R$.

3.26 THEOREM. *Every nondegenerate set K which is a set C_p is a connected and lc point set which has no cut point.*

PROOF. Since C is connected, $C \cap K = K$ is connected by Theorem 3.25. Also, if $x \in K$ and R is an ϵ-region of x, then again by Theorem 3.25, $R \cap K$ is connected and consequently K is lc. Finally, if $x \in K$, then $K - x$ is connected. For suppose $K - x = A \cup B$ separate, and let $a \in A$, $b \in B$. Then a and b are q-equivalent in $C - x$; this is certainly the case if x is the non-cut point p, and also when $x \neq p$, otherwise the point x would separate p from one of the points a, b. Hence by Theorem II 3.6, a and b are in the same component L of $C - x$. Since by Theorem 3.25 the set $L \cap K$ is connected, although $x \notin L$, a contradiction results.

We may now state, as a result of preceding theorems:

3.27 THEOREM. *Every nondegenerate set C_p has all the properties* (1) – (4) *of the space C itself.*

3.28 THEOREM. *If K is a set C_p and A, B are disjoint, nondegenerate closed subsets of K, then there exist in K two disjoint arcs ab, cd such that $A \cap (ab \cup cd) = a \cup c$, $B \cap (ab \cup cd) = b \cup d$.*

PROOF. Since K has properties (1)-(4) by Theorem 3.27, it follows from Corollary 3.11 that K contains an arc ab such that $A \cap ab = a$ and $B \cap ab = b$. Let $p \in A - a$, and R a region containing p and no points of ab (all regions will be regions of K throughout the proof). Then every $x \in R$ is joined to p by an arc px of R not meeting ab. Let S denote the set of all points x of K such that either (1) $x \in A$ or (2) there exist such arcs as ab and px above. Then we assert that $S = K$.

To prove this, we shall show that S is both open and closed in K. That S is open in K follows from Theorem 3.27 and the local arcwise connectedness of K. Let y be a limit point of S—we may suppose $y \notin A$, since all points of A are in S. Since, by Theorem 3.26, y is a non-cut point of K, there exists in $K - y$ an arc $a'b'$ meeting A and B only in a' and b' respectively. Let R be a region containing y such that $\bar{R} \cap (A \cup a'b') = 0$, and let $x \in R \cap S$. There exist arcs ab and px satisfying (1) and (2) above.

Now if $R \cap ab = 0$, y is certainly a point of S. If $R \cap ab \neq 0$, then ab and px contain arcs aw and pr respectively such that $aw \cap \bar{R} = w$, $pr \cap \bar{R} = r$. Let $H = A \cup aw \cup pr$. Then $a'b'$ contains an arc uv (possibly degenerate) such that $uv \cap H = u$ and $uv \cap B = v$. Let T denote one of the sets aw, pr which does not contain u and let Z denote the other one of these sets. Let Q be a region containing the point $T \cap \bar{R}$ and containing no point of $Z \cup uv$. Then $Z \cup uv$ contains an arc st and $T \cup Q \cup R$ contains an arc qy such that $st \cap A = s$, $st \cap B \supset t$, $qy \cap A = q$, and $st \cap qy = 0$. Consequently $y \in S$.

3.29 Lemma. *If K is a nondegenerate C_p, D a domain of K, and p a non-cut point of D, then the set C_p of D determined by p is nondegenerate.*

Proof. Since every domain itself satisfies conditions (1)-(4) defining C, Theorem 3.20 applies. Hence if the set C_p of D determined by p were degenerate, then p would be an end point of D. Then p would be an end point of K. But in order to have an end point, a space such as C must at least have cut points (cf. Definition 3.17), whereas by Theorem 3.26, K has no cut points.

3.30 Theorem. *If K is a nondegenerate set C_p and $x \in K$, then there exists an arc axb in K such that $x \in axb$ and $a \neq x \neq b$.*

Proof. By Theorem 3.27, K has all the properties (1)-(4) defining C. If x is a cut point of some region of K, then by use of Theorem 3.9 the existence of an arc of the desired type is easily shown. Hence we may suppose x to be a cut point of no region of K.

Let a, b be distinct points of $K - x$. By Theorem 3.3, x lies in a ulc domain D_1 of K of diameter <1 such that $\overline{D}_1 \cap (a \cup b) = 0$. By Theorem 3.6, $\overline{D}_1$ is lc. By Lemma 3.29, the set C_x of D_1 determined by x is nondegenerate, so that the same holds for the set C_x of $\overline{D}_1$ determined by x; denote this C_x by C_1. By Theorem 3.28 there exist in K disjoint arcs aa_1 and bb_1 meeting C_1 only in a_1 and b_1 respectively. If $a_1 = x$ or $b_1 = x$, an arc of the type desired is immediately obtained from the arcwise connectedness of C_1. Hence we suppose $a_1 \neq x \neq b_1$.

Again, x lies in a ulc domain D_2 of C_1 of diameter $<1/2$ such that $\overline{D}_2 \cap (a_1 \cup b_1) = 0$ and $\overline{D}_2$ is lc. We may again suppose that x cuts no region of C_1, so that the C_x of $\overline{D}_2$ determined by x is a nondegenerate set C_2. By Theorem 3.28 there exist in C_1 disjoint arcs a_1a_2 and b_1b_2 meeting C_2 only in a_2 and b_2 respectively.

Continuing, let D_3 be a ulc domain in C_2 containing x, of diameter $<1/3$, etc. The inductive definition of the process indicated should be clear. Then the point set

$$x \cup aa_1 \cup bb_1 \cup a_1a_2 \cup b_1b_2 \cup a_2a_3 \cup b_2b_3 \cup \cdots$$

contains an arc of the type desired.

3.31 Definition. A space C is called *cyclicly connected* or *cyclic* if x, $y \in C$ implies that there exists a 1-sphere in C which contains both x and y. A space C which contains no 1-sphere whatsoever will be called *acyclic.*

3.32 Theorem. *Every nondegenerate set C_p is cyclicly connected.*

Proof. Let K be a set C_p and a, $b \in K$. Now every point x of K lies on some 1-sphere of K. For by Theorem 3.30 there exists an arc pxq in K, and by Theorem 3.10 there exists an arc pyq in $K - x$. The set $pxq \cup pyq$ contains a 1-sphere which contains x. Hence we may assert that there exist 1-spheres J_a, J_b containing a and b, respectively.

If $J_a \cap J_b = 0$, there exist in K by Theorem 3.28 disjoint arcs uv and rt such that $uv \cap J_a = u$, $uv \cap J_b = v$, $rt \cap J_a = r$, $rt \cap J_b = t$. A 1-sphere containing a and b is now easily shown to exist in the set $J_a \cup J_b \cup uv \cup rt$.

If $J_a \cap J_b = p$, a single point, then by Theorem 3.10, $K - p$ contains an arc uv meeting J_a and J_b only in u and v, respectively, and the point set $J_a \cup J_b \cup uv$ contains a 1-sphere of the desired type.

Finally, if $J_a \cap J_b$ is nondegenerate, then J_b contains an arc pbq which contains b and such that $pbq \cap J_a = p \cup q$. Then $J_a \cup pbq$ contains a 1-sphere which contains both a and b.

3.32a Corollary. *If C has no cut points, then C is cyclicly connected.*

3.33 Theorem. *If K is a set C_p and a, b, c are three distinct points of K, then K contains an arc from a to c which contains b.*

Proof. Let J and L be 1-spheres of K such that $J \supset a \cup b$ and $L \supset b \cup c$ (Theorem 3.32). If either J or L contains all three of the points a, b, c, the theorem follows. If $J \cap L = b$, then arcs ab and bc on J and L, respectively, combine to give the desired arc. And if $J \cap L$ is nondegenerate, it is easy to show that if A is that arc of L which contains c and has only its end points on J, then $J \cup A$ contains an arc of the desired type.

We recall at this point that (Theorem II 3.6) if two points are q-equivalent in an open subset U of an lc space S, then they are c-equivalent in U; i.e., they lie in the same domain $D \subset U$. Consequently we can state:

3.34 Theorem. *If two points of an open subset U of C are q-equivalent in U, then they are joined by an arc of U.*

3.35 Lemma. *If K is a connected subset of a separable, connected space S, then the set of cut points of S that lie in K and are non-cut points of K is countable.*

Proof. Let $\{x_\nu\}$ be a collection of points $x_\nu \in K$ such that (1) $S - x_\nu = A_\nu \cup B_\nu$ separate, and (2) the set $K - x_\nu$ is connected. Then $K - x_\nu \subset B_\nu$, say. And if ν', ν'' are different values of the index ν, then $A_{\nu'} \cap A_{\nu''} = 0$. For $x_{\nu''} \in B_{\nu'}$, and since $x_{\nu'} \notin A_{\nu''}$, the connected set $A_{\nu''} \cup x_{\nu''}$ must lie in $B_{\nu'}$.

Now let M be a countable subset of S that is dense in S. Then since each A_ν is open, the set $A_\nu \cap M \neq 0$, and since M is countable, the collection $\{A_\nu\}$, and consequently $\{x_\nu\}$, is countable.

3.36 Theorem. *If M is a nondegenerate subset of C such that no point of C separates points of M in C, then M lies in a single set C_p of C.*

Proof. Let $x, y \in M$, and A be an arc from x to y. Let $a \in A - x - y$. Then, since a does not separate x and y in C, there exists by Theorem 3.34 an arc B from x to y in $C - a$. Then there exists in $A \cup B$ a 1-sphere, J, which contains a subarc A' of A. By Lemma 3.35 there exists $p \in A'$ such that p is a non-cut point of C. Then the set $K = C_p$ contains M.

In order to show this, note that K certainly contains x, since a point q sepa-

rating p and x in C would of necessity lie between p and x on A and therefore separate x and y in C. And similarly $y \in K$. Finally, if $z \in M$ and $z \notin K$, then the boundary point u of the domain of $C - K$ that contains z must separate z from either x or y, contradicting the hypothesis. Hence $M \subset K$.

3.37 Theorem. *If p is a non-cut point of C, then the set C_p may be defined as the maximal, cyclicly connected subset of C that contains p.*

Proof. Let p be a non-cut point of C, and let K be the maximal cyclicly connected subset of C that contains p. Then $K \subset C_p$. For if $C_p = p$, then, by Theorem 3.20, p is an end point, and since from its definition it is clear that no end point lies on a 1-sphere of C, $K = p = C_p$. If C_p is nondegenerate, then by virtue of Theorem 3.24, no point of a component M of $C - C_p$ could lie on the same 1-sphere of C as a point of C_p distinct from the boundary of the component. Hence all points that lie on a 1-sphere with p must lie in C_p.

Conversely, $C_p \subset K$, since by Theorem 3.32 the set C_p is cyclicly connected.

3.38 Theorem. *If M is an lc, closed, connected subset of C, then the union of M and any collection of components of $C - M$ is again an lc, closed, connected subset of C.*

Proof. Let $\{C_\nu\}$ be any collection of components of $C - M$, and let $H = \bigcup C_\nu \cup M$. Then H is connected, since every C_ν has a boundary point in M by Corollary II 3.4. The set $C - H$ is of the form $\bigcup C_\mu$, where C_μ is a component of $C - M$ and hence $C - H$ is open by Corollary II 3.2; consequently H is closed.

The set $\bigcup C_\nu$ is open and therefore H is lc at every point of $\bigcup C_\nu$. If $p \in M$, and $\epsilon > 0$, let $\delta > 0$ be such that if $x, y \in M \cap S(p, \delta)$, then there is an arc from x to y in $M \cap S(p, \epsilon)$; and let $\eta > 0$ be such that if $x, y \in S(p, \eta)$, then there is an arc of C from x to y in $S(p, \delta)$. Consider any $x, y \in H \cap S(p, \eta)$. Let A be an arc from x to y in $S(p, \delta)$. In the order from x to y on A let x' be the first point of M, and in the order from y to x let y' be the first point of M. Then since $x', y' \in M \cap S(p, \delta)$, there is an arc A' in $M \cap S(p, \epsilon)$ from x' to y'. The arc A' together with the portions of A from x to x' and from y' to y forms an arc of H in $S(p, \epsilon)$.

4. Recognition of the 2-sphere.

The 2-sphere may be recognized among the Peano continua by a variety of conditions, a number of which we shall give below.

4.1 Lemma. *If M_1 and M_2 are subsets of the countably compact metric spaces S_1 and S_2, respectively, such that $\overline{M}_i = S_i$, $i = 1, 2$, and there exists a (1-1) mapping f of M_1 onto M_2 such that both f and f^{-1} are uniformly continuous,[4] then f may be extended to a homeomorphism between S_1 and S_2.*

[4] As in the case of uniform continuity of real functions, a mapping of a metric space A onto a metric space is called uniformly continuous if $\epsilon > 0$ implies the existence of a $\delta > 0$ such that if $x, y \in A$ and $\rho(x, y) < \delta$, then $\rho[f(x), f(y)] < \epsilon$.

We leave the proof of this lemma to the reader.

4.2 Theorem (Zippin). *A Peano continuum C which satisfies the following three conditions is a 2-sphere:*

4.2a. *C contains at least one 1-sphere.*

4.2b. *Every 1-sphere of C separates C.*

4.2c. *No arc which lies on a 1-sphere of C separates C.*

The reader will be interested in comparing Theorem 4.2 with Theorem II 2.12. If in conditions 4.2a and 4.2b "1-sphere" is replaced by "0-sphere", and 4.2c is replaced by "no point of C disconnects C", then the resulting conditions are exactly those which characterize the 1-sphere among the separable, connected, lc spaces. We did not need compactness in the case of the 1-sphere, however; as a matter of fact, we shall see later that the 1-sphere may be characterized among the Peano continua by a very weak condition.

In Theorem II 2.16 we found that among the separable, connected, lc spaces the 1-sphere is also characterized by the fact that it is nondegenerate ("contains a 0-sphere") and is separated by every 0-sphere into just two components. For the 2-sphere, the analogue of this is as follows:

4.3 Theorem. *A Peano continuum which contains at least one 1-sphere and which satisfies the Jordan curve theorem is a 2-sphere.*

By "satisfies the Jordan curve theorem" is meant that the complement of a 1-sphere is exactly two domains which have the 1-sphere as common boundary. In Theorem 4.3 the Jordan Curve Theorem appears in its true light as a basic property of the 2-sphere.

We first notice the equivalence of Theorems 4.2 and 4.3. That Theorem 4.2 implies Theorem 4.3 is almost self-evident. As for the converse, it is easily seen that under conditions 4.2a-c, every 1-sphere of C is the common boundary of all its complementary domains. The burden of the proof must be devoted to showing that there are not more than two such domains.

Suppose, then, J is a 1-sphere of C and that there exist (at least) three components D_1, D_2, D_3 in $C - J$. Let us make the following definition:

4.4 Definition. If K is a point set and ab is an arc with end points a and b, then by the statement that *ab spans K* is meant that $K \cap ab = a \cup b$.

Since J is the boundary of each of the domains D_i, there exists in $\overline{D}_i$ an arc T_i that spans J and whose end points are in any two preassigned open arcs of J. However, we can prove:

4.5 *Under conditions* 4.2a-c, *if T, T_1, T_2 are three arcs spanning a 1-sphere J of C such that $\langle T_1 \rangle$ and $\langle T_2 \rangle$* (I 11.1) *are in different components of $C - J$ and $T \cap T_i = 0$, $i = 1, 2$, then the end points of T cannot be separated on J by the end points of T_1 as well as by the end points of T_2.*

For in the contrary case, $J \cup T_1 \cup T_2$ contains a 1-sphere J' that does not

meet T, such that the set $H = J \cup T_1 \cup T_2 \cup T - J'$ consists of T and two open arcs of J each containing an end point of T. Each domain, D, of $C - J'$ must meet the domains D_1, D_2 of $C - J$ that contain $\langle T_1 \rangle$, $\langle T_2 \rangle$ respectively, and hence D must meet J and therefore meet H. But as H is connected, this implies that $C - J'$ is connected.

We may now proceed to prove Theorems 4.2 and 4.3 simultaneously. For later purposes, successive parts of the proof will be numbered.

4.6 *Under conditions* 4.2a-c, *if* X, Y *and* Z *are three arcs with common end points, but otherwise disjoint, then* $C - X - Y - Z$ *is the union of three domains having* $X \cup Y$, $Y \cup Z$ *and* $X \cup Z$ *as their respective boundaries.*

The domain E of $C - (X \cup Y)$ not containing $\langle Z \rangle$, and analogous domains of $C - (Y \cup Z)$, $C - (X \cup Z)$, are separated in $C - X - Y - Z$ and have the boundaries designated above. Suppose D is a fourth domain of $C - X - Y - Z$. Then D necessarily has boundary points in $\langle X \rangle$, $\langle Y \rangle$ and $\langle Z \rangle$, since otherwise one of the 1-spheres designated in 4.6 would separate C into at least three components. A contradiction of 4.5 is now obtainable using $X \cup Y$ as J, an arc in D with end points on $\langle X \rangle$ and $\langle Y \rangle$ as T, Z as T_1, and T_2 an arc spanning $X \cup Y$ such that $\langle T_2 \rangle$ lies in E.

4.7 *Under conditions* 4.2a-c, C *is cyclic.*

For by 4.2a, C contains at least one 1-sphere J, and by Theorem 3.36, J lies in a nondegenerate set C_p which we shall denote by K. If $K \neq C$, then a component A of $C - K$ has only one limit point, x, in K by Theorem 3.24, and by Theorem 3.32, x lies on a 1-sphere J' of K. But then an arc of J' containing x separates C, contradicting 4.2c. Hence $K = C$ and C is cyclic by Theorem 3.32.

4.8 From now on we let J denote a fixed 1-sphere of C and E denote one of the components of $C - J$. The remainder of the proof will be devoted to showing that $\bar{E}$ is a closed 2-cell (I 11.16) whose boundary 1-sphere is J. Theorem 4.2, and hence 4.3, will then follow. The mode of proof will be to set up a process closely analogous to that of subdivision of an S^2 such as is described in II 5.1. It will be helpful to adopt, for the purposes of the present proof, a nomenclature bearing out this analogy, and accordingly we call any finite collection of arcs meeting, in pairs, at most in their end points a 1-complex; and by an E-complex $[E]$ will be designated any subdivision of E by a 1-complex G provided $G = J \cup \bigcup_{i=1}^{n} T_i$ can be constructed by performing n times the following operation: Augment $J \cup \bigcup_{i=1}^{m-1} T_i$ by an arc T_m spanning $J \cup \bigcup_{i=1}^{m-1} T_i$ in $\bar{E}$. The 0-, 1- and 2-cells of $[E]$ are the end points of the arcs forming G, the open arcs of the latter, and the components of $\bar{E} - G$, respectively. The boundary of a 2-cell M consists of the closure of all 1-cells T such that M has a limit point in T.

Given any $[E]$, *there exists a subdivision of the closed 2-cell* $\bar{E}^2$ (*the euclidean* $\bar{E}^2$, *that is*) *by means of polygons such that the resulting complex* $[E^2]$ *is isomorphic*[5] *with* $[E]$.

[5] Two complexes are isomorphic if there exists a (1-1)-correspondence between their cells which preserves dimensionality and incidence [I 6] both ways.

This is readily proved on the basis of 4.6, using induction on the number of arcs T_i employed in constructing G.

Conversely, if there is given a subdivision of $\bar{E}^2$ yielding a complex $[E^2]$ such that each spanning arc used in the construction of $[E^2]$ has its end points in the interiors of two 1-cells of the complex at that stage of the construction, then there exists an isomorphic E-complex $[E]$.

The proof is as in the above case; the provision regarding the end points of arcs used in the construction of $[E^2]$ is made to avoid having to prove accessibility properties of $\bar{E}$.

4.9a For later purposes, we note the following, which may be proved by the sort of argument used in 4.7:

The closure of each domain of $C - G$ is cyclic.

Also, we have:

4.9b *Each domain of $C - G$ is cyclic.*

To prove this, it is sufficient to consider E itself. Suppose E has a cut point p; $E - p = E_1 \cup E_2$ separate. Then $\bar{E}$ contains an arc $Z = apb$ spanning J such that the portion $Z_1 = ap - p - a$ of Z lies in E_1 and the portion $Z_2 = pb - p - b$ of Z lies in E_2. Denoting the two arcs of J with end points a and b by X and Y respectively, and applying 4.6, $C - X - Y - Z = \bigcup_{i=1}^{3} D_i$ where the sets D_i are domains such that D_2 and D_3 have $X \cup Z$ and $Z \cup Y$ as their respective boundaries. Then since $D_2 \subset E_1 \cup E_2$, and D_2 has limit points in Z_1 and Z_2, it meets both E_1 and E_2. But this is impossible since D_2 is connected.

4.10 *If $x, y \in \bar{E}$, $x \neq y$, then x and y can be separated in $\bar{E}$ by an arc spanning J.* If $x \notin J$, then, using Theorem 3.33 and 4.9a, we may show the existence of an arc axb of $\bar{E}$ spanning J and containing x. If $y \in axb$, then, using 4.6, axb may be altered so that $y \notin axb$. And then if $y \notin J$, by use of 4.6, 4.9b and Theorem 3.33 it may be shown that there exists an arc cyd of $\bar{E}$ spanning J and containing y which does not meet axb. Then there exists a simple closed curve J' containing x and y as well as two arcs J_1, J_2 of J which separate x and y on J'; and there exists an arc of $\bar{E}$ spanning J' with end points on J_1 and J_2 which separates x and y in $\bar{E}$. Cases where x or y, or both, lie on J we leave to the reader.

We next show the existence of a finite number of arcs, not necessarily forming a 1-complex, which separate $\bar{E}$ into arbitrarily small components. For any given $\delta > 0$ let F_r, $r = 1, \cdots, n$, be closed sets of diameter $< \delta/2$ such that $\bar{E} = \bigcup F_r$. Since each F_r is compact we may assume for each $p \in \bar{E} - F_r$ the existence of a finite set of arcs spanning J such that for any $x \in F_r$, p and x are separated in $\bar{E}$ by at least one of these arcs. Accordingly, given F_r and F_s such that $F_r \cap F_s = 0$, we may assume the existence of a finite set of arcs spanning J such that if $x \in F_r$, $y \in F_s$, then x and y are separated in E by an arc of the new set. Hence, finally, there exists a finite set of arcs spanning J, say T_1, $\cdots$, T_i, $\cdots$, T_m, such that if x and y are points of disjoint sets F_r, then some T_i separates x and y in $\bar{E}$.

4.11 *We now construct a sequence of E- complexes* $[E]_i$ *defined by 1-complexes* G_i *contained in* $J \cup \bigcup T_j$ *and approximating* $J \cup \bigcup T_j$. Let $G_0 = J = G_0^0$. If G_i^j has been defined and $j < m$, we select for G_i^{j+1} the union of G_i^j and all sub-arcs of T_{j+1}, spanning G_i^j, such that the end points of each sub-arc cannot be joined in G_i^j by an arc of diameter $<1/i$. Since G_i^j is lc, the number of such arcs must be finite and therefore G_i^{j+1} is actually a 1-complex. If G_i^j has been defined and $j = m$, we let $G_i^m = G_{i+1} = G_{i+1}^0$.

For some k, G_k *contains arcs spanning J which separate in* $\overline{E}$ *every possible pair x, y such that* $x \in F_r$, $y \in F_s$, $F_r \cap F_s = 0$. For suppose this not to be the case. Then for each i, there exists such a pair, say x_i, y_i, with the property that G_i contains no arc spanning J which separates x_i and y_i in $\overline{E}$. Without loss of generality we may assume that all points x_i are in a fixed F_r and all points y_i in a fixed F_s such that $F_r \cap F_s = 0$. There exist x lp $\{x_i\}$ and y lp $\{y_i\}$ such that $x \in F_r$, $y \in F_s$. No G_i contains an arc spanning J and separating x and y, but some T_j (which we consider as fixed from now on) does. Let X and Y be arcs not meeting T_j joining x and y respectively to J, and let k be a positive integer such that $1/k < \rho\,(T_j, X \cup Y)$. Let $U = \overline{E} \cap \overline{S}(T_j, 1/k)$. Then the end points x' and y' of T_j are in a connected subset of the set $U' = U \cap G_{k+1}$. For if $U' = U_1 \cup U_2$ separate, $x' \in U_1$, $y' \in U_2$, then there exist $p, q \in T_j$ such that p is the last point of $T_j \cap U_1$, and q the first point of $T_j \cap U_2$ after p, in the order from x' to y'. But then G_{k+1} must contain an arc from p to q of diameter $<1/k$, and this arc must therefore lie in U'—which is impossible. Thus U' must contain an arc Z from x' to y' and Z contains an arc spanning J separating the end points of X and Y on J, and accordingly separating x and y in E, which is impossible.

4.12 *The E-complex* $[E]_k$ *defined by* G_k *has no cells of diameter* $\geqq \delta$, since two points on the closure of a 2-cell of $[E]_k$ would have to be in the same or intersecting sets F_r.

Let $[E^2]_k$ be a polygonal subdivision of $\overline{E}^2$ isomorphic to $[E]_k$, $[E^2, \delta]_k$ a polygonal subdivision of $[E^2]_k$ whose cells are all of diameter $<\delta$ (such that any new 0-cell is interior to some old 1-cell), and $[E, \delta]_k$ a subdivision of $[E]_k$ isomorphic to $[E^2, \delta]_k$. In this manner we see, then, that we may construct a sequence of isomorphic pairs $[E, \delta_i]$, $[E^2, \delta_i]$ such that δ_i approaches zero with increasing i, inasmuch as the process defined as above for $\overline{E}$ may be repeated for the closure of any 2-cell of $[E, \delta]$.

4.13 Let A and B be the sets of 0-cells of all $[E, \delta_i]$'s and of all $[E^2, \delta_i]$'s respectively. Then $\overline{A} = \overline{E}$ and $\overline{B} = \overline{E}^2$. Moreover, there exists a (1-1)-mapping f of A onto B such that both f and f^{-1} are uniformly continuous. The correspondence f is obtained from the above-mentioned isomorphisms. As for the uniform continuity: Let $\epsilon > 0$ be given, and let n be such that $2\delta_n < \epsilon$. It is easily shown that there exists a $\delta > 0$ such that if x, y are a pair of points of $\overline{E}$ or of $\overline{E}^2$, such that $\rho(x, y) < \delta$, then x and y are in or on the boundary of the same or intersecting 2-cells of $[E, \delta_n]$ or $[E^2, \delta_n]$. If two 0-cells of any subdivision of $\overline{E}$ $(\overline{E}^2)$ are of distance apart $<\delta$, the 0-cells corresponding to them

under f (f^{-1}) in $\bar{E}^2$ ($\bar{E}$) are in or on the boundary of the same or intersecting 2-cells of $[E^2, \delta_n]$ ($[E, \delta_n]$), so that their distance apart is $<\epsilon$. Hence f and f^{-1} are uniformly continuous. Since by Lemma 4.1 the mapping f may be extended to a homeomorphism between $\bar{E}$ and $\bar{E}^2$, it follows that $\bar{E}$ is a closed 2-cell, which is what we set out to prove. As noted above, Theorems 4.2 and 4.3 follow immediately, since the space C and the 2-sphere S^2 are readily shown to be homeomorphic, in view of the existence of the homeomorphism f defined above.

5. **Recognition of the closed 2-cell.** As a basis for later (VII 9.5; IX 7.11) characterizations in terms of homology, we shall give here a characterization of the closed 2-cell.

5.1 THEOREM (ZIPPIN). *A Peano continuum C containing a* 1*-sphere J and satisfying the following three conditions is a closed* 2*-cell with boundary J*:

5.1a *C contains an arc that spans J.*

5.1b *Every arc of C that spans J separates C.*

5.1c *No closed proper subset of an arc spanning J separates C.*

As in the case of the above proof of Theorems 4.2 and 4.3, we shall sectionalize the proof.

5.2 From conditions 5.1b and 5.1c, and the properties of lc spaces, we have: *If $X = xy$ is an arc spanning J in C, then every component of $C - X$ has every point of X as a limit point.*

5.3 Let X be as in 5.2. Then *C does not contain three arcs $T = ab$, $T_1 = a_1b_1$, $T_2 = a_2b_2$ having the following properties: T_1 and T_2 span $J \cup X$ and $\langle T_1 \rangle$, $\langle T_2 \rangle$ lie in different components of $C - X$; T spans X and meets neither T_1 nor T_2; furthermore, the end point a_i ($i = 1, 2$) is between a and b on X and separated from b_i in $J \cup X$ by $a \cup b$, and if b_1, b_2 are on X, then they are separated on X by both a and b.*

If such arcs as T, T_1 and T_2 existed, then there would exist an arc Y spanning J consisting of T_1, T_2 and 0, 1, 2 or 3 arcs on X, depending on whether $a_1 = a_2$ or not and whether b_1 and b_2 are on $\langle X \rangle$ or not. Every component D of $C - Y$ would intersect X but all points of $X - Y$ would lie in one component of $C - Y$ so that $C - Y$ would be connected.

5.4 It follows from 5.3 that *$C - X$ consists of exactly two components.*

5.5 We shall now suppose that $C - X$ has a component A which does not meet J, and show that this supposition leads to contradiction.

The set $A' = \bar{A} - x$ is cyclicly connected. We note that $\bar{A}$ is a Peano continuum by Theorem 3.38, and x is a non-cut point of $\bar{A}$. The set $\langle X \rangle$ contains no cut point of either $\bar{A}$ or A' because of 5.2, and therefore $X - x$ must lie wholly in one cyclic element M of A' and X in one cyclic element of $\bar{A}$. If $M \neq A'$, then M contains a cut point z of A' through which passes an arc xzy of $\bar{A}$ (by virtue of Theorem 3.33). But then the subset $x \cup z$ of xzy disconnects C, contradicting 5.1c.

We shall say that *an arc Z covers a point p* if Z has x and y as end points, contains a subarc xz of X, is contained in $\bar{A}$, and if p lies in the component of $C - Z$ containing $C - \bar{A}$. *Each point p of $A' - y$ is covered by at least one arc.* For A' contains an arc zpy spanning $\langle \underline{X} \rangle \cup y$ by Theorem 3.33. In the component of $C - xzpy$ not containing $C - \bar{A}$, there exists an open arc $\langle sr\underline{y} \rangle$, where $s \in \langle zp \rangle$. Now the component H of $C - xzpy$ that contains $C - \bar{A}$ has p as limit point by 5.2 and consequently $\bar{H} \cup p \subset C - xzsry$. Hence the component of $C - xzsry$ that contains $C - \bar{A}$ also contains p and $xzsry$ covers p. Now if U is an open subset of $\bar{A}$ containing x and $y \notin \bar{U}$, then each point of $F(U)$ is covered by some arc, and this arc covers an open subset of $F(U)$. Hence there exists a finite set of arcs, $T_1, \cdots, T_j, \cdots, T_m$, covering all points of $F(U)$.

We next define a sequence of 1-complexes G_i approximating $\langle X \rangle \cup \bigcup T_j$ analogous to the sequence of 4.11. Then there must be an integer k such that for each point p of $F(U)$ there is an arc in G_k that covers p. Otherwise we could show as in 4.11 the existence of a $p \in F(U)$ covered by a T_j but not covered by an arc of a G_i. Let T' be an arc joining p to a point of X but not meeting T_j. Then some G_i contains an arc xy not meeting T', and this arc must cover p.

The 1-complex G_i contains an arc B_i that covers all points of $G_i - B_i$. In view of how G_i was constructed, in order to prove this we need only show that if any 1-complex G contained in $X \cup \bigcup T_j$ contains an arc B covering all points of $G - B$, and if N is an arc in $X \cup \bigcup T_j$ spanning G, then $G' = G \cup N$ contains an arc B' covering all points of $G' - B'$. If B covers all points of $\langle N \rangle$, we may let $B' = B$. Otherwise, $\langle N \rangle$ is contained in the component of $C - B$ not meeting J and N must span B. Its end points determine a subarc N' of B and we may let $B' = B \cup \langle N \rangle - \langle N' \rangle$.

The arc B_k of G_k covers all points that are covered by any arc B' of G_k. If D_k (D') is the domain of $C - B_k$ ($C - B'$) not meeting J, we must prove that $D_k \subset D'$. Since $C - D_k \supset B'$, the only alternative is that $D_k \cap D' = 0$ (for $D_k \cap D' \neq 0$ and $D_k \not\subset D'$ would imply $D_k \cap B' \neq 0$ by Theorem I 7.8). The arcs B_k, B' must have some subarc xz of X in common, inasmuch as all the arcs T_j have a subarc in common. Hence three disjoint domains have xz on their boundaries, namely D_k, D' and $C - \bar{A}$, and in each of these there exists an open arc with end points on any two open subarcs of xz. But this implies a contradiction of 5.3.

The arc B_k would now of necessity cover all points of $F(U)$, and this is impossible because B_k must contain points of $F(U)$.

We conclude, then, that each component of $C - X$ meets J. As a corollary, we have:

5.6 *There do not exist disjoint arcs T_1 and T_2 spanning J such that the end points of T_1 separate the end points of T_2 on J.*

5.7 *If M is a component of $C - J$ containing an arc spanning J, then $\bar{M} \supset J$.* For suppose not. Then there exists a maximal open subarc $\langle T \rangle = \langle pq \rangle$ of J such that no point of it is a limit point of M. The set $M \cup J - \langle T \rangle$ is lc by

Theorem 3.38, and both p and q lie in a cyclic element N of this set—for a point separating p and q in $M \cup J - \langle T \rangle$ would of necessity lie on $J - T$ and its existence would imply nonconnectedness of M. By Theorem 3.32, there exists a 1-sphere J' in N containing $p \cup q$, and J' contains an arc pq spanning J. But then $p \cup q$ would be a subset of an arc spanning J and separating C, which is impossible.

It follows, because of 5.6, that every component of $C - J$ different from M has exactly one limit point on J. If all such components are deleted, C will satisfy the condition that $C - J$ is connected. We shall prove that C is a closed 2-cell under this condition, and it will then follow that no components other than M can have formed part of $C - J$. We note before proceeding that under the additional condition, *C is cyclic.*

5.8 If A is a component of $C - X$, it follows from what we have already proved that the portion of J which it contains is an open arc $\langle T \rangle$. Then $\bar{A}$ has all the properties that were assumed for C, with $X \cup T$ replacing J. That $\bar{A}$ contains a spanning arc of $X \cup T$ and is cyclic is easily proved. We have to show:

(a) *If B is an arc in $\bar{A}$ spanning $X \cup T$, then $\bar{A} - B$ is not connected.* If both end points of B are on J, the separation of C by B defines a separation of $\bar{A}$ by B. Suppose, however, that at least one end point of B is on X. There is an arc Y spanning J, contained in $X \cup B$ and containing B. The portion of X not on Y is an open subarc $\langle D \rangle$ of X. Suppose B does not separate $\bar{A}$. Then there exists an arc pq in $\bar{A}$ from a point p on $\langle D \rangle$ to a point q of $X - D$. The arc B, the arc pq, and an arc in $C - \bar{A}$ from a point on X between p and q to a point of J are three arcs contradicting 5.3. We conclude, then, that (a) holds.

(b) *If E is a subarc of an arc B in $\bar{A}$ spanning $X \cup T$, then $B - E$ does not separate $\bar{A}$.* Defining Y and D as before, any two points of $\bar{A} - Y$ are joined by an arc of $C - (Y - \langle E \rangle)$. All portions of that arc in $C - \bar{A}$ can be replaced by subarcs of D, so that the two points mentioned are joined by an arc in $\bar{A} - (B - \langle E \rangle)$.

From here on, the proof of Theorem 5.1 is like that of Theorems 4.2 and 4.3 from 4.8 on, with $\bar{E}$ replaced by $\bar{A}$, and may be omitted.

As an immediate corollary of Theorem 5.1 we have:

5.9 SCHOENFLIES EXTENSION THEOREM. *If D is a domain (bounded domain) complementary to a simple closed curve J in S^2 (E^2), then $\bar{D}$ is a closed 2-cell with boundary J.*

6. **Recognition of the 2-manifolds.** Up to this point, our attention has been fixed mainly on the *sphere*, of 1 and 2 dimensions. It is time, however, to generalize to the more general configurations known under the name *2-manifolds.* It is not our intention to give an exhaustive treatment of these—they have been adequately treated elsewhere; in particular the reader is referred to Kerékjártó [K]. But for the sake of completeness, and in order to make connections with the general n-dimensional case in which we are mainly interested,

we shall give the properties which serve to characterize them among the Peano spaces.

Generally, by a 2-manifold, we shall mean a connected, separable metric space, every point of which has a neighborhood that is homeomorphic with the euclidean plane; in other words, every point has a 2-cell neighborhood. A 2-manifold is obviously a special type of Peano space. And in case a 2-manifold is compact, we call it a *closed 2-manifold*[6]—otherwise, an *infinite 2-manifold*. Thus the 2-sphere is a special type of closed 2-manifold, and the 2-cell E^2 an infinite 2-manifold. The torus (surface of the anchor ring) is another example of a closed 2-manifold, as are also the projective plane and "Klein bottle." Other simple examples of closed 2-manifolds are obtainable through the device of cutting nonintersecting tunnels through a spherical block, and considering the surfaces of the resulting solids (Alexander [e]); these are frequently described as "spheres with handles," since they are also obtainable by attaching "handles" to the 2-sphere. Other examples of infinite 2-manifolds, topologically distinct from E^2, may be obtained by deleting nonintersecting closed 2-cells from E^2.

We shall prove the following theorems:

6.1 THEOREM. *A Peano continuum B which, for some positive number ϵ, satisfies the following three conditions is a closed 2-manifold:*

6.1a *B contains at least one 1-sphere of diameter $<\epsilon$.*

6.1b *Every 1-sphere of diameter $<\epsilon$ in B separates B.*

6.1c *No arc of a 1-sphere of diameter $<\epsilon$ in B separates B.*

6.2 THEOREM. *If B is a noncompact Peano space such that for every compact subset F of B there exists a positive number $\epsilon(F)$ such that the following three conditions are satisfied, then B is an infinite 2-manifold.*

6.2a *B contains a 1-sphere J of diameter $<\epsilon(J)$.*

6.2b *If F is any compact subset of B and J is a 1-sphere of F of diameter $<\epsilon(F)$, then J separates B.*

6.2c *With F as before, no subarc of a 1-sphere of F of diameter $<\epsilon(F)$ separates B.*

The practically identical proofs of these two theorems are given below in sectionalized form.

6.3 Suppose B has a non-cut point p and that δ is an arbitrary positive number. There exists, by Corollary 3.16, a positive number $\eta(\delta)$ such that all points of $B - S(p, \delta)$ lie in one component of $B - S(p, \eta(\delta))$. Then *if H is a 1-sphere in $S(p, \eta(\epsilon/2))$, the set $B - H$ has exactly two components, the closure of one of which is a closed 2-cell of diameter $<\epsilon$.* (In the case of Theorem 6.2, we use $\epsilon(F)$ instead of ϵ, where F is the compact closure of a sufficiently small neighborhood of p). To see this, note that $B - S(p, \epsilon/2)$ lies in one component A of $B - H$, and hence all 1-spheres in $B - A$ are of diameter $<\epsilon$ and separate B. The method used in 4.5 shows that the number of components

[6] Frequently called, in the literature, a *manifold without boundary.*

of $B - H$ is two (let T lie in A). The method of 4.6 shows that an arc spanning H in $B - A$ separates $B - \bar{A}$ into two components (let H play the part of $X \cup Z$ in 4.6). The method of 4.8-4.13 may then be applied to show that the set $B - A$ is a closed 2-cell.

6.4 We shall need, for present purposes, a notion like that of the 2-complex of II 5.2, which is somewhat different from the latter, however. By a *B-complex* K we shall mean a point set consisting of a finite or countable set of arcs (closed 1-cells) and closed 2-cells, called basic, such that (1) two basic closed i-cells ($i = 1$ or 2) meet at most in their boundaries, and no basic closed 1-cell meets a basic closed 2-cell except in the boundary of the latter; (2) each closed cell of K has a neighborhood in B that meets only a finite number of the cells of K; and (3) no basic arc is on the boundary of more than two basic 2-cells. The collection of all points in a B-complex K that lie in open 2-cells (not necessarily basic) of K is the union of infinite 2-manifolds, each component being an infinite 2-manifold; we may call it the interior of K.

We shall call a point of B *regular* if it has a neighborhood which is a 2-cell in B.

We first construct a B-complex K such that the set C of all regular points of B is a subset of the interior of K.

There exists a countable set of open 2-cells $M_1, \cdots, M_n, \cdots$ covering C and a similar set of 2-cells $N_1, \cdots, N_n, \cdots$ covering C such that for each n, $\bar{N}_n \subset M_n$. Let K_1 be the B-complex consisting of $\bar{N}_1$ as basic closed 2-cell, and suppose that a B-complex K_{m-1} has been constructed covering $\bigcup_{i<m} \bar{N}_i$. We shall construct a B-complex K_m covering $\bigcup_{i \leq m} \bar{N}_i$.

If K_{m-1} does not meet $\bar{N}_m$, let T be an arc spanning $K_{m-1} \cup \bar{N}_m$—then $K_{m-1} \cup T \cup \bar{N}_m$ forms a B-complex K_m with T and $\bar{N}_m$ as basic closed cells augmenting the basic cells of K_{m-1}. If K_{m-1} meets $\bar{N}_m$, then owing to the fact that K_{m-1} is lc, only a finite number of components of $M_m \cap (B - K_{m-1})$ meet N_m; and of these, the intersection of each with $M_m - \bar{N}_m$ has only a finite number of components U_i having limit points on the boundaries of both M_m and N_m. In each of the sets U_i there exists an arc T_i spanning both K_{m-1} and the boundary of U_i, and separating in $\bar{U}_i$ the boundaries of M_m and N_m. Let $L_m = K_{m-1} \cup \bigcup T_i$, and let K_m consist of L_m augmented by the components of $M_m - L_m$ separated by L_m from $F(M_m)$, after suitable subdivision by arcs spanning L_m if necessary. That K_m is actually a B-complex will follow from 5.9.

As no basic 2-cell of K_{m-1} is subdivided, and each point of C is regular, this process leads to a B-complex whose interior covers C.

6.5 The remainder of the proof will consist in showing that $B = C$.

Every point of the given 1-sphere J (6.1a or 6.2a) is a limit point of $B - J$ and is a non-cut point of B; this follows at once from the given conditions.

If $p \in J$, every neighborhood of p contains 1-spheres. For let $\eta > 0$ be such that $\bar{S}(p, \eta)$ is compact. There exists a sequence of arcs from a point not on J to a sequence of distinct points on J converging to p, and hence a subsequence of subarcs $p_i q_i$ of these arcs such that: $p_i \in J$ and $\{p_i\}$ converges to p; for a certain δ, $0 < \delta < \eta$, q_i is the first point of $F(p_i, \delta/3)$ in the order

from p_i on the arc with end point p_i ; $\{q_i\}$ converges to a point q. If $q \notin J$, there exists m such that for $i \geqq m$ there is an arc pp_i on J of diameter $\leqq \delta/3$, and an arc qq_i in B of diameter $\leqq \delta/3$ not meeting J. Then for $i > m$, $p_m p_i \cup p_i q_i \cup q_i q \cup qq_m \cup q_m p_m$ contains a 1-sphere that lies in $S(p, \eta)$. If $q \in J$, δ must satisfy an additional condition: The set $J \cap S(p, \delta/3)$ must lie on a subarc X of J such that $X \subset S(p, \eta/3)$. There is a number m such that for $i = m$ there exists an arc qq_i of diameter $\leqq \delta/3$ not meeting p_i. Then $p_i q_i \cup q_i q \cup Z$, where Z is the subarc qp_i of J, will contain a 1-sphere that lies in $S(p, \eta)$.

From 6.3 and 6.5 it follows that B *contains regular points*, i.e., $C \neq 0$.

6.6 Suppose B contains a point that is not regular. Then since C is open, there exists an arc X which lies in C except for one end point p, which is a nonregular point. If U is a connected neighborhood of p, then p is in the cyclic element of U containing the component $X(U)$ of $X \cap U$ determined by p. Consequently by Theorem 3.32, p lies on arbitrarily small 1-spheres, and by 6.1c or 6.2c is not a cut point of B.

There exists a neighborhood V of p such that every 1-sphere in V is the boundary of a 2-cell in V. Let U be a neighborhood of p of diameter $<\eta(\epsilon/2)$ (see 6.3) and let V be the union of U and all 2-cells that are bounded by 1-spheres of U and that contain no points of $B - S(p, \epsilon/2)$ (we assume $\epsilon <$ diameter B). Then if S is a 1-sphere contained in the union of U and a finite number of such 2-cells, evidently S also bounds a 2-cell in V. That such is the case for every 1-sphere S of V follows from the compactness of S.

6.7 The component R of $C \cap V$ determined by $X(V)$ (see 6.6) is a 2-cell. This follows from 6.4 and the fact that the interior of a B-complex which is "simply connected", in the sense that every 1-sphere in it is the boundary of a 2-cell in it, is a 2-cell.

There exists an open set W containing nonregular points, such that $R \subset W \subset \overline{R}$. The cyclic element of V which contains R is identical with $\overline{R} \cap V$ and every point of $(\overline{R} - R) \cap V$ is accessible by arcs from R. For each point y of that cyclic element is on a 1-sphere of V containing a point of R so that, by definition of V, y is on a closed 2-cell whose interior meets R. Now let us delete from V the cut points of V on $\overline{R} \cap V$ as well as all points of V that are separated from points of $\overline{R} \cap V$ by such cut points. The resulting set W is open, satisfies the relations $R \subset W \subset \overline{R}$, and contains at least one nonregular point. For $\overline{R} - R$ is a connected set containing at least one continuum of nonregular points, and the set of cut points on $\overline{R} \cap V$ has no limit point in V. (The latter statement follows from the fact that the cut points on $\overline{R} \cap V$ cannot be cut points of B, so that the portions separated from points of $\overline{R} \cap V$ must have limit points on the boundary of V.)

6.8 *The existence of the set W implies a contradiction.*

If Z is an arc with end points q and r in $\overline{R} - R$ and $\langle Z \rangle$ in R, then $W - Z = W_1 \cup W_2$ separate, $\overline{W}_1 \cap \overline{W}_2 = Z$. This follows from the fact that $\overline{R}$ is a closed 2-cell and that $R \subset W \subset \overline{R}$. If q and r are in the same component of

nonregular points in W, then Z is part of a 1-sphere in W, so that Z cannot separate B by 6.1c or 6.2c. But if both W_1 and W_2 had limit points in $\bar{R} - W$ ($\subset \bar{R} - Z$), there would exist an arc Z' meeting Z in one point with end points in $\bar{R} - W$, and $\langle Z' \rangle$ would separate q and r in W. And $\langle Z' \rangle$ would contain no nonregular points, whereas q and r are on the same component of nonregular points in W. Hence $\bar{W}_1 - Z$, say, is a subset of W, and it follows that $B - Z$ is not connected, contradicting the fact that Z cannot separate B.

This completes the proof of Theorems 6.1 and 6.2.

It will be noted that in 6.4 it was shown that there exists a B-complex K such that the set C of all regular points of B is interior to K. And since, as is easily shown, every closed or infinite 2-manifold satisfies the conditions of Theorems 6.1 or 6.2, we have:

6.9 ("TRIANGULATION THEOREM"). *If B is a closed or infinite 2-manifold, then there exists a B-complex K covering B. In case B is closed, then K has only a finite number of cells.*

BIBLIOGRAPHICAL COMMENT

§1. For references regarding Peano's solution of the space-filling curve problem and the work of Hahn and Mazurkiewicz, see R. L. Moore [d]. Lemma 1.11 was proved by P. Urysohn [b], as was also Theorem 1.14 [c].

§2. Theorem 2.1 was proved by P. Alexandroff [a]. Theorem 2.5 is the Hahn-Mazurkiewicz Theorem, regarding which see the citations in Moore [d]; also Whyburn [g].

§3. The material in 3.1–3.5, 3.7, was abstracted in Wilder [A_2], but not published heretofore. The history of Theorem 3.9 (and 3.10), the "arc theorem," as well as the extension to complete spaces, may be traced through the comment and citations in Moore [d] and Whyburn [h]. The proof given above is taken from the latter paper, but there are numerous proofs in the literature, several of which make use of the mapping from the real number interval [0, 1].

The development of the cyclic element theory utilized in the proof of Theorem 3.32 ("cyclic connectivity theorem"), as well as the latter proof, are based on Whyburn [f]. For later developments and more complete details regarding cyclic element theory, see Whyburn [Wh, Chapter IV].

§§4, 5, 6. The proofs given here follow closely van Kampen [a], to which the reader is referred for citations to sources of the characterizations given above.

Since this chapter was written, there have appeared new characterizations of the 2-sphere and 2-manifolds; see Bing [a] and Young [a].

CHAPTER IV

NON-METRIC LC SPACES, WITH APPLICATIONS TO SUBSETS OF THE 2-SPHERE

The preceding chapter was devoted to the study of metric (chiefly Peano) spaces. In the present chapter we return to the general Hausdorff space without any assumption of a metric, with some applications in the closing sections to euclidean spaces.

The Hahn-Mazurkiewicz theorem (III 2.5) gave rise to a new development in the set-theoretic topology, namely the investigation of Peano continua. A number of new characterizations of these continua were found in the search for their topological properties. The chief tools in the investigation were the arc theorem (III 3.10) and the Whyburn cyclic element theory (III 3.18-3.32), and while generalizations were made to noncompact spaces, the basic space was always metric. There seems no reason, a priori, why the structure study of locally connected, compact (or locally compact) connected spaces should be confined to the metric case. (While in the book of Moore [Mo], Chapter II, there is no explicit assumption of a metric in the continuous curve characterizations (for example, Theorems 50, 52, 53), the continua involved are nevertheless metrizable (as proved in loc. cit. I, Theorem 19).) It is not the purpose of the present chapter to settle the problem as to whether such a study could be carried through to a satisfactory conclusion, but some results are obtained that might be taken to indicate that it could. Even such basic configurations as the arc and 1-sphere, occupying a central position in the classical theory, permit of satisfactory generalizations (see Chapters IX and X), so that their retention is not precluded by complete exclusion of a metric. (One may ask, for example, does the arc theorem (III 3.10) generalize to nonmetric spaces, when for "arc" is substituted the case $n = 1$ of the generalized closed n-cell of Chapter IX?)

It will be necessary first to establish further properties of locally compact Hausdorff spaces.

1. Components of locally compact Hausdorff spaces.

1.1 THEOREM. *In a compact Hausdorff space, quasi-components and components are identical.*

PROOF. Let Q be a quasi-component of a compact Hausdorff space S, and suppose that $Q = A \cup B$ separate. Since (I 8.8) Q is closed, the sets A and B are closed. And as every compact Hausdorff space is normal (III 1.27), there exist disjoint open sets U, V containing A, B respectively. Denote the complement of $U \cup V$ by C. Then C is compact, since it is a closed subset of a compact space. (I 12.11).

For each $x \in C$, $S = A(x) \cup B(x)$ separate, where $x \in A(x)$, $Q \subset B(x)$, since Q is a quasi-component of S. As C is compact, a finite number of the open sets $A(x)$ cover C—say $A(x_1), \cdots, A(x_k)$. Let $G = \bigcup A(x_i)$, $E = \bigcap B(x_i)$. Then $S = G \cup E$ separate, where $G \supset C$ and $E \supset Q$.

Now $G \cup U$ and $E \cap V$ are disjoint open sets. And since every point of V that is not in E is in G, the union of $G \cup U$ and $E \cap V$ is S. Furthermore, $G \cup U \supset U \supset A$ and $E \cap V \supset B$. But this implies that A and B are not in the same quasi-component of S.

1.2 Theorem. *If M is a compact component of a locally compact Hausdorff space S, and P is an open set containing M, then S is the union of disjoint open sets U, V such that $M \subset U \subset P$.*

Proof. Since S is locally compact, each $x \in S$ is in an open set $E(x)$ whose closure is compact. And inasmuch as M is compact, a finite number of such sets $E(x)$, each of which may be assumed to lie in P, cover M. Consequently we may replace P by a union of the sets $E(x)$—or what amounts to the same thing, we may assume that $\bar{P}$ is compact.

Since M is a component of S, it is certainly a component of $\bar{P}$. And as $\bar{P}$ is compact, M is a quasi-component of $\bar{P}$ by the preceding theorem. Since $M \cap F(P) = 0$ (I 7.5), for each $x \in F(P)$ there exists a separation $\bar{P} = A(x) \cup B(x)$, where $x \in A(x)$, $M \subset B(x)$. As $F(P)$ is a closed subset of a compact Hausdorff space $\bar{P}$, it is itself compact, and therefore a finite set of the $A(x)$'s, say $A(x_1), \cdots, A(x_k)$, cover $F(P)$. Let $U = \bigcup A(x_i)$, $V = \bigcap B(x_i)$. Then U and V are disjoint open subsets of $\bar{P}$ such that $\bar{P} = U \cup V$, $F(P) \subset U$, $M \subset V$. Evidently $S = [(S - \bar{P}) \cup U] \cup V$ separate, where $V \subset P$.

1.3 Theorem. *Let A be a compact subset of a locally compact Hausdorff space S such that for each $x \in A$ the component, $C(x)$, of S that contains x is compact. Let $C = \bigcup_{x \in A} C(x)$, and let B be a closed subset of S lying in $S - C$. Then $S = S_1 \cup S_2$ separate, where $C \subset S_1$ and $B \subset S_2$.*

Proof. For each $x \in A$ there exists, by Theorem 1.2, a decomposition $S = U(x) \cup V(x)$ separate such that $C(x) \subset U(x) \subset S - B$. Since A is compact, there exists a finite number of points $x_1, \cdots, x_k \in A$ such that the sets $U(x_i)$, $i = 1, \cdots, k$, form a covering of A. Then $S_1 = \bigcup_{i=1}^k U(x_i)$, $S_2 = \bigcap_{i=1}^k V(x_i)$ satisfy the conclusion of the theorem.

1.4 Corollary. *In a locally compact Hausdorff space, the compact components and the compact quasi-components are identical.*

Remark. The necessity of the qualifying "compact" in the statement of this corollary is shown by the following example (cf. Hausdorff [H, 249]): In the cartesian plane let R_n be the set of points on the perimeter of the rectangle determined by the lines $x = \pm n/(n + 1)$, $y = \pm n$, $n = 1, 2, 3, \cdots$. Let S denote the union of the sets R_n and of the lines $x = \pm 1$. Then the two lines $x = \pm 1$ lie in the same quasi-component of the space S.

And as a corollary of 1.4 and Theorem I 9.7b:

1.4a Corollary. *If $x_1, x_2, \cdots, x_k$ are a finite number of points of a locally compact Hausdorff space S, no two of which lie in the same component of S but such that the components containing them are all compact, then S is the union of pairwise separated open sets S_i, $i = 1, 2, \cdots, k$ such that $x_i \in S_i$.*

The following corollaries of Theorem 1.3 are also worth noting:

1.5 Corollary. *If A_1 and A_2 are disjoint closed subsets of a compact Hausdorff space S, and no point of A_1 is c-equivalent to a point of A_2, then $S = S_1 \cup S_2$ separate, where $S_i \supset A_i$, $i = 1, 2$.*

1.6 Corollary. *If K is a compact subset of a locally compact Hausdorff space S, and every component of S that intersects K is compact, then the set, M, composed of all points of S that are c-equivalent to points of K, is closed.*

1.7 Corollary. *If K is a closed subset of a compact Hausdorff space S, and M is the set composed of all points of S that are c-equivalent to points of K, then M is closed.*

We recall that in 12.14 we defined *continuum* to mean a nondegenerate, compact, connected space. From now on, we shall always assume that a continuum is a Hausdorff space.

1.8 Theorem. *If K is a closed subset of a continuum S, and C is a component of $S - K$, then $K \cap F(C) \neq 0$.*

Actually, we may prove a more general theorem than this, for which we need the following lemma:

1.9 Lemma. *If S is locally compact, and A, B are disjoint closed subsets of S such that A is compact, then there exist disjoint open subsets of S containing A and B respectively.*

Proof. Each $x \in A$ is in an open set U_x such that $\overline{U}_x$ is compact and contained in $S - B$; and as A is compact, a finite number of these open sets covers A, the union of these yielding an open set U'. Evidently $\overline{U}'$ is compact and $\overline{U}' \cap B = 0$; hence U', $S - \overline{U}'$ are disjoint open sets containing A and B respectively.

1.10 Theorem. *If K is a closed subset of a locally compact, connected space S and C is a component of $S - K$ such that $\overline{C}$ is compact, then $K \cap F(C) \neq 0$.*

Proof. Suppose $K \cap F(C) = 0$, so that $\overline{C} = C$ is a subset of $S - K$. By the above lemma, there exist disjoint open sets U, V containing C, K, respectively. By Theorem 1.2, $S - K = A \cup B$ separate, where $C \subset A \subset U$. But then $S = A \cup (B \cup K)$ separate, contradicting the fact that S is connected.

Remark. In general, $F(C)$ is not a subset of K. For example, in the cartesian plane, let $K = \{(x, y) \mid (0 \leqq x \leqq 1)\&(y = 0)\}$, $C = \{(x, y) \mid (x = 0)\&$

$(0 < y \leqq 1)\}$, $L_n = \{(x, y) \mid (x = 1/n)\&(0 \leqq y \leqq 1)\}$ for $n = 1, 2, 3, \cdots$. Let $S = \bigcup L_n \cup K \cup C$. Then C is a component of $S - K$ and $F(C) = C$.

That the conclusion of Theorem 1.10 no longer holds if C is not compact, even when K is compact, is shown by the following example in the cartesian plane: Let $K = (1, 0)$; $C = \{(x, y) \mid (x = -1)\&(y \geqq 0)\}$; $L = \{(x, y) \mid (x = 1)\& (y \geqq 0)\}$; and for each natural number n, $p_n = (0, n)$, $q_n = (-n/(n + 1), 0)$. Let C_n be the set of all points on the straight line intervals p_nq_n and Kp_n, except for the point K. Finally, let $S = C \cup L \cup \bigcup C_n$.

1.11 Definition. If $\{M_\nu\}$ is a collection of sets M_ν, then by lim sup $\{M_\nu\}$ we denote the set of all points x such that every open set which contains x also contains points of infinitely many of the sets M_ν. The following lemma is evident from the definition:

1.12 Lemma. *If $\{M_\mu\}$ is a subcollection of $\{M_\nu\}$, then* lim sup $\{M_\mu\} \subset$ lim sup $\{M_\nu\}$.

1.13 Lemma. Lim sup $\{M_\nu\}$ *is a closed point set.*

(We leave the proof of this lemma to the reader.)

1.14 Theorem. *In a compact Hausdorff space S, let $\{M_\nu\}$ be a collection of sets M_ν, each component of M_ν being a continuum that meets a fixed closed set A. Then every component of* lim sup $\{M_\nu\}$ *meets A.*

Proof. Let C be a component of lim sup $\{M_\nu\}$ and suppose $C \cap A = 0$. Then by Theorem 1.2, lim sup $\{M_\nu\} = G \cup H$ separate such that $C \subset G \subset S - A$. Since G and H are closed, there exist disjoint open sets U', V' containing G and $H \cup A$ respectively. But there is an infinite subcollection $\{M_\mu\}$ of $\{M_\nu\}$ such that each set $M_\mu \cap U' \neq 0$, since every open set containing C must contain points of infinitely many sets M_ν. Let M'_μ be a component of M_μ such that $M'_\mu \cap U' \neq 0$. But $M'_\mu \cap A \subset M'_\mu \cap (S - U')$. Hence each set $M'_\mu \cap F(U') \neq 0$, and as $F(U')$ is compact, $F(U') \cap$ lim sup $\{M'_\mu\} \subset F(U') \cap$ lim sup $\{M_\mu\} \subset F(U') \cap$ lim sup $\{M_\nu\} \neq 0$.

2. A characterization of locally compact, connected spaces that fail to be lc.

2.1 Theorem. *If the locally compact, connected Hausdorff space S is not lc, then there exist $x \in S$, and open sets P, Q such that $x \in Q \Subset P$ and such that (1) infinitely many components, M_ν, of $\bar{P} - Q$ contain points of both $F(P)$ and $F(Q)$, (2) there exists a component M_ω of* lim sup $\{M_\nu\}$, *such that M_ω meets both $F(P)$ and $F(Q)$ and such that (3) if K is the component of $\bar{P} - Q$ that contains M_ω, then some component of $S - K$ contains infinitely many of the sets M_ν having a component of their* lim sup *in $M_\omega \cap (\bar{P} - Q)$, this component meeting $F(P)$ and $F(Q)$.*

Proof. Since S is not lc, there exist [II 1.7] $x \in S$ and open set P' containing

x such that no open set Q such that $x \in Q \subset P'$ lies in one component of P'. We may assume $\overline{P'}$ compact. Let Q be an open set such that $x \in Q \Subset P'$. Then the collection $\{C'_\nu\}$ of components of P' that have points in Q is infinite. Let P be an open set such that $Q \Subset P \Subset P'$, and let $x'_\nu \in C'_\nu \cap Q$. Then by Theorem 1.10 the component C_ν of $\overline{C'_\nu} \cap P$ that contains x'_ν has limit points in $S - P$—i.e., in $F(P)$. Let $x_\nu \in \overline{C}_\nu \cap F(P)$. Again by Theorem 1.10, the component K_ν of $\overline{C}_\nu \cap (S - \overline{Q})$ that contains x_ν has limit points in $F(Q)$. Let M_ν be the component of $C_\nu \cap (\overline{P} - Q)$ that contains K_ν.

Let P_1, P_2, P_3, P_4 be open sets such that $P \Supset P_1 \Supset P_2 \Supset P_3 \Supset P_4 \Supset Q$. By an argument similar to that just employed to show the existence of the sets M_ν, we may show that for each ν there exists a subcontinuum M'_ν of M_ν that is a component of $\overline{P}_1 - P_2$ and meets both $F(P_1)$ and $F(P_2)$. Let M'_ω be a component of lim sup $\{M'_\nu\}$. By Theorem 1.14, M'_ω meets both $F(P_1)$ and $F(P_2)$. Let T be the component of $\overline{P}_1 - P_2$ that contains M'_ω. In $S - T$ let L_ν be the component which contains M'_ν. If only finitely many sets L_ν are distinct, then at least one of them contains a subcollection $\{M'_\mu\}$ of $\{M'_\nu\}$ having a component M''_ω of its lim sup in M'_ω. Then if in the statement of the Theorem we replace P and Q by P_1 and P_2, respectively, M_ν and M_ω by M'_μ and M''_ω respectively, and K by T, the statement of the theorem follows.

If on the other hand there exist infinitely many sets L_ν, let H_ν be the union of all components of $M_\nu \cap (\overline{P}_3 - P_4)$ that meet both $F(P_3)$ and $F(P_4)$. Let W be a component of lim sup $\{H_\nu\}$ and let W' be the component of $\overline{P}_3 - P_4$ containing W. If any set L_ν contains W', we delete it from the set of L'_ν's and change $\{H_\nu\}$ accordingly—the lim sup $\{H_\nu\}$ is unaffected. Now if each L_ν has limit points in T, $\bigcup L_\nu \cup T$ is a connected subset of $S - W'$ which contains infinitely many of the sets H_ν. Denote the components of the H_ν's by K_μ's. Then lim sup $\{H_\nu\} \subset$ lim sup $\{K_\mu\}$, so that $W \subset$ lim sup $\{K_\mu\}$. Hence the statement of the theorem now holds if we replace P and Q by P_3 and P_4 respectively, M_ν and M_ω by K_μ and W respectively, and K by W'.

[If S is compact, each L_ν has limit points in T by Theorem 1.8, and the proof of the theorem is complete at this point.]

Suppose, however, that for some index $\nu = i$, L_i has no limit point in T. Then, since S is connected, L_i contains a limit point, p, of $S - (T \cup L_i)$. But no one component of $S - (T \cup L_i)$ can have p as a limit point, and hence there exists a collection of such components, $\{C_\alpha\}$, such that $p \in$ lim sup $\{C_\alpha\}$; each C_α is, furthermore, a component of $S - T$. Let U be an open set containing p such that $\overline{U}$ is compact and $\overline{U} \cap T = 0$. Let V be an open set such that $p \in V \Subset U$. We may delete from the collection $\{C_\alpha\}$ all elements that do not meet V, and continue to denote the new collection by $\{C_\alpha\}$. Now since T is compact and C_α is a component of $S - T$, each $\overline{C}_\alpha$ is either not compact and hence meets $S - \overline{U}$, or $\overline{C}_\alpha$ is compact and by Theorem 1.10 has a limit point in T, and thus again meets $S - \overline{U}$. And by the same method as used above to establish the existence of the sets C_ν, we can show that there exist components C'_α of $C_\alpha \cap U$ such that $\overline{C'_\alpha} \cap F(U) \neq 0$. Then by Theorem 1.14,

the component C of lim sup $\{\overline{C}'_\alpha\}$ which contains p lies in $\overline{U}$ and meets $F(U)$. Evidently $C \subset L_i$.

Let H_α be the union of all components of $\overline{C}'_\alpha \cap (\overline{U} - V)$ that meet both $F(U)$ and $F(V)$. Every component of $H =$ lim sup $\{H_\alpha\}$ meets both $F(U)$ and $F(V)$ by Theorem 1.14. Suppose no such component lies in C. Then H and C are closed, disjoint sets, and as C was a component of lim sup $\{C'_\alpha\}$, the latter is the union of disjoint closed sets E and F such that $E \supset C$, $F \supset H$. But let P_1 and P_2 be disjoint open sets with compact closures such that $P_1 \supset E$, $P_2 \supset F$. There do not exist infinitely many sets $\overline{C}'_\alpha$ that meet both P_1 and P_2; and all but a finite number must lie in $P_1 \cup P_2$. On the other hand, infinitely many must lie in P_1 since $P_1 \supset C$, and each $\overline{C}'_\alpha \cap (\overline{U} - V)$ contains a nonempty H_α. Then H has points in P_1, contrary to the choice of P_1 and P_2.

Thus some component H_ω of H lies in C. Now $S - L_i$ is connected by Theorem I 9.11. The statement of the theorem now becomes valid if P and Q are replaced by U and V, respectively, $\{M_\nu\}$ by the set of all components of the sets H_α, and M_ω by H_ω.

Remark. It may help, in considering the meaning of Theorem 2.1, to consider some examples of continua that are not lc. In this connection, Example I 10.13 might be recalled, with x any point $(1, \theta)$ and P, Q interiors of circles with center x and radii 1/2, 1/4 respectively. The sets M_ν will be arcs cut from the spiral $\rho = (\theta - 1)/\theta$, M_ω an arc of the curve $\rho = 1$.

In the examples of the "Remark" following Theorem 1.10 above, the spaces S are not lc for any point of C for which $y > 0$. In the second of these examples S is not compact, and the component whose existence is asserted in (3) of Theorem 2.1 will be an unbounded subset of the plane.

It will be noticed that the space S of Theorem 2.1 is not lc at any point of $M_\omega \cap (P - \overline{Q})$. We can therefore state:

2.2 Corollary. *A locally compact, connected Hausdorff space cannot fail to be lc at only one point; indeed, if such a space is not lc, it contains a continuum of points at which it is not lc.*

3. Some characterizations of locally compact lc spaces.

3.1 Theorem. *In order that a locally compact, connected Hausdorff space S should be lc, it is necessary and sufficient that if two points x, y lie in a component of an open subset G of S, then x and y lie in a subcontinuum of G.*

Proof of the necessity. Since S is lc, the component, C, of G which contains x and y is open (Theorem II 3.1). In order to obtain a continuum in C containing x and y we may cover each point p of C by a domain D_p such that $\overline{D}_p \subset C$, $\overline{D}_p$ compact, and select a simple chain of these domains (Theorem I 12.3) from x to y. The union of the closures of the sets D_p which make up this simple chain will be the continuum desired.

Proof of the sufficiency. Suppose a continuum S is not lc. Then we

may assert the existence of the sets P, Q, M_ν, etc., as in the statement of Theorem 2.1 above; let H be the component of $S - K$ mentioned in (3), and x a limit point of H in $M_\omega \cap (P - \overline{Q})$. Then $H \cup x$ is a connected set.

Now let D be an open set containing x such that $\overline{D} \subset P - \overline{Q}$, and let $K \cap D = Q'$. Then $K - Q'$ is closed, and $H \cup x$ lies in the open set $S - (K - Q')$. Consequently, by the hypothesis of the theorem, x and y lie in a subcontinuum C of $S - (K - Q')$. By Theorem 1.8, $C \cap D$ has a component C' which contains x and has limit points on $F(D)$; let p be one such limit point. Then since $C' \cup p$ is a connected subset of S in $P - \overline{Q}$ that contains $x \in K$, so must $C' \cup p \subset K$. But $C' \cup p \subset C$, implying that $p \in S - K$ inasmuch as C meets K only in x.

3.2 Definition. A space S is called *regular* at $x \in S$ if for every open set P containing x there exists an open set Q such that $x \in Q \Subset P$. If S is regular at every $x \in S$, then S is called regular.

Remark. Obviously a space S is regular at $x \in S$ if and only if for every closed set M not containing x there exists an open set Q such that $x \in Q \Subset S - M$. Phrased in this manner, it is clear that regularity of a Hausdorff space is a weaker separation property than normality (II 4.6).

The property stated in the above theorem—that the points of a component of an open set lie, in pairs, in subcontinua of that component—is a special case of a general and important property of lc spaces which is needed in the sequel.

3.3 Theorem. *If S is regular and lc, U a connected open subset of S, and A is a compact subset of U, then A lies in a closed (rel. S) connected subset L of U. And if S is locally compact, then L may be taken so as to be compact.*

Proof. Each $x \in A$ is in a domain $R(x) \Subset U$, and a finite collection, $\mathfrak{U}$, of these covers A. Denoting the elements of $\mathfrak{U}$ by $R(x_1), \cdots, R(x_k)$, let $\mathfrak{C}_i$ denote a simple chain of sets $R(x)$ from x_1 to x_i, $i = 2, \cdots, k$. The closure of the set of all points in elements of $\mathfrak{U}$, as well as in elements of $\mathfrak{C}_i$'s, is a closed connected subset L of U containing A.

If S is locally compact, the sets $R(x)$ may be taken with compact closures, and in this case the set L will be compact.

3.4 Corollary. *If S is regular and lc, U is an open subset of S, and K is a compact subset of U, then K lies in a finite number of closed (rel. S) connected subsets of U, equal in number to the number of components of U that intersect K. And if S is locally compact, then these subsets may be taken so as to be compact.*

Proof. Since, by Theorem II 3.1, the components of U are open and K is compact, K must lie in a finite number of components of U—say $U_1, U_2, \cdots, U_k$. Each set $K \cap U_i$, $i = 1, 2, \cdots, k$, is compact, and by Theorem 3.3 lies in a closed connected subset L_i of U_i.

3.5 Definition. If $\mathfrak{C}$ is a covering of a space S, then a point set M is said to be of *diameter* $< \mathfrak{C}$ if some element of $\mathfrak{C}$ contains M.

3.6 Definition. A subset M of a space S is said to have *property* S if for arbitrary covering $\mathfrak{C}$ of S by open sets there exists a finite number of connected sets M_i each of which is of diameter $< \mathfrak{C}$ and such that $M = \bigcup M_i$.

3.7 Theorem. *In order that a continuum S should be lc, it is necessary and sufficient that it have property* S.

Proof. To prove the necessity, all we need notice is that if $\mathfrak{C}$ is any covering of S by open sets, each $x \in S$ is in a domain $R(x)$ which lies in an element of $\mathfrak{C}$; and as S is compact, a finite number of the sets $R(x)$ covers S.

To prove the sufficiency, suppose that S is a continuum that is not lc. Then the statements (1) and (2) of Theorem 2.1 hold. Let $\mathfrak{C}$ be a covering of S obtained as follows: As $S - P$ and $\overline{Q}$ are disjoint closed sets and S is normal (III 1.27) there exist disjoint open sets U and V such that $U \supset S - P$ and $V \supset \overline{Q}$. Let $\mathfrak{C}$ consist of the three open sets U, V, $P - \overline{Q}$.

Since for subsets of a compact space, the lc property implies a finite number of components, and property S also implies this, we have:

3.8 Corollary. *In order that a compact subset of a Hausdorff space S should be lc, it is necessary and sufficient that it have property* S.

3.9 Corollary (Sierpinski). *In order that a metric continuum S should be lc it is necessary and sufficient that for each $\epsilon > 0$, S be the union of a finite number of continua of diameter $< \epsilon$.*

The necessity follows from utilization of a covering of S by spherical neighborhoods $S(x, \epsilon/2)$; the sufficiency follows from the following lemma, whose proof we leave to the reader.

3.10 Lemma. *If M is a compact metric space and $\mathfrak{C}$ a covering of M by open sets, then there exists $\epsilon > 0$ such that if K is a subset of M of diameter less than ϵ, then K is of diameter $< \mathfrak{C}$.*

In Theorem III 3.23 we proved that if D is a domain in a locally compact, metric, lc, connected space C, then the set of all boundary points of D which are accessible from D is dense in the boundary of D. The same argument shows that for any open set D whatsoever, the set of points of $F(D)$ that are accessible from D is dense in $F(D)$. It is an interesting and easy consequence of Theorem 2.1 that this property is characteristic of such spaces.

3.11 Theorem. *If C is a locally compact, metric, connected space, then in order that C should be lc, it is necessary and sufficient that for every open subset D of C, the points of $F(D)$ that are accessible from D be dense in $F(D)$.*

Proof of sufficiency. With C replacing S, the sets P, Q, M_ν , M_ω and K exist as in the statement of Theorem 2.1 if C be assumed not lc. Let $D =$

$C - K$. Then D is an open set such that $F(D) \subset K$. Moreover, since $M_\omega \subset \limsup \{M_\nu\}$ and $\bigcup M_\nu \subset D$, so must $M_\omega \subset F(D)$. Let $p \in M_\omega \cap (P - \overline{Q})$, and let R be an open set such that $p \in R \Subset P - \overline{Q}$.

Since p is a boundary point of D and the accessible points of $F(D)$ are dense in $F(D)$, there exists $x \in R \cap F(D)$ such that x is accessible from D. Let xy be an arc from x to a point y of D, having only x on $F(D)$. In the order from x to y let z be the first point of $F(R)$. Then since the portion xz of the arc xy lies in $\overline{R}$, which is a subset of $P - \overline{Q}$, so must the arc xz lie in K. But this is impossible since all of the arc except x lies in $D = C - K$.

Remark. The principal reason for assuming a metric in the above theorem is that in the definition of accessibility the arc is employed. One may expect that the theorem can be extended to more general spaces, and this might be done in a number of ways. One might, for example, generalize the notion of arc (see the generalized closed 1-cell of chap. IX). Or one might generalize the notion of accessibility in other ways, such as will be done in chap. XII. It is evident from the method of proof used alone, that the sufficiency of Theorem 3.11 holds for the nonmetric case if accessibility by continua instead of arcs be employed (see §5).

3.12 Theorem. *In order that a locally compact, connected space S should be lc, it is necessary and sufficient that every two disjoint closed sets, at least one of which is compact, be separated by a finite set of continua of S.*

Proof. To prove the necessity, let A and B be closed sets, A being compact. There exists an open set P such that $A \subset P \Subset S - B$ and such that $\overline{P}$ is compact. Each $x \in F(P)$ is in a domain $R(x)$ such that $\overline{R}(x)$ is compact and $\overline{R}(x) \cap (A \cup B) = 0$. As $F(P)$ is compact, a finite set $\mathfrak{U}$ of the sets $R(x)$ covers $F(P)$. The closures of the elements of $\mathfrak{U}$ yield the finite set of continua which separate A and B.

To prove the sufficiency, we use Theorem 2.1. Let $A = \overline{Q}$, with $\overline{Q}$ compact, and $B = S - P$—where P and Q are the open sets of Theorem 2.1. Then a finite set of continua separating A and B would necessarily lie in a finite set of the sets M_ν ; and as the sets M_ν are infinite in number and each meets both A and B, there could not exist such a finite set of separating continua.

The above theorems do not exhaust, by any means, the enumeration of properties of general compact, or locally compact, connected spaces that are equivalent to the lc property. They suggest, however, that such equivalences do not depend, in general, on separability or metric assumptions, or assumptions regarding the existence of special neighborhood systems, such as have generally been used in proving the equivalences to the lc property.

4. **Relations between lc, S and ulc properties.** Because of the special role played by property S and its higher-dimensional extensions later on, we point out here that the sufficiency part of Theorem 3.7, the theorem that characterizes

the lc continua by means of the S property, can be proved in much more general form.

4.1 Lemma. *In a regular space S, let M be a set having property* S. *Then if $x \in S$ and P is an open set containing x, there exists an open set Q such that $x \in Q \subset P$ and $M \cap Q$ lies in a finite number of components of $M \cap P$.*

Proof. Let U be an open set containing $S - P$ such that $x \notin \bar{U}$. Let $\mathfrak{C}$ be the covering of S consisting of P and U. Let $Q = S - \bar{U}$. Since M has property S, there exists a finite number of connected sets M_i of diameter less than $\mathfrak{C}$ such that $M = \bigcup M_i$. A set M_i which meets Q must lie in P, and therefore $M \cap Q$ lies in a finite number of components of $M \cap P$.

4.2 Theorem. *If a subset M of a regular space S has property* S, *then M is lc.*

Proof. Let $x \in M$, and P an open set containing x. With Q as in Lemma 4.1, and M_j that component of $M \cap P$ that contains x, there exists an open subset R of Q such that $x \in R$ and $M \cap R \subset M \cap M_j$.

4.3 Remark. That a set may be lc but not have property S may be seen if we let S be the set of all points in the polar coordinate plane for which $\rho \leqq 1$, S' denote the set S of Example I 10.13, and $M = S - S'$. Here M is lc but does not have property S.

4.4 This also seems the appropriate time to remark that although property S is a topological invariant of a space, it is not necessarily topologically invariant for *subsets* of a space. Thus, in the example of 4.3, the set M is homeomorphic to the set $\{(\rho, \theta) \mid \rho < 1\}$ (although this is not obvious) and the latter has property S. A more obvious example is afforded by the sets $\{(x, y) \mid (0 < x \leqq 1) \& (y = \sin 1/x)\}$, $\{(x, y) \mid (0 < x \leqq 1) \& (y = 0)\}$ in E^2. We are dealing, then, with a property that is only a *positional invariant* for certain subsets of a space (see I 6), inasmuch as it is defined relative to the open sets of the imbedding space.

4.5 Theorem. *If a subset M of a regular space S has property* S, *then for arbitrary covering $\mathfrak{C}$ of S by open sets, M is the union of a finite collection of domains of M—i.e., open (rel. to M) connected subsets of M—of diameter $< \mathfrak{C}$.*

Proof. We first express M as the union of a finite number of connected sets M_i of diameter $< \mathfrak{C}$. Each M_i is in some $E_j \in \mathfrak{C}$, and since by Theorem 4.2, M is lc, each $x \in M_i$ lies in a domain $R(x)$ of M that lies in E_j. The set $\bigcup_{x \in M_i} R(x)$ is an open connected subset of M containing M_i and lying in E_j.

4.6 Corollary. *If a subset M of a regular space S has property* S, *then for arbitrary covering $\mathfrak{C}$ of S by open sets, the set M is the union of a finite number of sets of diameter $< \mathfrak{C}$, each of which is the closure in M of an open, connected subset of M.*

In II 5.32 we defined uniform local connectedness for metric spaces. For nonmetric spaces we may paraphrase this definition as follows:

4.7 DEFINITION. A subset M of a space S is called ulc if for arbitrary covering $\mathfrak{C}$ of S by open sets there exists a covering $\mathfrak{D}$ of S by open sets such that if $x, y \in M$ and $x \cup y$ is of diameter $< \mathfrak{D}$, then $x \cup y$ lies in a connected subset of M which is of diameter $< \mathfrak{C}$.

By way of justification of the definition just given we may prove:

4.8 THEOREM. *In a compact metric space, a set which is ulc in the sense of the above definition is ulc in the sense of* II 5.32 *and conversely.*

(We leave the proof to the reader.)

4.9 THEOREM. *If a subset M of a space S is ulc, then M is lc.*

(Hint: If U is an open (rel. M) set containing $x \in M$, let V be an open subset of S such that $V \cap M \subset U$. Let $\mathfrak{C}$ be the covering of S consisting of V and $S - x$, and apply Definition 4.7.)

REMARK. For a noncompact lc subset of a compact space that is not ulc, see 4.14 below. For compact subsets of a Hausdorff space, ulc and lc are equivalent:

4.10 THEOREM. *Every compact lc subset of a space S is ulc.*

(By Corollary 3.8 and Theorem 4.5, if $\mathfrak{C}$ is a covering of S by open sets then S is the union of a finite collection $\mathfrak{D}$ of domains of diameter $< \mathfrak{C}$.)

4.11 LEMMA. *If a subset M of a regular space S is ulc, and $x \in S$, then for any open set P containing x there exists an open set Q such that $x \in Q \subset P$ and $M \cap Q$ lies in one component of $M \cap P$.*

PROOF. Let U be an open set containing $S - P$ such that $x \notin \overline{U}$. Let $\mathfrak{C}$ be the covering of S consisting of P and U. Since M is ulc, there exists a covering $\mathfrak{D}$ of S by open sets such that if $x, y \in M \cap D$, $D \in \mathfrak{D}$, then there exists $E \in \mathfrak{C}$ such that $x \cup y$ lies in a connected subset of $M \cap E$. Let $D \in \mathfrak{D}$ such that $x \in D$, and Q an open set such that $x \in Q \subset D \cap (S - \overline{U})$.

4.12 THEOREM. *If a subset M of a regular space S is ulc, then $\overline{M}$ is lc.*

PROOF. Let $x \in \overline{M}$ and P an open set containing x. By Lemma 4.11, there exists an open set Q such that $x \in Q \subset P$ and $M \cap Q$ lies in one component N of $M \cap P$. Since S is regular, we may assume $\overline{Q} \subset P$. Now $\overline{M} \cap Q$ is a subset of $\overline{N} \cap Q$, and as $N \cup (\overline{N} \cap Q)$ is a connected subset of $\overline{M} \cap P$, it follows that $\overline{M}$ is lcw at x. As x was an arbitrary point of $\overline{M}$, the latter set is lc.

Of importance for later purposes is the following theorem:

4.13 THEOREM. *If a subset M of a compact space S is ulc, then M has property* S.

PROOF. Let $\mathfrak{C}$ be any covering of S by open sets. For each $x \in S$, let $E(x) \in \mathfrak{C}$ such that $x \in E(x)$. By Lemma 4.11, there exists an open set $D(x)$ such that $x \in D(x) \subset E(x)$ and $M \cap D(x)$ lies in one component $C(x)$ of $M \cap E(x)$. Let $\mathfrak{D}$ be a finite collection of the sets $D(x)$ covering S. The sets $C(x)$ corresponding to the elements of $\mathfrak{D}$ form a finite collection of connected sets of diameter $< \mathfrak{C}$ whose union is M.

4.14 REMARK. That ulc is actually stronger than property S, and hence than lc, in a compact space, is shown by the following example in the cartesian plane: Let $S = \{(x, y) \mid x^2 + y^2 \leqq 1\}$, $M = \{(x, y) \mid x^2 + y^2 < 1\} - \{(x, 0) \mid 0 \leqq x < 1\}$. Then M has property S but is not ulc.

4.15 Note that the ulc property, like property S (4.4), is only a positional invariant for subsets of a space. As will be seen below, the domains complementary to an S^1 in S^2 are ulc—a positional invariant of the S^1 in S^2—but the ulc property is not a topological invariant of such domains, since of themselves they are homeomorphic with the set M of 4.14.

5. **Accessibility.** We saw in Theorem II 5.37 that the k-sphere in S^n is arcwise accessible from each of its complementary domains; also, in II 5.38, that the S^1 in S^2 may be characterized by this property. And in Theorem 3.11 above the density of accessible points on domain boundaries was employed to characterize the lc property in the locally compact, metric, connected spaces. In the latter connection it was remarked that the principal reason for assuming a metric was the use of the arc in the definition of accessibility. In nonmetric spaces accessibility by continua, or, more generally, by closed, connected sets, may be substituted. Thus, a point x may be called accessible from a point set M by continua (or by closed, connected sets) if for each $y \in M$ there exists a continuum (or a closed, connected set) K which contains x and y and such that $K - x \subset M$. For euclidean spaces, or, more generally, Peano spaces, accessibility by continua is equivalent to arcwise accessibility:

5.1 THEOREM. *If S is a Peano space, then a necessary and sufficient condition that a point p of the boundary B of a domain D in S should be arcwise accessible from D is that p be accessible by closed, connected sets from D.*

PROOF OF SUFFICIENCY. Let N be a closed, connected subset of $D \cup p$ that contains p. Let η_1 be a positive number such that $N - S(p, \eta_1) \neq 0$, and $\bar{S}(p, \eta_1)$ is compact. Then by Theorem 1.10, the component C_1 of $N \cap S(p, \eta_1)$ that contains p has limit points in $F(p, \eta_1)$; let p_1 be one such point. If K_1 is the component of $\bar{C}_1 - p$ that contains p_1, then $N_1 = K_1 \cup p$ is a subcontinuum of $N \cap \bar{S}(p, \eta_1)$ containing $p \cup p_1$ and $N_1 - p$ is connected. In general, having defined, for $n > 1$, continua $N_1, \cdots, N_{n-1}$, let $\eta_n = \eta_{n-1}/2$, and by the same procedure as was used to obtain N_1 from N, we may show the existence of a subcontinuum N_n of $N_{n-1} \cap \bar{S}(p, \eta_n)$ containing p and a $p_n \in F(p, \eta_n)$, and such that $N_n - p$ is connected.

For each $n > 1$ and $x \in N_n - p$, let η_x be a positive number such that $S(x, \eta_x) \subset D \cap S(p, \eta_{n-1})$. By Theorems III 3.3 and 4.12 there exists for each such x a domain R_x such that $x \in R_x \subset S(\eta_x, x)$, and $\overline{R}_x$ is peanian. Since $N_n - p$ is connected, there exists a simple chain of the sets R_x from p_n to p_{n+1} (Theorem I 12.3); let the closure of the set of all points in such a simple chain be denoted by A_n. Then $\bigcup_2^\infty A_n \cup p$ is a Peano continuum (cf. Corollary 2.2) which contains an arc from p_2 to p. It follows that p is arcwise accessible from D.

5.2 Lemma. *In an lc regular space S, let D be a domain and $x \in F(D)$ a point of countable character such that if P is an open set containing x, then there exists an open set Q such that $x \in Q \subset P$ and $D \cap Q$ is contained in a finite number of components of $D \cap P$. Then x is accessible from D by closed* (*rel. S*), *connected sets.*

Proof. Since x is of countable character, it follows from the hypothesis that there exists a sequence $U_1, U_2, \cdots, U_n, \cdots$ of open sets such that $\bigcap U_n = x$, $U_n \supset U_{n+1}$, and $D \cap U_{n+1}$ lies in the union of a finite number of domains $D(n, i)$, $i = 1, \cdots, i(n)$, which lie in $D \cap U_n$.

Let $p(n, i) \in U_{n+1} \cap D(n, i)$. For each j, $1 \leqq j \leqq i(n + 1)$, there is an i and (Theorem 3.3) a closed (rel. S), connected subset $C(n, i, j)$ of $D(n, i)$ containing $p(n, i) \cup p(n + 1, j)$. If $y \in D$, let $C(i)$ be a closed, connected subset of D containing $y \cup p(1, i)$, $i = 1, 2, \cdots, i(1)$. Then $C = x \cup \bigcup C(i) \cup \bigcup C(n, i, j)$ is a closed, connected subset of $D \cup x$ containing $x \cup y$.

As a consequence of Lemma 5.2 and Lemmas 4.1, 4.11 we have:

5.3 Theorem. *If D is a domain in a regular space S, and D either* (1) *has property* S, *or* (2) *is ulc, then every point of $F(D)$ of countable character is accessible from D by closed* (*rel. S*), *connected sets.*

And from Theorems 5.1 and 5.3 we have:

5.4 Theorem. *If a domain D in a Peano space has property S, then every point of $F(D)$ is accessible from D by arcs.*

5.5 Theorem. *If D is a ulc domain in a locally compact, metric space S, then every point of $F(D)$ is accessible from D by arcs.*

[By Theorem 4.12, $\overline{M}$ is a Peano space.]

Remark. That neither the ulc nor the S property is necessary for the arcwise connectedness of a domain boundary in general is shown by the following example in the coordinate plane: Let S be the first example of the Remark following Theorem 1.10.* To S add an arc T joining (0, 0) to (1, 0) such that $\langle T \rangle \subset E^2 - S$ and the simple closed curve $T \cup K$ has all sets $\langle L_n \rangle$ in its bounded complementary domain D. Every point of $F(D)$ is arcwise accessible from D.

It will be noted that Theorem II 5.37 is a corollary of Theorem 5.5. Another interesting application of the above to the euclidean case follows below:

5.6 Theorem. *In S^n, let D be a domain and F a closed subset of B, the boundary of D, such that $p^1(S^n - F, 2) = 0$. Then if F separates $x, y \in B$ in B, there exists a subcontinuum N of $D \cup F$ that separates x and y in S^n.*

Proof. By hypothesis, $B - F = K \cup L$ separate, $x \in K$, $y \in L$. Since S^n has Property V' of Chapter II (Theorem II 5.19), there exists a continuum $M \subset S^n - (K \cup L)$ which separates x and y in S^n. Let $M \cap D = M'$ and $M - (M' \cup F) = M''$. Then the set $A = M' \cup F \cup M''$ contains M, separates x and y in S^n, and lies in $S^n - (K \cup L)$.

The set $M' \cup F$ separates x and y in S^n. For suppose not. Let η be a positive number such that $S(x, \eta) \cap A = 0 = S(y, \eta) \cap A$. Let x' and y' be 0-cells of some subdivision s_i of S^n that lie in $D \cap S(x, \eta)$ and $D \cap S(y, \eta)$, respectively. Since x and y are not separated by $M' \cup F$ in S^n, the same holds for x' and y', and hence $x' + y' \sim 0$ in $S^n - (M' \cup F)$. And since x' and y' both lie in D, they are not separated by $M'' \cup F$ and therefore $x' + y' \sim 0$ in $S^n - (M'' \cup F)$. It follows from Theorem II 5.18 that x' and y', and hence x and y, are not separated in S^n by A, contradicting the fact that A was so chosen as to separate x and y.

By Lemma II 5.20, x and y are separated in S^n by a subcontinuum N of $M' \cup F$.

As consequences of Theorem 5.6 we can state:

5.7 Theorem. *If D is a domain in S^2 and F a subcontinuum of the boundary B of D which separates $x, y \in B$ in B, then some subcontinuum of $D \cup F$ separates x and y in S^2.*

[Cf. Theorem II 5.25.]

5.8 Theorem. *If D is a domain in S^n and K, L are disjoint components of the boundary of D, then there exists a subcontinuum of D which separates K and L in S^n.*

[Cf. Theorem 1.1.]

5.9 Theorem. *If D is a domain in S^n and p is a cut point of the (connected) boundary of D, then p is arcwise accessible from D.*

[This is an immediate consequence of Theorems II 5.21, 5.1 and 5.6.]

6. **More properties of the 2-sphere.** As a special case of the n-sphere, the S^2 partakes of the properties of the S^n already found in §2 of Chapter II. Besides the Properties I—V of II 1, it was shown, for instance, that the domains complementary to an S^1 in S^2 are ulc (Theorem II 5.35). Also, as a special case of an lc space, the S^2 has the properties of the latter, such as for instance the openness of the components of open sets (Theorem II 3.1), the arcwise

connectedness of the domains (Theorem III 3.10), and the density of the accessible points of the boundary of a domain (Theorem III 3.23). The present section is interpolated to indicate more fully the role of the ulc and S properties in the investigation of positional properties of the 2-sphere, and to point the way to similar properties and their role in the higher-dimensional manifolds. It is not intended to be exhaustive, however; further properties of the 2-sphere will come out as special cases of the studies made of the n-dimensional case in the sequel. And the theorems below are themselves special cases of theorems in Chapter XII 1, 2.

6.1 THEOREM. *Let M be a Peano continuum in S^2. Then if D is a domain complementary to M, D has property* S.

PROOF. Suppose D does not have property S. Then there exists $\epsilon > 0$ such that D is not the union of a finite number of connected sets of diameter $< \epsilon$. Now for each $x \in D$, let $H(x)$ denote the set of all points y of D such that $x \cup y$ lies in a connected subset of D of diameter $< \epsilon/2$; obviously each set $H(x)$ is a connected subset of D of diameter $< \epsilon$. Then there exists a sequence $x_1, x_2, \cdots, x_i, \cdots$ of points of D having a sequential limit point $x \in \overline{D}$, such that no two points $x_i, x_j, i \neq j$, lie in the same set $H(x)$. Let $U = S(x, \epsilon/4)$, $V = S(x, \epsilon/8)$. Since all but a finite number of the points x_i lie in V, we may suppose they all do. For any $i > 1$, there exists an arc t_i in D with end points x_1 and x_i. Let y_i be the first point of this arc on $F(x, \epsilon/4)$ in the order from x_i to x_1, and z_i the first point of $\underline{F}(x, \epsilon/8)$ in the order from y_i to x_i. Then the portion y_iz_i of t_i is wholly in $\overline{U} - V$, and only its end points are not in $U - \overline{V}$. No two arcs $y_iz_i, y_jz_j, i \neq j$, lie in a connected subset of $D \cap (\overline{U} - V)$, and therefore if C_i is the component of $D \cap (\overline{U} - V)$ containing y_iz_i, every two components $C_i, C_j, i \neq j$, are disjoint.

Let $K = F(x, 3\epsilon/16)$. Since $y_iz_i \cap K \neq 0$, let $q_i \in y_iz_i \cap K$. In some cyclic order on K, there exists an infinite subsequence $\{q_n\}$ of $\{q_i\}$ such that $q_n < q_{n+1}$. As the arc q_nq_{n+1} of K that contains no other q_n cannot lie wholly in D, there exists a $p_n \in M$ such that $q_n < p_n < q_{n+1}$. Now despite the fact that (1) because M is lc, almost all the points p_n are joined by arcs of M in $U - \overline{V}$, it is easily shown that (2) it is impossible to join any two points p_n, $p_{n'}$, $n \neq n'$, by an arc of M in $U - \overline{V}$, since any such arc must meet y_nz_n or $y_{n+1}z_{n+1}$.

As a corollary of Theorems 5.3 and 6.1 we have:

6.2 COROLLARY. *If M is a Peano continuum in S^2 and D is a domain complementary to M, then $F(D)$ is accessible from D.*

6.3 THEOREM. *If a domain D complementary to a continuum M in S^2 has property* S, *then the boundary of D is peanian.*

PROOF. Since S^2 has Property II of Chapter II, the boundary of D is a continuum, and we may therefore suppose that $M = F(D)$. Suppose M not

lc. Then by virtue of Theorem 2.1, there exist $x \in M$, and $\epsilon > \delta > 0$ such that if $P = S(x, \epsilon)$, $Q = S(x, \delta)$, then infinitely many components M_ν of $M \cap (\bar{P} - Q)$ contain points of both $F(P)$ and $F(Q)$. Let $K = F(x, (\epsilon + \delta)/2)$. Then there exist points $p_1, p_2, \cdots, p_n, \cdots$ of sets M_ν on K such that in some cyclic order $p_n < p_{n+1}$ and no two points p_n belong to the same M_ν.

Now there exists a number m such that D is the union of m connected sets $D_1, \cdots, D_m$ of diameter $< (\epsilon - \delta)/8$. One of these, say D_1, must have infinitely many of the points p_n as limit points, and we may suppose for the sake of brevity that every p_n lp D_1. But $D_1 \subset P - \bar{Q}$, and any connected subset of $P - \bar{Q}$ containing p_2 and p_4, such as $D_1 \cup p_2 \cup p_4$, must meet $M_1 \cup M_3 \cup M_5$. (This is easily shown, for example, by the fact that such a connected set must meet any set of broken lines from $F(P)$ to $F(Q)$ having vertices on the sets M_1, M_3 and M_5 and approximating these sets sufficiently closely.)

6.4 Corollary. *In order that the boundary of a domain D of S^2 such that $p^1(D, 2) = 0$ should be lc, it is necessary and sufficient that D have property* S.

Remark. The domain D defined in the Remark following Theorem 5.5 does not have property S.

6.5 Corollary. *If M is a Peano continuum in S^2, then the boundaries of the domains complementary to M are all peanian.*

6.6 Theorem. *If the common boundary of (at least) two domains in S^2 is lc, then it is an S^1.*

Proof. If M is a common boundary of two domains A and B in S^2 and is lc, then M is a Peano continuum (cf. Theorem II 4.12). Hence M is accessible from both A and B by Corollary 6.2, and is an S^1 by Theorem II 5.38.

6.7 Theorem. *If two points of S^2 are separated by a Peano continuum M in S^2, then they are separated by some simple closed curve of M.*

Proof. Denoting the points separated by M by x and y, let D denote the component of $S - M$ that contains x, E the component of $S - \bar{D}$ that contains y, and D' the component of $S - \bar{E}$ that contains $\bar{D}$. Then D' and E have a common boundary, and this common boundary is lc (as is shown by successive applications of Corollary 6.5).

6.8 Lemma. *If p is a cut point of the boundary, B, of a domain D in S^n, and x, y are points separated by p in B, then there exists a Peano continuum C in $D \cup p$ that separates x and y in S^n.*

Proof. By Theorem 5.6, there exists a subcontinuum N of $D \cup p$ that separates x and y in S^n. Let η be a positive number such that $S(p, \eta) \supset N$. For each natural number k let $N \cap [\bar{S}(p, \eta/k) - S(p, \eta/(k + 1))] = N_k$. Then N_k is a compact subset of D which in an obvious manner can be covered by a finite set of spherical neighborhoods $S(x_{ki}, \delta_k) = S_{ki}$, $x_{ki} \in N_k$, whose closures

lie in D. Let $C = \bigcup_{k,i} \overline{S}_{ki} \cup p$. That C is connected follows from the fact that it contains N and each $\overline{S}_{ki}$ is connected (cf. Theorem I 7.3a). And since each $\overline{S}_{ki}$ is lc, the only point at which C could fail to be lc is p, and this is impossible since a continuum cannot fail to be lc at a single point (Corollary 2.2).

6.9 COROLLARY. *If p is a cut point of the boundary, B, of a domain D in S^2, and x, y are points separated by p in B, then some S^1 in $D \cup p$ separates x and y in S^2.*

6.10 DEFINITION. Let J be an S^1 in S^2, $p, q \in J$, and $\eta > 0$ such that $q \notin \overline{S}(p, \eta)$. Note that since J is lc there exists $\delta > 0$ such that $J \cap S(p, \delta)$ is a subset of H, the component of $J \cap \overline{S(p, \eta)}$ containing p. Let T be an arc in $S(p, \delta)$ with end points on H. Since $\overline{H}$ is itself an arc cd, there exist first and last points, e and f, of T on $\overline{H}$ in the order from c to d. Then the arc eqf of J, together with the portion of T from e to f, is an S^1 which we call an *η-alteration of J at p*.

6.11 LEMMA. *In S^2, let J be an S^1, A and B the domains complementary to J, and $x \in A$, $y \in B$. Then if $p \in J$, there exists a positive number η such that every η-alteration of J at p also separates x and y in S^2.*

(Roughly speaking, Lemma 6.11 means simply that a simple closed curve which separates two points x and y will still separate x and y if altered only in the neighborhood of one of its points.)

PROOF. Let $q \in J - p$. By Theorem II 5.38, there exist arcs xq and yq in $A \cup q$ and $B \cup q$, respectively. Let η be such that $S(p, \eta) \cap (xq \cup yq) = 0$. Let K be an η-alteration of J at p as defined in 6.10. If K does not separate x and y in S^2, then $xq - q$ and $yq - q$ lie in the same domain D complementary to K in S^2. Let $\epsilon > 0$ be such that $S(q, \epsilon) \cap S(p, \eta) = 0$. Then, since D is ulc, there exists $\delta > 0$ such that if $x', y' \in D \cap S(q, \delta)$, then there exists an arc $x'y'$ in $D \cap S(q, \epsilon)$. In particular, let $x' \in xq$, $y' \in yq$. Then in $xy \cup x'y'$ there is an arc Q from x to y that fails to meet the arc cqd of J as defined in 6.10. But $Q \cap S(p, \eta) = 0$ and hence $Q \cap H = 0$, where H is as in 6.10. But then $Q \cap J = 0$, contradicting the fact that x and y are separated by J.

6.12 THEOREM. *Let D be a ulc domain in S^2 such that $p^1(D, 2) = 0$. Then B, the boundary of D, is either degenerate or an S^1.*

PROOF. Suppose B is nondegenerate. Then $S^2 - D$ is a continuum by Theorem II 5.25, and hence B is a continuum by Theorem II 4.12. Furthermore, since D has property S by Theorem 4.13, B is lc by Theorem 6.3. Thus B is a Peano continuum.

Suppose p is a cut point of B. Then by Corollary 6.9 there exists in $D \cup p$ a simple closed curve J that separates points x and y of B in S^2. By Lemma 6.11 there exists a positive number η such that every η-alteration of J at p also separates x and y in S^2. Since D is ulc, there exists $\delta > 0$ such that any

two points of $D \cap S(p, \delta)$ are joined by an arc of $D \cap S(p, \eta)$. Hence there exists an η-alteration K of J that lies in D. But then K cannot separate x and y since $B \cap K = 0$. We conclude that B has no cut point.

By Theorems III 3.32 and III 3.37, B is cyclicly connected. Hence B contains a simple closed curve J. Let D' denote that domain complementary to J that contains D. Then $\overline{D} = D \cup B \subset \overline{D'} = D' \cup J$, so that if $B \neq J$, there exists a point x of B in D'. Since B is cyclicly connected, there exists an arc ab of B which contains x and lies, except for a and b, in D'—where $a, b \in J$. However, by III 4.6, if $X = ab$ and Y, Z are the two arcs of J having end points a and b, then $S^2 - X - Y - Z$ is the union of three domains having $X \cup Y$, $Y \cup Z$ and $X \cup Z$ as their respective boundaries. And since $X \cup Y \cup Z \subset B$, D must lie in one of these domains and cannot, therefore, have B as its boundary. We must conclude, then, that $B = J$.

Remark. The example in 4.14 might be recalled here.

7. Recognition of Peano continua in S^2 by accessibility properties. It was pointed out in I 6 that Schoenflies [S] gave a characterization of planar Peano continua by means of accessibility properties. The type of accessibility employed by Schoenflies was shown by Whyburn [see W, Theorem VI 4.2] to be equivalent, for domains complementary to continua in S^2, to regular accessibility: A point x is *regularly accessible* from a domain D if for arbitrary positive number η there exists a positive number δ such that if $y \in D \cap S(x, \delta)$ then there exists an arc xy such that $\langle xy \rangle \subset D \cap S(x, \epsilon)$. (If all points of $F(D)$ are regularly accessible from D, we say simply that $F(D)$ is regularly accessible from D.) Since this formulation of the accessibility property allows of immediate generalization (see Chapter XII) to higher dimensions, we shall utilize it instead of the "all-sided accessibility" of Schoenflies. Note that, as stated above, the definition of regular accessibility is meaningful for any metric space.

[If D is a domain in S^2 with connected boundary B, then $p \in B$ was said by Schoenflies to be *accessible from all sides* from D if, for any arc ab spanning B in $\overline{D}$, the point p is arcwise accessible from each domain of $D - ab$ having p on its boundary. Schoenflies showed that in order that a continuum in S^2 should be peanian, it is necessary and sufficient that for arbitrary positive number η at most a finite number of domains complementary to M be of diameter $> \eta$ and that each point of the boundary of a complementary domain D be accessible from all sides from D. Compare Theorem 7.7 below.]

By the methods used in proving 5.1 and 5.2 above it may be shown:

7.1 Lemma. *Let D be an open subset of a Peano space. Then in order that $x \in F(D)$ should be regularly accessible from D, it is necessary and sufficient that for arbitrary $\eta > 0$ there exist $\delta > 0$ such that each point of $D \cap S(x, \delta)$ lie in some component of $D \cap S(x, \eta)$ which has x as a limit point.*

In view of Lemma 4.1 we have:

7.2 COROLLARY. *If an open set D in a Peano space has property* S, *then $F(D)$ is regularly accessible from D.*

7.3 THEOREM. *In order that a domain D with connected boundary in S^2 should have property* S, *it is necessary and sufficient that the boundary of D be regularly accessible.*

The necessity follows immediately from Corollary 7.2. To prove the sufficiency, suppose D does not have property S. Then as in the proof of Theorem 6.1 we may obtain the sets U, V, K and points q_n. The sequence $\{q_n\}$ converges to a point q of $K \cap F(D)$, and if $\eta < \epsilon/16$ and $\delta < \eta$, the component of $D \cap S(q, \eta)$ that contains a $q_n \in S(q, \delta)$ fails to have q as a limit point.

REMARK. The example in the Remark following Theorem 5.5 might be recalled here. In this example $F(D)$ is not regularly accessible from D.

As a corollary of 6.4 and 7.3 we have:

7.4 COROLLARY. *In order that the boundary, B, of a domain D in S^2 such that $p^1(D, 2) = 0$ should be lc, it is necessary and sufficient that B be regularly accessible from D.*

And in view of Theorem 6.1:

7.5 COROLLARY. *If M is a Peano continuum in S^2 and D is a domain complementary to M, then $F(D)$ is regularly accessible from D.*

Since by Theorem II 5.35 the k-sphere in S^n has a ulc complement unless $k = n - 1$, and in the latter case the complementary domains are ulc, and since by Theorem 4.13 every ulc subset of S^n has property S, we may state as a corollary of 7.2

7.6 COROLLARY. *If M is a k-sphere in S^n, then M is regularly accessible from its complement.*

7.7 THEOREM. *In order that a continuum in S^2 should be peanian, it is necessary and sufficient that each of its complementary domains have property* S, *and that for arbitrary positive number η at most a finite number of these domains be of diameter greater than η.*

PROOF. Let M be a Peano continuum in S^2. The domains complementary to M all have property S by Theorem 6.1. Suppose $\{D_i\}$ is a denumerable collection of such domains D_i such that $\delta(D_i) > \eta$ for some fixed η and all i. Then each D_i contains an arc T_i such that $\delta(T_i) > \eta$. From this situation and the compactness of S^2 follows readily that we may assume the existence of a point p and a subcollection $\{T_n\}$ of $\{T_i\}$ such that each T_n meets both $F(p, \eta/2)$ and $F(p, \eta/8)$. Each such T_n contains a subarc $A_n = a_n b_n$ such that $a_n \in F(p, \eta/2)$, $b_n \in F(p, \eta/8)$ and $\langle a_n b_n \rangle \subset S(p, \eta/2) - \bar{S}(p, \eta/8)$. Each T_n meets $F(p, \eta/4)$; let $x_n \in T_n \cap F(p, \eta/4)$. Then there exists a subsequence

$\{x_j\}$ of $\{x_n\}$ such that in some cyclical order on $F(p, \eta/4)$, $x_j < x_{j+1}$ for all j. And there exists x slp $\{x_j\}$, $x \in M \cap F(p, \eta/4)$. Now by consideration of points $y_j \in M \cap x_jx_{j+1}$, where x_jx_{j+1} is the arc of $F(p, \eta/4)$ not containing x_{j+2}, it is readily shown that M cannot be lc at x.

Conversely, suppose M a continuum in S^2 satisfying the conditions stated in the theorem. If M is not peanian, then by Theorem 2.1 there exist sets P, Q, M_ν satisfying the conditions of that theorem, with ν replaced by n (representing a natural number). Since S^2 is metric we may assume a $p \in M$ and $\eta > 0$, $\delta > 0$ such that $P = S(p, \eta)$, $Q = S(p, \delta)$. Let $\sigma = (\eta + \delta)/2$. Without loss of generality, we may assume $x_n \in M_n \cap F(p, \sigma)$ such that $x_n < x_{n+1}$ in a cyclic order on $F(p, \sigma)$. Since, by Theorem II 5.19, S^2 has property V of II 4, there exists in $S^2 - M \cap [\bar{S}(p, \eta) - S(p, \delta)]$ a continuum C_n separating x_n and x_{n+1} in S^2. Then $C_n \cap x_nx_{n+1} \neq 0$, and by Theorem 1.8, C_n contains a continuum K_n which meets both $F(p, \sigma)$ and $F(p, \eta) \cup F(p, \delta)$ and lies in $\bar{S}(p, \eta) - S(p, \delta)$.

Now from the hypothesis concerning diameters of domains complementary to M it follows that we may assume the K_n's all to lie in one such domain, say D. But D has property S and must therefore contain a connected subset of diameter $<(\eta - \delta)/2$ which contains points of infinitely many of the sets $K_n \cap F(p, \sigma)$; this is impossible.

In view of Theorem 7.3 we may also state:

7.8 Theorem. *In order that a continuum in S^2 should be peanian, it is necessary and sufficient that if D is a domain complementary to M, $F(D)$ be regularly accessible from D, and that for arbitrary positive number η, at most a finite number of such domains be of diameter $> \eta$.*

7.9 Remark. The necessity for the condition placed on the diameters of the complementary domains in Theorems 7.7 and 7.8 is shown by the following example: In the coordinate plane, let S be the set defined in the Remark following Theorem 1.10, and let N be the set $\{(x, y) \mid (0 \leqq x \leqq 1)\&(y = 1)\}$. Then if $M = S \cup N$ the continuum M can be considered as lying on an S^2 and satisfying the conditions of 7.7 and 7.8 so far as the individual complementary domains are concerned. However, the condition on the diameters of these domains is not satisfied, and hence M is not peanian.

8. **Remarks.** The restriction of some of the noteworthy theorems of this chapter to S^2 (e.g., 6.4, 6.6, 6.12, 7.3) raises the question as to whether such restriction is necessary. That this seems to be the case may be shown by examples. For instance, that the connected boundary of a domain D in S^3 may be peanian without D having property S may be seen by a glance at the figure accompanying Example XI 5.1; and no point of the subcontinuum E of $F(D)$ is accessible from D. By extending the plane rectangles R_i of this example down to the base of the wedge and the S_i's to the top of the wedge, there is obtained a Peano continuum in S^3 whose complementary domains all have diameters

$> \eta = h/2$, where h is the height of the wedge (see also Example XI 5.17). If Example XI 5.2 is considered as lying in S^3, then it is a common boundary of two domains one of which is ulc—but the boundary is not lc. As regards Theorem 6.6, a Peano continuum in S^3 that is the common boundary of two domains may not only fail to be a sphere, but may be the common boundary of infinitely many domains, all of which may be ulc. [Wilder [k]]. Nevertheless, Theorem 6.6 has a generalization: If a Peano continuum in S^n is irreducibly linked (II 5.22) by an $(n - 2)$-cycle, then it is an S^1—the case $n = 2$ is precisely Theorem 6.6. (See Wilder [j, Corollary 4].) And Theorem 6.7 has a generalization to higher dimensions, viz: If a cycle γ^{n-2} links a Peano continuum in S^n, then γ^{n-2} links some S^1 in M (the case $n = 2$ is exactly Theorem 6.7). (See Wilder [j].)]

The restriction to S^2 of the theorems cited, however, is due not to necessity, but to limitations in the concepts involved. As we shall see in the sequel, the lc property, property S and the ulc property are only the cases $n = 2$ of general n-dimensional concepts; we have already noticed this (Theorem II 5.33) in the case of the ulc property so far as open subsets of S^n are concerned. When such concepts as these are seen in the proper perspective, the theorems cited fall into their places in the general theory of manifolds. Before this can be achieved, however, it is necessary to digress into the domain of algebraic topology in order to set up the needed tools. This will be the function of Chapter V.

Bibliographical Comment

§2. For the metric case of parts (1) and (2) of Theorem 2.1 see Moore [e, 296-297]; of part (3), Wilder [a, III, Lemma I].

§3. For the compact metric case, the necessity part of Theorem 3.1 was proved by Moore [c], and the sufficiency by Wilder [a, Theorem 18]. Corollary 3.9 is due to Sierpinski [a]; it was this result which suggested to Moore [d] the metric case of Definition 3.6 and the designation thereof as "property S". Regarding the metric case of Theorem 3.12 see Whyburn and Ayres (Ayres [a, Theorem 4]). For other theorems of this type see Moore [Mo, e] and Kline [b].

§4. Relative to Theorems 4.2, 4.12, 4.13, see Moore [d].

§5. A theorem similar to Theorem 5.1 was proved by Knaster and Kuratowski [b, 38, footnote 3)] and Whyburn [e, Theorem 1].

§6. Corollary 6.4 was proved by Moore [d, Theorem 4]. Corollary 6.5 is the "Torhorst Theorem"; see Torhorst [a]. (As a matter of fact, *every subcontinuum* of the boundary of a domain complementary to a Peano continuum in S^2 is peanian; see Wilder [a, Theorem 11].) Theorem 6.7 was proved by Moore [c, Theorem 5], as was also Theorem 6.12 [f].

§7. See Whyburn [W, VI 4].

CHAPTER V

BASIC ALGEBRAIC TOPOLOGY[1]

1. **Complexes.** Let S be a collection, finite or infinite, in which certain finite subsets, to be called *simplexes*, have been specified. If a simplex consists of $n + 1$ elements of S, it will be called an *n-dimensional simplex*, or simply an *n-simplex*. The elements of the simplex are called *vertices*. In all applications, if $v_0, v_1, \cdots, v_n$ denote vertices of an n-simplex E^n, then all (nonempty) subcollections $v_{i_0}, \cdots, v_{i_r}$ of E^n will also be simplexes and will be called r-dimensional faces of E^n, or simply *r-faces* of E^n. Thus the vertices become "0-faces" of E^n, although the latter term is rarely used. As an example, one may think of the four vertices of a tetrahedron as constituting a 3-simplex E^3; the faces correspond to the subcollections of three vertices which correspond to the triangular faces of the tetrahedron, the subcollections of two vertices which form pairs of endpoints of edges, and the individual 0-faces or vertices. (Every triangulated polyhedron similarly yields an example.)

Any collection of simplexes, which, if it contains a simplex E^n, also contains all faces of E^n, will be called an *unrestricted complex*. If it consists only of n-simplexes and their faces, we call it an *unrestricted n-dimensional complex*, or *unrestricted n-complex*. Of fundamental importance are the *incidences* between simplexes of a complex: If E^n is a simplex and E^{n-1} is a face of E^n, we say that E^{n-1} is *incident to* E^n and that E^{n-1} and E^n are *incident*. We shall use the symbol St E^n—"star of E^n"—to denote the collection of simplexes consisting of E^n and all simplexes of which E^n is a face.

2. **Algebraic apparatus.** We next associate with S and its simplexes an algebraic apparatus. To each n-simplex E^n, $n > 0$, we first assign *positive* and *negative orientations* as follows: A certain linear ordering $v_0, v_1, \cdots, v_n$ of its vertices is selected, and the symbol $v_0v_1 \cdots v_n$, or any even permutation of the v's therein, regarded as an algebraic symbol associated with the simplex and determining its positive orientation. The symbol derived from $v_0v_1 \cdots v_n$ by any odd permutation of the v's therein is similarly related to negative orientation. The symbols associated with the simplex E^n and determining its positive orientation are considered as equivalent, any of them being denoted by σ^n, and we call σ^n the *n-cell* associated with E^n.[2] The symbol $-\sigma^n$ will

[1]The reader may compare the notions introduced below with the material of II 5. A knowledge of the latter is not presumed below, however.

[2]It will be noted that although in II 5 the term "cell" stood for a euclidean geometric element, here it seems to have only algebraic significance. However, if one wishes to do so, he may imagine that to each n-simplex there is made to correspond a euclidean geometric element with

denote any of the symbols associated with negative orientation of E^n. With a 0-simplex E^0 we also associate, for sake of completeness, symbols σ^0, $-\sigma^0$ denoting its "positive and negative orientations," but make the convention that in the algebra $v^0 = \sigma^0$, $-v^0 = -\sigma^0$.

Let ∂ be an operator, which may be called a *boundary operator*, such that, if $n > 0$,

$$\partial(v_{i_0}v_{i_1} \cdots v_{i_n}) = \sum_{s=0}^{s=n} (-1)^s v_{i_0} \cdots v_{i_{s-1}}v_{i_{s+1}} \cdots v_{i_n}, \tag{2.1}$$

and more generally,

$$\partial(\sum \eta^i \sigma_i^n) = \sum \eta^i \partial(\sigma_i^n), \tag{2.2}$$

where the η's are integers, finite in number, and the laws of ordinary algebra hold—$(-1)\sigma^n = -\sigma^n$, $-(-\sigma^n) = \sigma^n$, etc In particular, for a single σ^n, $n > 1$, it easily follows that

$$\partial^2(\sigma^n) = \partial(\partial(\sigma^n)) = 0. \tag{2.3}$$

The coefficient of a σ^{n-1} in the expression $\partial(\sigma^n)$ is called the *incidence number* of σ^n and σ^{n-1}, and for the sake of completeness one may call the incidence number of a σ^n and σ^{n-1} whose corresponding simplexes are not incident, zero.

For $\partial\sigma^0$, either of two conventions may be made: (1) Define $\partial\sigma^0 = 0$, or (2) *augment* the complex under consideration by an ideal simplex E^{-1} with corresponding cell σ^{-1}, and for *all* 0-cells σ^0 of the complex let $\partial\sigma^0 = \sigma^{-1}$. (Then if σ_1^0 and σ_2^0 are distinct 0-cells, $\partial(\sigma_1^0 - \sigma_2^0) = 0$.) In case (2) is used, the complex is called *augmented*, case (1) then being identified by the adjective *non-augmented*. In either case, relation (2.3) continues to hold when $n = 1$.

A complex to all of whose simplexes have been assigned positive and negative orientations constitutes, together with its cells, an *oriented complex*. Let K be a fixed oriented complex, $\{\sigma^n\}$ any collection of n-cells of K, and let G be any abelian group. By an *unrestricted n-chain of K over G* is meant a single-valued function defined over the n-cells σ^n of K with values in G. Usually we shall treat the latter values as coefficients, directly associating each n-cell with the functional value thereon by means of the expression $g\sigma^n$, $g \in G$. We may then indicate an n-chain as a collection $\{g\sigma^n\}$, or $\{g^\rho\sigma_\rho^n\}$ when a distinguishing index is needed for the cells. The number n is called the *dimension* of the n-chain.

If $\{g^\rho\sigma_\rho^n\}$ is an unrestricted n-chain of K over G, $n \geqq 0$, and if no $(n-1)$-cell σ_ν^{n-1} of K appears in more than a finite number of the polynomials $\partial(\sigma_\rho^n)$ for which $g^\rho \neq 0$ (the identity of G), then we may define what may be called a *boundary chain* $\partial\{g^\rho\sigma_\rho^n\}$ as follows: Consider the collection $\{g^\rho\partial(\sigma_\rho^n)\}$; assum-

---

$n + 1$ vertices, and that the symbol σ^n stands for this geometric element together with some orientation of the same. As we use the term, "simplex" encompasses more than the euclidean element, but allows of interpretation by means of the latter.

ing the distributive law and making the convention that $g \cdot 0 = 0$, the identity of G, $g \cdot (-1) = -g$, the inverse of g in G, etc., then each $g^\rho \partial(\sigma_\rho^n)$ yields a finite collection $\{g_\rho^\nu \sigma_\nu^{n-1}\}$ (in the nonaugmented case, if $n = 0$, this becomes 0, of course). As a given σ_ν^{n-1} occurs in only a finite number of such collections, we may assemble all the coefficients g_ρ^ν with which it occurs, obtain the group element $g^\nu = \sum_\rho g_\rho^\nu$, and hence the form $g^\nu \sigma_\nu^{n-1}$. The collection $\{g^\nu \sigma_\nu^{n-1}\}$ thus obtained for all $(n - 1)$-cells σ_ν^{n-1} of K is called the *boundary-chain*, or simply *boundary*, of the n-chain, and denoted by $\partial\{g^\rho \sigma_\rho^n\}$.

The machinery of unrestricted complexes and chains is more general than we shall need in the applications. As a matter of fact, we may hereafter assume that all complexes considered contain at most a denumerable number of simplexes and are *star-finite*—i.e., no n-simplex, $n \geqq 0$, is incident with more than a finite number of $(n + 1)$-simplexes. Most of the time we shall be dealing with finite complexes, and throughout we shall use only *finite chains*, i.e., chains having only a finite number of nonzero coefficients. Hence we shall be able to represent each chain $\{g^\rho \sigma_\rho^n\}$ as a polynomial

$$g_1\sigma_1^n + g_2\sigma_2^n + g_3\sigma_3^n + \cdots = \sum g_\rho \sigma_\rho^n . \tag{2.4}$$

The advantage in so doing is that when we wish to find the boundary of the chain we may simply write

$$\partial(g_1\sigma_1^n + g_2\sigma_2^n + g_3\sigma_3^n + \cdots) = g_1\partial(\sigma_1^n) + g_2\partial(\sigma_2^n) + \cdots . \tag{2.5}$$

With the algebraic conventions adopted above and addition of group elements, this yields a polynomial on the right of the form $g_1^1\sigma_1^{n-1} + g_2^1\sigma_2^{n-1} + \cdots$. Thus both chains and their boundaries may be conveniently written in polynomial form, and we shall continue to do so throughout.

From (2.3) it follows that for any chain C^n of the form (2.4), with $n > 0$, we shall have—making the convention of denoting the chain all of whose coefficients are zero by 0—

$$\partial^2 C^n = \partial(\partial C^n) = 0. \tag{2.6}$$

Usually, in writing any particular chain in polynomial form, it is the practice to omit all terms with coefficients zero, although in most general discussions we retain the complete form (2.4).

In case the complex is nonaugmented, $\partial C^0 = 0$ for all 0-chains C^0; and as only finite chains are used, ∂C^0 is defined and may or may not be zero in the augmented case.

For future purposes we record here:

2.1 Definition. If $C^0 = \sum g_\rho \sigma_\rho^0$, then by $\mathrm{Ki}(C^0)$, or *Kronecker index* of C^0, is meant the sum $\sum g_\rho$.

3. **Chain groups.** Given a complex K and a dimension n, we arrange the n-cells in a certain indexed array σ_i^n, $i = 1, 2, 3, \cdots$, and represent each n-

chain as a polynomial (2.4), which we abbreviate to $\sum_{i=1}^{\infty} g_i\sigma_i^n$. If $\sum_{i=1}^{\infty} g_i'\sigma_i^n$ is another chain, we define the sum of the two chains as

$$\sum_{i=1}^{\infty} g_i\sigma_i^n + \sum_{i=1}^{\infty} g_i'\sigma_i^n = \sum_{i=1}^{\infty} (g_i + g_i')\sigma_i^n, \tag{3.1}$$

where the "+" in "$g_i + g_i'$" denotes the group operation in G. Hence the sum of two chains is again a chain, and this addition of chains is commutative and associative. Consequently the operation (3.1) defines an abelian group which we denote by $C^n(K; G)$, and which we call the *group of n-chains of K over G*. The identity of $C^n(K; G)$ is obviously the chain all of whose coefficients are 0, and which we denote by 0. The inverse of $\sum_{i=1}^{\infty} g_i\sigma_i^n$ is $\sum_{i=1}^{\infty}(-g_i)\sigma_i^n$ which we abbreviate to $\sum_{i=1}^{\infty} -g_i\sigma_i^n$.

Since by our algebraic conventions, for $n \geqq 0$,

$$\partial(C_1^n + C_2^n) = \partial(C_1^n) + \partial(C_2^n), \tag{3.2}$$

for all C_1^n, $C_2^n \in C^n(K; G)$, the linear operator ∂ induces a homomorphism of $C^n(K; G)$ into $C^{n-1}(K; G)$ (or into 0, the identity of G, in the nonaugmented case when $n = 0$). And since the images of the elements of $C^n(K; G)$, $n > 0$, under ∂ are always chains C^{n-1} such that $\partial(C^{n-1}) = 0$, we are naturally led to a consideration of the collection $Z^{n-1}(K; G)$ of $(n - 1)$-chains whose boundary-chains are all 0.

It follows from (3.2) that the collection $Z^n(K; G)$, $n \geqq 0$, is a subgroup of $C^n(K; G)$. We call it the *group of n-cycles of K over G*. (It is defined only for $n \geqq 0$, since the operator ∂ was defined only for this case.)

Finally, we define a group $B^n(K; G)$; which is that subgroup of $Z^n(K; G)$ consisting of all boundary chains or *bounding cycles*. We note that by (2.3) every boundary chain is a cycle when $n > 0$; every 0-chain is a cycle in the nonaugmented case, and in the augmented case every boundary 0-chain is a cycle since $\partial[\partial\sigma_\rho^1] = \partial[\sigma_{\rho 1}^0 - \sigma_{\rho 2}^0] = \sigma^{-1} - \sigma^{-1} = 0$. And by (3.2) the sum of bounding cycles is again a bounding cycle. The chain 0 of dimension n bounds the chain 0 of dimension $n + 1$, so that $B^n(K; G)$ contains the identity.

Frequently, in the sequel, we may use the terms "augmented cycle," "augmented chain," etc., to indicate that the complex under discussion is augmented.

4. **Homology groups.** We have defined a descending series of abelian groups,

$$C^n(K; G) \supset Z^n(K; G) \supset B^n(K; G), \qquad n \geqq 0.$$

For $n > 0$, the linear operator ∂ applied to $C^n(K; G)$ effects a homomorphism of that group into the subgroup $B^{n-1}(K; G)$ of $C^{n-1}(K; G)$. The kernel of this homomorphism is the group $Z^n(K; G)$.

If now we form the factor group $Z^n(K; G)/B^n(K; G)$, we obtain the *n-dimensional* (or *nth*) *homology group of K over G*, $H^n(K; G)$, also commonly called the *nth Betti group of K over G*. Evidently two elements of $Z^n(K; G)$, say Z_1^n,

Z_2^n are in the same element of $H^n(K; G)$ if and only if $Z_1^n - Z_2^n$ is a bounding cycle. We express this fact by the relation

$$Z_1^n \sim Z_2^n, \tag{4.1}$$

called a *homology* or *homology relation* (compare II 5.5). The fact that a cycle Z^n is an element of $B^n(K; G)$ may therefore be expressed by the relation $Z^n \sim 0$, and (4.1) may be written in the form $Z_1^n - Z_2^n \sim 0$. In view of (3.2) this implies that relations such as (4.1) may be treated like ordinary algebraic equations so far as addition and transposition of elements are concerned. And a useful corollary in the sequel will be that when we desire to show that a single-valued mapping φ of one group of cycles into another induces a homomorphism in the corresponding homology groups it is sufficient to show that $Z^n \sim 0$ implies that $\varphi(Z^n) \sim 0$.

(A relation such as (4.1) may be read "Z_1^n is homologous to Z_2^n;" Z_1^n and Z_2^n are also called *homologous cycles.*)

In case the group $H^n(K; G)$ has finite rank, then the minimum number in a complete set of linearly independent generators is called the *n-dimensional Betti number of K over G* and is denoted by $p^n(K; G)$. (Also sometimes called "connectivity number". For the case where G is an algebraic field, a somewhat different definition is given below.) Otherwise we may write $p^n(K; G) = \infty$. Evidently for $p^n(K; G)$ to be a positive integer k means that there exist n-cycles $Z_1^n, \cdots, Z_k^n$ such that no homology of the form $a_1Z_1^n + \cdots + a_kZ_k^n \sim 0$ exists, where the a's are integers not all zero, and such that if Z^n is an arbitrary n-cycle, then there exists a homology $aZ^n + a_1Z_1^n + \cdots + a_kZ_k^n \sim 0$. In other words, if we call a set of cycles $Z_1^n, \cdots, Z_k^n$ *linearly independent relative to homology* (=lirh) provided there does not exist any relation of the form $a_1Z_1^n + \cdots + a_kZ_k^n \sim 0$, then $p^n(K; G)$ is the maximum number of n-cycles of K over G that are lirh.

Frequently, in the sequel, when augmented complexes are used, this fact is indicated by a subscript "a"; for example, $H_a^n(K; G)$, $B_a^n(K; G)$, $p_a^n(K; G)$.

5. **Important special cases and geometric interpretations.** The simplest case of the theory outlined above is illustrated by the finite complex and the additive group of integers mod 2 (cf. II 5). In all chains only the coefficients 0, 1 occur; thus the effect of orientation is eliminated inasmuch as $-\sigma^n = \sigma^n$. Furthermore a geometric interpretation of the algebra is quite immediate, in that each n-chain may be considered as a selection or collection of certain simplexes E^n, and in a sum of chains the cells that have coefficient 1 represent exactly those simplexes that have been "selected" in an odd number of the chains. In particular, relation (3.2) shows that the boundary of an n-chain, being the sum, mod 2, of the boundaries of the individual n-cells, is simply the selection of those $(n - 1)$-simplexes that occur in an odd number of the individual simplexes. This modulo 2 "combinatorial topology", originally explored by

Tietze, Veblen and Alexander, has consequently great advantages due to its simplicity, but as will be pointed out below does not suffice to detect certain topological properties, and historically it contributed to some confusion in that the obviousness of the geometric interpretation of chain, bounding, etc., modulo 2, retarded the necessary distinction between what is geometric and what is of purely algebraic character

As for the finite complex, evidently any ordinary geometric surface or solid which has been "triangulated" or cut up into a finite number of nonintersecting tetrahedra furnishes an illustration, if one selects the vertices of each edge, triangle and tetrahedron as the vertices of the 1-, 2- and 3-simplexes respectively. The classical combinatorial topology was essentially a study of such configurations and their higher-dimensional analogues, and the proof that the Betti groups obtained are topologically invariant, not dependent upon the particular mode of subdivision, was a central problem (and a difficult one). For exposition of the modulo 2 topology of finite complexes, the reader may be referred to the Colloquium volume of Veblen [V]. A brief but satisfactory account of the homology theory of finite complexes may also be found in Alexandroff [g].

Historically, the additive group of integers took precedence as coefficient group (vide Poincaré [a]; in this paper "homologies" as used above were introduced, but the relations between chains and their bounding cycles were represented by congruences $C^n \equiv C^{n-1}$). Here orientation plays an important and significant role, and each symbol σ^n as well as the polynomial $\partial\sigma^n$ becomes a chain. The orientation can be easily interpreted in an inductive manner, for $n = 1$ considering it as indicating direction from one vertex to another. The appearance of $c\sigma^n$, c a positive integer, in a chain can be interpreted as a selection or collection of the corresponding simplex c times; $-c\sigma^n$ as the occurrence c times of the oppositely oriented simplex. The geometric significance of the integer coefficients is well exemplified in such an elementary complex as that obtained by a triangulation of the projective plane. Here it will be found that the cycle mod 2 associated with a "straight line through infinity" (topologically a circle) is unbounding, yielding $p^1(K; G) = 1$, whereas the analogous cycle with integers as coefficients has a "multiple" that bounds—$2Z^1 \sim 0$—rendering $p^1(K; G) = 0$. This circumstance is described as the presence of *torsion* in the surface, the coefficient 2 being called a 1-*dimensional coefficient of torsion* of the projective plane (see Veblen [V, p. 119]). One may also compare the mod 2 and integer Betti groups of the torus and Klein bottle. For a discussion of these matters, especially as related to such 2-dimensional surfaces, the reader may consult the notes of Tucker [T]; also Hilbert and Cohn-Vossen [H-C, Chap. 6] and Kerékjártó [K, IV].

It is a classical theorem that the 2-dimensional closed manifolds are characterized topologically by their 1-dimensional Betti numbers mod 2 and their "orientableness" (in the sense that a surface is called orientable when its simplexes E_i^2 can be so assigned orientations that $\partial \sum \sigma_i^2 = 0$. For surfaces in

euclidean 3-space, this conception coincides with "two-sidedness", so that all such surfaces are orientable.) Cf. Kerékjártó [K, IV].

It is now well known that for any finite complex the additive group of integers forms a "universal" coefficient group, in that knowledge of the groups $H^n(K; I)$, where I is the group of integers, is sufficient to calculate the groups $H^n(K; G)$ for any coefficient group G (cf. Lefschetz [L, 109]).

It was shown by Alexander [b] that use of the integers mod m for arbitrarily large m is sufficient for the determination of torsion in a finite complex. And as shown by Steenrod [a], the additive group of real numbers modulo 1 (isomorph of the group of rotations of a circle) forms a universal coefficient group for the homology theory of a compact metric space.

The last mentioned group, that of the real numbers modulo 1, has a topological character not evidenced in the previously mentioned discrete coefficient groups—mod 2, integers, etc. It lacks the fundamental distributive property of algebraic rings, but possesses a continuity which can be defined in an obvious manner by a fundamental neighborhood system. Indeed, when so set up as a topological space, it becomes both compact and metric (having a denumerable fundamental neighborhood system), as well as connected.

From the point of view of the present work, the case where the group G is an algebraic field (discrete), $\mathfrak{F}$, such as that of the rational numbers (originally introduced by Lefschetz [b]), and chains are finite is of paramount importance. The chain groups $C^n(K; \mathfrak{F})$ then become vector spaces whose generators are the cells σ_i^n. The groups $Z^n(K; \mathfrak{F})$, $B^n(K; \mathfrak{F})$ are subspaces of $C^n(K; \mathfrak{F})$. Cycles $Z_1^n, \cdots, Z_k^n$ are called *linearly independent relative to homology* (*lirh*) if there does not exist any relation of the form $a_1Z_1^n + \cdots + a_kZ_k^n \sim 0$, where the a's are now elements of $\mathfrak{F}$ not all zero. Then, with the usual definition of dimension of a vector space, $p^n(K; \mathfrak{F}) =$ dimension $Z^n(K; \mathfrak{F}) -$ dimension $B^n(K; \mathfrak{F})$ for finite K. We shall go into these matters more fully later on.

For an example of an infinite complex, we may consider any open subset P of the euclidean plane (compare II 5.12). Let $T_1, \cdots, T_k, \cdots$ be a sequence of triangulations of the plane, each T_{k+1} being obtained from T_k by further subdivision, such that if G_k is the collection of simplexes in T_{k+1} but not in T_k, then the maximum diameter of simplexes in G_k approaches zero as $k \to \infty$. For each k, let K_k be the complex obtained from the simplexes of T_k which lie wholly in P. Then $K = \bigcup_k K_k$ is an infinite complex. Of simpler structure, however, and topologically as useful, is the complex obtained by selecting for each K_{k+1} only those simplexes which were present in K_k, and those simplexes of T_{k+1} not occurring in K_k.

6. **Some fundamental lemmas.** Suppose K is a finite oriented complex with cells σ_i^r, $i = 1, 2, \cdots$. Let us add to K a new vertex v_0 and form a new complex $\hat{K}$, the *cone-complex* of K, as follows: If $v_1, v_2, \cdots, v_r$ are vertices of a simplex of K, then $v_0, v_1, v_2, \cdots, v_r$ form a simplex of $\hat{K}$. For each cell σ_i^r of K, define a new cell $\sigma_i^{r+1} = v_0\sigma_i^r$; i.e., if $\sigma_i^r = v_1v_2 \cdots v_{r+1}$, then $\sigma_i^{r+1} = v_0v_1 \cdots v_{r+1}$.

The oriented complex $\hat{K}$ consists of the new simplexes with corresponding cells so formed, together with the new vertex v_0 and the original simplexes with corresponding cells of K; K is a subcomplex of $\hat{K}$ which may be called the *base* of $\hat{K}$. (The complex $\hat{K}$ may also be referred to as the *join* of K and v_0 .)

6.1 LEMMA. *For any finite oriented complex K, the groups $H^r_a(\hat{K}; G)$ all reduce to the identity.*

PROOF. Treating $\hat{K}$ as an augmented complex, we define a mapping, f, of the r-chains of $\hat{K}$ into the $(r + 1)$-chains of $\hat{K}$ as follows: $f(\sigma^{-1}) = v_0$; if σ^r is a cell having v_0 as a vertex, then $f(\sigma^r) = 0$, but otherwise $f(\sigma^r) = v_0\sigma^r$. The extension to arbitrary chains is made linearly. Then for any chain C^r,

$$\partial f(C^r) = C^r - f(\partial C^r), \qquad r \geqq 0. \tag{6.1a}$$

Since ∂ is linear, it suffices to prove (6.1a) for the case where C^r is a cell σ^r. (Of course σ^r may not be an *r-chain*, but this does not invalidate the argument.) Suppose $\sigma^r = v_{i_0}v_{i_1} \cdots v_{i_r}$, $0 < i_0 < i_1 < \cdots < i_r$. Then $f(\sigma^r) = v_0v_{i_0} \cdots v_{i_r}$ and $\partial f(\sigma^r) = \sigma^r - \sum (-1)^i v_0 \cdots \hat{v}_{i_j} \cdots v_{i_r} = \sigma^r - f(\partial\sigma^r)$. (By the symbol $\hat{v}_{i_j}$ we indicate that v_{i_j} is deleted.) On the other hand, if $\sigma^r = v_0v_{i_1} \cdots v_{i_r}$, $0 < i_1 < \cdots < i_r$, then $f(\sigma^r) = 0 = \partial f(\sigma^r)$. Hence we must show that $\sigma^r = f(\partial\sigma^r)$. Now $\partial\sigma^r = v_{i_1} \cdots v_{i_r} - \sum_{j=1}^{r} (-1)^j v_0 \cdots \hat{v}_{i_j} \cdots v_{i_r}$; hence by definition, $f(\partial\sigma^r) = v_0v_{i_1} \cdots v_{i_r} = \sigma^r$.

To complete the proof of the lemma we need only notice that if in (6.1a) the chain C^r is a cycle, then $\partial f(C^r) = C^r$.

6.2 COROLLARY. *The (augmented) homology groups of the complex consisting of a single oriented simplex are all zero.*

REMARK. The mapping f used in the proof of Lemma 6.1 is a special case of a *chain-mapping*; i.e., a homomorphism of one chain-group into another. Usually the chain-groups involved belong to different complexes, however. The case (exemplified above) where the homomorphism is into a group of one dimension higher is a common one, and the simplest case therein occurs when the mapping commutes with the boundary operator: $\partial f = f\partial$. This commutativity also occurs in simplicial mappings:

6.3 DEFINITION. Let K and K' be oriented complexes, and to each vertex v of K let correspond a vertex $v' = f(v)$ of K' in such a way that if $v_0 , \cdots , v_r$ are vertices of a simplex of K, then $v'_0 , v'_1 , \cdots , v'_r$, whether distinct or not, are vertices of a simplex of K'. Now suppose $\sigma^r = v_0v_1 \cdots v_r$ is a cell of K. If the v'_i , $i = 0, 1, \cdots , r$, are not all distinct, define $f(\sigma^r) = 0$; if they are distinct, they form a cell σ'^r (or $-\sigma'^r$) of K' and we define $f(\sigma^r) = \sigma'^r$ (or $-\sigma'^r$, as the case may be), and $f(-\sigma^r) = -\sigma'^r$ (or σ'^r). Then f is called a *simplicial mapping* of K into K'. We may use the symbols $f : K \to K'$ to express the fact that f is a simplicial mapping of K into K'. Finally, if $C^r = \sum g^i\sigma^r_i$ is a *chain* of K, we define $\varphi(C^r) = \sum g^i f(\sigma^r_i)$, where by convention $g \cdot 0 = 0$ and $g \cdot (-\sigma') =$

$-g\sigma'$. Such a mapping φ is called the *chain-mapping* of $C^r(K; G)$ into $C^r(K'; G)$ induced by the simplicial mapping f. Such a mapping may be denoted by the symbols $\varphi : C^r(K; G) \to C^r(K'; G)$.

6.4 LEMMA. *If K and K' are complexes, and f a simplicial mapping of K into K', then the chain-mapping $\varphi : C^r(K; G) \to C^r(K'; G)$ induced by f commutes with ∂.*

We leave the details of the proof to the reader.

As a corollary, *such a mapping φ induces a homomorphism of $H^r(K; G)$ into $H^r(K'; G)$.*

In the sequel, we shall ordinarily use f throughout instead of f and φ; it will be understood that if f is a simplicial mapping of K into K', then $f(C^r)$ really means the $\varphi(C^r)$ defined above, etc.

With a simplicial mapping $f : K \to K'$ we may associate a third complex, the *deformation-complex*. Let the vertices of K' be ordered, and then order the vertices of K so that if $v_1 < v_2$, then $f(v_1) \leqq f(v_2)$ in K'. Then if $v_{i_0} \cdots v_{i_n}$, $i_0 < \cdots < i_n$, is a simplex of K, the collection of simplexes $v_{i_0} \cdots v_{i_i} v'_{i_i} \cdots v'_{i_n}$ (where the v' symbol indicates a vertex of K') together with their faces forms an $(n + 1)$-complex, and the totality of these over K forms the deformation-complex $\mathfrak{D}K$. Thus $\mathfrak{D}K$ contains simplexes of both K and K' (all of the former), as well as new simplexes. The new simplexes may be oriented, and with the old orientation of the simplexes of K and K', $\mathfrak{D}K$ becomes an oriented complex.

If $\sigma^n = v_{i_0} \cdots v_{i_n}$ is a cell of K, then $\mathfrak{D}\sigma^n = \sum_{i=0}^n (-1)^i v_{i_0} \cdots v_{i_i} v'_{i_i} \cdots v'_{i_n}$ (with the convention that cells with nondistinct vertices become zero) is called the *deformation-chain* of σ^n induced by f (it is not necessarily an element of $C^{n+1}(\mathfrak{D}K; G)$, of course).

6.5 LEMMA. *If f is a simplicial mapping of K into K', σ^n is a cell of K, and $\mathfrak{D}\sigma^n$ the deformation-chain of σ^n induced by f, then $\partial\mathfrak{D}\sigma^n = f(\sigma^n) - \sigma^n - \mathfrak{D}\partial\sigma^n$.*

The extension of $\mathfrak{D}$ to a mapping of chains is made in linear fashion, and thereby a chain-mapping of $C^n(K; G)$ into $C^{n+1}(\mathfrak{D}K; G)$ induced. And we then have:

6.6 LEMMA. *Under the hypothesis of Lemma 6.5, if $C^n \in C^n(K; G)$, then $\partial\mathfrak{D}C^n = f(C^n) - C^n - \mathfrak{D}\partial C^n$.*

In particular, if C^n is a cycle, $\partial C^n = 0$ and we have:

6.7 LEMMA. *If f is a simplicial mapping of K into K' and Z^n is a cycle of K, then $Z^n \sim f(Z^n)$ on $\mathfrak{D}K$.*

REMARK. It may occur to the reader that a relation such as that in Lemma 6.6 is probably a special case of a relation involving two mappings, f and g. If we consider f as a mapping of K into $\mathfrak{D}K$, and g as the identity mapping on K, then the relation mentioned becomes $\partial\mathfrak{D}C^n = f(C^n) - g(C^n) - \mathfrak{D}\partial C^n$. In

the sequel we encounter just such relations (see the proof of Theorem 7.2 below).

In general, if f and g are mappings of $C^n(K; G)$ into $C^n(K'; G)$, where K and K' are any two complexes, and if there exists a mapping $\mathfrak{D}$ of $C^n(K; G)$ into $C^{n+1}(K'; G)$ such that $\partial\mathfrak{D}C^n = f(C^n) - g(C^n) - \mathfrak{D}\partial C^n$, then f and g are called *chain-homotopic*. Evidently from this definition follows:

6.8 Lemma. *If f and g are chain-homotopic mappings (that commute with ∂) of $C^n(K; G)$ into $C^n(K'; G)$, then f and g induce the same homomorphism of $H^n(K; G)$* into $H^n(K'; G)$.

7. Čech cycles and homology groups.

We next define the type of cycle introduced by Čech for general spaces. We first give the mode of definition due to Čech.

In general, spaces will be Hausdorff. And throughout the present section, a *covering* of a space will be a covering by a finite number of open sets (fcos). Coverings will be denoted by German capitals $\mathfrak{U}$, $\mathfrak{V}$, $\cdots$, and their elements (the individual open sets) will be denoted by italic capitals U, V, $\cdots$. If every element of a covering $\mathfrak{V}$ is a subset of some element of $\mathfrak{U}$, then $\mathfrak{V}$ will be called a *refinement* of $\mathfrak{U}$; this will be symbolized $\mathfrak{V} > \mathfrak{U}$. If $\mathfrak{V}$ is a refinement of every covering of a collection $\{\mathfrak{U}_\nu\}$, then $\mathfrak{V}$ will be called a common refinement of the coverings $\mathfrak{U}_\nu$. That $\mathfrak{V}$ is a common refinement of the coverings $\mathfrak{U}_1, \cdots, \mathfrak{U}_k$ will be denoted by the symbols $\mathfrak{V} > (\mathfrak{U}_1, \cdots, \mathfrak{U}_k)$. We denote the collection of all coverings of the space by Σ. If $\mathfrak{U}, \mathfrak{V} \in \Sigma$, then by $\mathfrak{U} \cap \mathfrak{V}$ will be meant the covering consisting of all nonempty sets $U \cap V$ such that $U \in \mathfrak{U}$, $V \in \mathfrak{V}$. Evidently $\mathfrak{U} \cap \mathfrak{V} > (\mathfrak{U}, \mathfrak{V})$. Thus

7.0 *Every finite set of coverings has a common refinement.*

7.1 *Čech cycles.* Let $\mathfrak{U} \in \Sigma$. We may think of $\mathfrak{U}$ as constituting a complex if we let each $U \in \mathfrak{U}$ be called a "vertex", and a collection $U_0, U_1, \cdots, U_n$ constitute an n-simplex if their *nucleus* $\bigcap_i U_i \neq 0$. We shall speak henceforth of chains of $\mathfrak{U}$, etc., meaning chains of the complex $\mathfrak{U}$. And we shall use $\mathfrak{U}$ to denote both the covering and the complex; the context should make clear in each case which of the two meanings is indicated.

If $\mathfrak{V} > \mathfrak{U}$, a *projection* $\pi_{\mathfrak{U}\mathfrak{V}}$ of $\mathfrak{V}$ into $\mathfrak{U}$ will be a simplicial mapping of $\mathfrak{V}$ into $\mathfrak{U}$ such that if $V \in \mathfrak{V}$, then $\pi_{\mathfrak{U}\mathfrak{V}} V \supset V$. Since projections are not usually unique, of fundamental importance is the theorem:

7.2 Theorem. *If $\mathfrak{V} > \mathfrak{U}$, then all projections of $\mathfrak{V}$ into $\mathfrak{U}$ induce the same homomorphism of $H^n(\mathfrak{V}; G)$ into $H^n(\mathfrak{U}; G)$. Hence if z^n is a cycle of $\mathfrak{V}$, then for every pair of projections π_1, π_2 of $\mathfrak{V}$ into $\mathfrak{U}$, $\pi_1 z^n \sim \pi_2 z^n$.*

Proof. Let the elements of $\mathfrak{V}$ be ordered: $V_1, V_2, \cdots$, and let $\pi_1 V$, $\pi_2 V$ be denoted by U, U', respectively. (Note that $V \subset U \cap U'$.)

If $\eta\sigma^n = V_{i_0} V_{i_1} \cdots V_{i_n}$, $i_0 < i_1 \cdots < i_n$, is a positively or negatively oriented cell of $\mathfrak{V}$, let $P(\eta\sigma^n)$ be defined by

$$(7.2a) \quad P(\eta\sigma^n) = P(V_{i_0} V_{i_1} \cdots V_{i_n}) = \eta \sum_i (-1)^i U_{i_0} \cdots U_{i_i} U'_{i_i} \cdots U'_{i_n},$$

with the convention that every term in the right-hand member of (7.2a) whose vertices are not all distinct is zero. Since $U_{i_0} \cap \cdots \cap U_{i_j} \cap U'_{i_j} \cap \cdots \cap U'_{i_n} \supset V_{i_0} \cap \cdots \cap V_{i_j} \cap \cdots \cap V_{i_n}$, each term on the right of (7.2a) is a positively or negatively oriented cell of $\mathfrak{U}$ or zero. If P is extended linearly to chains of $\mathfrak{B}$, there is obtained a chain-mapping of $C^n(\mathfrak{B})$ into $C^{n+1}(\mathfrak{U})$. This is a homomorphism such that

$$\partial P(C^n) = \pi_2 C^n - \pi_1 C^n - P(\partial C^n). \tag{7.2b}$$

Hence π_1 and π_2 are chain-homotopic and the theorem follows from Lemma 6.8.

7.3 Definition. An n-dimensional Čech cycle, or *C-cycle* of a space S is a collection $\{z^n(\mathfrak{U})\}$, $\mathfrak{U} \in \Sigma$, where $z^n(\mathfrak{U})$ is a cycle of $\mathfrak{U}$ called the *coordinate* of the cycle on $\mathfrak{U}$, and such that if $\mathfrak{B} > \mathfrak{U}$, then $\pi_{\mathfrak{U}\mathfrak{B}} z^n(\mathfrak{B}) \sim z^n(\mathfrak{U})$ on $\mathfrak{U}$. By Theorem 7.2 this homology is independent of the particular projection employed. Instead of "n-dimensional Čech cycle" we may say "Čech n-cycle". *It must be emphasized that a C-cycle has a coordinate on every* $\mathfrak{U} \in \Sigma$.

If $Z_1 = \{Z_1^n(\mathfrak{U})\}$, $Z_2 = \{Z_2^n(\mathfrak{U})\}$ are two C-cycles of S, then we let $Z_1 + Z_2 = \{Z_1^n(\mathfrak{U}) + Z_2^n(\mathfrak{U})\}$, and in terms of this addition we obtain the group $Z^n(S; G)$ of n-dimensional C-cycles of S. In order to obtain a homology group $H^n(S; G)$, we define $Z_1 \sim 0$ on S to mean that for *every* $\mathfrak{U}$, $Z_1^n(\mathfrak{U}) \sim 0$ on $\mathfrak{U}$. Then if $B^n(S;G)$ is the group of all C-cycles that are ~ 0 on S, $H^n(S;G)$ is $Z^n(S;G)/B^n(S;G)$. The corresponding Betti number (§4) is denoted by $p^n(S;G)$.

7.4 Definition. Let $\Sigma' \subset \Sigma$. Then Σ' will be called a *complete family of coverings* of S if for every $\mathfrak{U} \in \Sigma$ there exists $\mathfrak{U}' \in \Sigma'$ such that $\mathfrak{U}' > \mathfrak{U}$.

Evidently by restricting the choice of coordinates to cycles on the elements of a complete family Σ', cycles similar to the C-cycles and their corresponding homology groups may be defined, and it is important to notice that the homology groups defined thus with respect to a complete family are isomorphic with those determined by Σ.

7.5 Theorem. *The homology group $H^n(S; G)$ and the n-dimensional homology group determined by a complete family of coverings are isomorphic.*

Proof. If $\{z^n(\mathfrak{U})\}$ is a C-cycle, then the collection $\{z^n(\mathfrak{U}')\}$, $\mathfrak{U}' \in \Sigma'$, is a cycle on Σ', and evidently if $\{z^n(\mathfrak{U})\} \sim 0$, then $\{z^n(\mathfrak{U}')\} \sim 0$ on Σ'. In this manner there is determined a homomorphism of $H^n(S; G)$ into the n-dimensional homology group determined by Σ'.

This is also a homomorphism "onto." For let the collection $\{z^n(\mathfrak{U}')\}$ be given. For $\mathfrak{U} \in \Sigma$, let $\mathfrak{U}' \in \Sigma'$ be such that $\mathfrak{U}' > \mathfrak{U}$, and let $z^n(\mathfrak{U}) = \pi_{\mathfrak{U}\mathfrak{U}'} z^n(\mathfrak{U}')$. By Theorem 7.2, $z^n(\mathfrak{U})$ is independent of the choice of $\pi_{\mathfrak{U}\mathfrak{U}'}$, so far as *homology* is concerned. To show independence of the choice of $\mathfrak{U}'$, let $\mathfrak{B}'$ be any other element of Σ' such that $\mathfrak{B}' > \mathfrak{U}$. Let $\mathfrak{W}' \in \Sigma'$ be such that $\mathfrak{W}' > (\mathfrak{U}', \mathfrak{B}')$ (7.0). By Theorem 7.2,

$$\pi_{\mathfrak{U}\mathfrak{U}'} \pi_{\mathfrak{U}'\mathfrak{W}'} z^n(\mathfrak{W}') \sim \pi_{\mathfrak{U}\mathfrak{B}'} \pi_{\mathfrak{B}'\mathfrak{W}'} z^n(\mathfrak{W}'), \tag{7.5a}$$

and by definition,

$$\pi_{\mathfrak{U}'\mathfrak{W}'}z^n(\mathfrak{W}') \sim z^n(\mathfrak{U}'), \tag{7.5b}$$

$$\pi_{\mathfrak{V}'\mathfrak{W}'}z^n(\mathfrak{W}') \sim z^n(\mathfrak{V}'). \tag{7.5c}$$

Applying $\pi_{\mathfrak{U}\mathfrak{U}'}$ to both sides of (7.5b) and $\pi_{\mathfrak{U}\mathfrak{V}'}$ to both sides of (7.5c), it follows from (7.5a) that $\pi_{\mathfrak{U}\mathfrak{U}'}z^n(\mathfrak{U}') \sim \pi_{\mathfrak{U}\mathfrak{V}'}z^n(\mathfrak{V}')$ on $\mathfrak{U}$, so that the homology class of $\mathfrak{U}$ determined is independent of $\mathfrak{U}'$.

The collection $\{z^n(\mathfrak{U})\}$ thus determined is a C-cycle. Suppose $\mathfrak{V} > \mathfrak{U}$, $\mathfrak{U}$, $\mathfrak{V} \in \Sigma$. Let $\mathfrak{U}' \in \Sigma'$ be such that $\mathfrak{U}' > \mathfrak{V}$. As shown in the previous paragraph $z^n(\mathfrak{V}) \sim \pi_{\mathfrak{V}\mathfrak{U}'}z^n(\mathfrak{U}')$, and therefore $\pi_{\mathfrak{U}\mathfrak{U}'}z^n(\mathfrak{U}') \sim \pi_{\mathfrak{U}\mathfrak{V}}\pi_{\mathfrak{V}\mathfrak{U}'}z^n(\mathfrak{U}') \sim \pi_{\mathfrak{U}\mathfrak{V}}z^n(\mathfrak{V})$. Again, by the preceding paragraph, $\pi_{\mathfrak{U}\mathfrak{U}'}z^n(\mathfrak{U}') \sim z^n(\mathfrak{U})$, hence $z^n(\mathfrak{U}) \sim \pi_{\mathfrak{U}\mathfrak{V}}z^n(\mathfrak{V})$. Thus $\{z^n(\mathfrak{U})\}$ is a C-cycle, and in the above homomorphism it maps into the class determined by the given $\{z^n(\mathfrak{U}')\}$.

Finally, if the cycle $\{z^n(\mathfrak{U}')\}$ on Σ' is ~ 0 on Σ', then the cycle $\{z^n(\mathfrak{U})\} \sim 0$. For if $\mathfrak{U}' > \mathfrak{U}$, and there exists a chain $c^{n+1}(\mathfrak{U}')$ on $\mathfrak{U}'$ such that $\partial c^{n+1}(\mathfrak{U}') = z^n(\mathfrak{U}')$, then (since $\partial\pi = \pi\partial$ by Lemma 6.4) $\partial\pi_{\mathfrak{U}\mathfrak{U}'}c^{n+1}(\mathfrak{U}') = \pi_{\mathfrak{U}\mathfrak{U}'}z^n(\mathfrak{U}')$. Thus $z^n(\mathfrak{U}) \sim \pi_{\mathfrak{U}\mathfrak{U}'}z^n(\mathfrak{U}') \sim 0$. It follows that the homomorphism defined above is one-to-one and therefore an isomorphism.

One application of Theorem 7.5 that is worthy of note is that to the homology theory of compact metric spaces. In this case, the group $H^n(S; G)$ is determined by the C-cycles defined on a *countable* set of coverings.

7.6 Definition. If $M \subset S$, $\mathfrak{U} \in \Sigma$, then a $\sigma^n = U_0U_1 \cdots U_n$ $(U_i \in \mathfrak{U})$, and the corresponding simplex, will be said to be *on M* if the nucleus $\bigcap_{i=0}^n U_i$ meets M. A chain $g^i\sigma_i$ will be said to be on M if each σ_i^n for which $g^i \neq 0$ is on M. Since the nucleus of a simplex is an open set, a simplex or cell is on M if and only if it is on $\overline{M}$.

An equivalent procedure is to let $\mathfrak{U} \wedge M$ be the complex consisting of all simplexes of $\mathfrak{U}$ that are on M, and to let a chain of $\mathfrak{U}$ be a chain on M if and only if it is a chain of $\mathfrak{U} \wedge M$. By $\mathfrak{U} \cap M$ we denote $\{U | (U \in \mathfrak{U})$ & $(U \cap M \neq 0)$.

7.7 Suppose that $L \subset M \subset S$, and that $z^n(\mathfrak{U})$ is a chain of $\mathfrak{U}$ which is also a chain on M, and suppose further that $\partial z^n(\mathfrak{U})$ is on L. Then we call $z^n(\mathfrak{U})$ a *cycle* mod L *on* M of $\mathfrak{U}$. Such a "cycle" may be called a *relative cycle.* And if $c^n(\mathfrak{U})$ is a chain on M such that $\partial c^n(\mathfrak{U}) = z^{n-1}(\mathfrak{U}) + \gamma^{n-1}$, where γ^{n-1} is on L, then we say that $\partial c^n(\mathfrak{U}) = z^{n-1}(\mathfrak{U})$ mod L and that $z^{n-1}(\mathfrak{U}) \sim 0$ mod L (or "bounds mod L") on M. Such a homology may be called a *relative homology* (i.e., relative to L). The n-cycles mod L on M of $\mathfrak{U}$ form a group $Z^n(M, L; G, \mathfrak{U})$, and those that are ~ 0 mod L on M form a group $B^n(M, L; G, \mathfrak{U})$. The factor group $Z^n(M, L; G, \mathfrak{U})/B^n(M, L; G, \mathfrak{U})$ is the nth homology group of M mod L on $\mathfrak{U}$, $H^n(M, L; G, \mathfrak{U})$.

7.8 In order to extend these "relative" ideas to the C-cycles, suppose $\mathfrak{V} > \mathfrak{U}$ and that $z^n(\mathfrak{V})$ is a cycle mod L on M. Noting that *projection has the effect of*

enlarging nuclei, it is clear that $\pi_{\mathfrak{U}\mathfrak{B}} z^n(\mathfrak{B})$ will be a chain on M and that $\pi_{\mathfrak{U}\mathfrak{B}} \partial z^n(\mathfrak{B})$ will be a chain on L, and consequently $\pi_{\mathfrak{U}\mathfrak{B}} z^n(\mathfrak{B})$ will be a cycle mod L on M.

7.9 THEOREM. *If $\mathfrak{B} > \mathfrak{U}$, then all projections of $\mathfrak{B}$ into $\mathfrak{U}$ induce the same homomorphism of $H^n(M, L; G, \mathfrak{B})$ into $H^n(M, L; G, \mathfrak{U})$. Hence if $z^n(\mathfrak{B})$ is a cycle* mod *L on M and π_1, π_2 are projections of $\mathfrak{B}$ into $\mathfrak{U}$, $\pi_1 z^n(\mathfrak{B}) \sim \pi_2 z^n(\mathfrak{B})$* mod *$L$ on M.*

PROOF. The proof follows that of Theorem 7.2 through relation (7.2b). Relation (7.2b) gives

$$\partial P(z^n(\mathfrak{B})) = \pi_2 z^n(\mathfrak{B}) - \pi_1 z^n(\mathfrak{B}) - P(\partial z^n(\mathfrak{B})). \tag{7.9a}$$

It follows from the definition of P (7.2a) that if a chain $C^n(\mathfrak{B})$ is on L, then $P(C^n(\mathfrak{B}))$ is on L. Hence $P(\partial z^n(\mathfrak{B}))$ is on L, and (7.9a) gives

$$\partial P(z^n(\mathfrak{B})) = \pi_2 z^n(\mathfrak{B}) - \pi_1 z^n(\mathfrak{B}) \qquad \text{mod } L$$

and hence

$$\pi_1 z^n(\mathfrak{B}) \sim \pi_2 z^n(\mathfrak{B}) \qquad \text{mod } L \text{ on } M.$$

7.10 DEFINITION. As a result of Theorem 7.9 we may define an n-dimensional *C-cycle* mod *L on M* as a collection of chains $\{z^n(\mathfrak{U})\}$, $\mathfrak{U} \in \Sigma$, such that: (1) $z^n(\mathfrak{U})$ is a cycle mod L on M, and (2) if $\mathfrak{B} > \mathfrak{U}$, then $\pi_{\mathfrak{U}\mathfrak{B}} z^n(\mathfrak{B}) \sim z^n(\mathfrak{U})$ mod L on M. And we define $\{z^n(\mathfrak{U})\} \sim 0$ mod L on M to mean that each $z^n(\mathfrak{U}) \sim 0$ mod L on M for all $\mathfrak{U}$. The additive group of C-cycles mod L on M, which we denote by $Z^n(S; M, L; G)$, is defined in an obvious manner, as is the group $B^n(S; M, L; G)$ of C-cycles mod L on M that are ~ 0 mod L on M. And the homology group $H^n(S; M, L; G) = Z^n(S; M, L; G)/B^n(S; M, L; G)$ is called the *n-dimensional homology group of S* mod *L on M*.

If $K \subset L$, then a cycle mod K on M is also a cycle mod L on M; and if a cycle mod K on M is ~ 0 mod K on M, it is necessarily ~ 0 mod L on M. This correspondence induces a homomorphism of $H^n(S; M, K; G)$ into $H^n(S; M, L; G)$. In general this homomorphism is not "onto," as simple examples show. In particular the group $H^n(S; M, 0; G)$ determined by cycles mod 0 (the empty set) on M, which we may call *absolute cycles* of S on M, maps homomorphically into every $H^n(S; M, L; G)$ in a natural way. Evidently $H^n(S; G)$ and $H^n(S; S, 0; G)$ are the same groups.

As in the case of the absolute C-cycles of S itself, we may, by restricting the choice of coordinates to those on the elements of a complete family of coverings Σ', define relative cycles and homology groups on Σ'. And the proof of the following theorem is the same as that of Theorem 7.5, except that all homologies, etc., are "mod L on M."

7.11 THEOREM. *There exists an isomorphism between $H^n(S; M, L; G)$ and the corresponding n-dimensional homology group determined by a complete family of coverings of S.*

7.12 REMARK. The groups defined above, although sufficiently general for

purposes of the present discussion, are not the most general that we shall need in later chapters. Frequently, as when the space S under discussion is fixed, we may abbreviate the parenthetical $(S; M, L; G)$ in the above group symbols to $(M, L; G)$; and if, in addition, G is fixed, it may be further abbreviated to (M, L). These conventions will be particularly valuable when the greater generality mentioned above makes necessary the introduction of more symbols for the sets involved.

8. **Covering lemmas.** Certain existence theorems concerning coverings of compact or locally compact spaces will be inserted at this point. The coverings considered in this section are not assumed to be finite unless so specified.

8.1 Definition. If $\mathfrak{U}$ and $\mathfrak{V}$ are coverings of a space S such that the *closure* of each element of $\mathfrak{V}$ is a subset of some element of $\mathfrak{U}$, then $\mathfrak{V}$ will be called a *closure refinement* of $\mathfrak{U}$; symbolically, $\mathfrak{V} \gg \mathfrak{U}$.

The proof of the following lemma should be obvious.

8.2 Lemma. *If S is a regular space, then every covering of S has a closure refinement.*

However, for compact spaces, a much stronger theorem is provable:

8.3 Lemma. *If $\mathfrak{U}$ is a finite covering of a normal space S, then for each $U_i \in \mathfrak{U}$ there exists an open set U_i' such that $U_i \Supset U_i'$ and the collection $\{U_i'\}$ covers S.*

Proof. Denote the elements of $\mathfrak{U}$ by $U_1, \cdots, U_m$. Since S is normal, there exists an open set U_1' such that $U_1 - \bigcup_{i=2}^m U_i \subset U_1' \Subset U_1$.

Suppose $U_1', \cdots, U_{k-1}'$, $k \leqq m$, have been defined so that $U_i \Supset \overline{U_i'}$, and the sets $U_1', \cdots, U_{k-1}', U_k, \cdots, U_m$ form a covering $\mathfrak{U}_{k-1}$ of S. Then we may define U_k' relative to $\mathfrak{U}_{k-1}$ just as U_1' was defined above relative to $\mathfrak{U}$.

Remark. In view of Theorem III 1.27, Lemma 8.3 holds for every compact Hausdorff space S.

8.4 Definition. If $\mathfrak{U}$ is any collection of sets and M an arbitrary set, then by $\mathrm{St}(M, \mathfrak{U})$ we denote the union of all elements of $\mathfrak{U}$ that meet M. If $\mathfrak{U}$ and $\mathfrak{V}$ are coverings of a space S, then by $\mathrm{St}(\mathfrak{V}, \mathfrak{U})$ we denote the covering whose elements are the sets $\mathrm{St}(V, \mathfrak{U})$, $V \in \mathfrak{V}$.

8.5 Definition. If $\mathfrak{U}$ and $\mathfrak{V}$ are coverings of a space S such that $\mathrm{St}(\mathfrak{V}, \mathfrak{V}) > \mathfrak{U}$, then we call $\mathfrak{V}$ a *star-refinement* of $\mathfrak{U}$; in symbols, $\mathfrak{V} >^* \mathfrak{U}$.

8.6 Lemma. *Every finite covering $\mathfrak{U}$ of a compact Hausdorff space has a star-refinement.*

Proof. Using the symbols of the statement of Lemma 8.3, for each $x \in S$ let

$$(8.6a) \qquad V_x = \bigcap_{x \in U_i'} U_i' \cap \bigcap_{x \in \overline{U_i'}} U_i \cap \bigcap_{x \notin (\overline{U_i'})} (S - \overline{U_i'}).$$

Let $\mathfrak{B}$ be a finite collection of the sets V_x covering S, and for any $V_x \in \mathfrak{B}$, suppose $V_y \in \mathfrak{B}$ such that $V_x \cap V_y \neq 0$, $V_x \neq V_y$. Let U'_i be such that $x \in U'_i$. By (8.6a), $V_x \subset U'_i$, and hence $y \in \overline{U'_i}$ else by (8.6a) $V_y \subset S - U'_i$ and $V_x \cap V_y = 0$. But $y \in \overline{U'_i}$ implies, by (8.6a), that $V_y \subset U_i$.

8.7 LEMMA. *If F is a compact subset of a locally compact space S and $\mathfrak{U}$ is a covering of S, then there exists a covering $\mathfrak{B}$ of S and an open set Q containing F such that* (1) $\mathfrak{B} > \mathfrak{U}$, (2) *only a finite number of elements of $\mathfrak{B}$ meet Q, and* (3) *if the nucleus of a simplex of $\mathfrak{B}$ meets $\overline{Q}$, then it meets F.*

PROOF. There exists an open set P containing F such that $\overline{P}$ is compact. Each point of $\overline{P}$ is in an open set with compact closure which is a subset of an element of $\mathfrak{U}$, and some finite collection $\mathfrak{U}'$ of these sets covers $\overline{P}$. The elements of $\mathfrak{U}'$ together with all sets $U - \overline{P}$, $U \in \mathfrak{U}$, form a collection $\mathfrak{W} > \mathfrak{U}$ covering S only a finite number of whose elements meet $\overline{P}$. If E_i^r is a simplex of $\mathfrak{U}'$, denote its nucleus by N_i^r. And if N_i^r meets F, let $p_i^r \in F \cap N_i^r$; otherwise, let p_i^r denote any point of N_i^r. Let M denote the finite point set $\bigcup p_i^r$.

Denoting the elements of $\mathfrak{U}'$ by $U_1, \cdots, U_i, \cdots, U_k$, we may replace, stepwise as in the proof of Lemma 8.3, each U_i by an open set U'_i such that (1) $U_i \supset \overline{U'_i}$, (2) $U'_i \supset (M \cap U_i) \cup (\overline{P} - \bigcup_{j=1}^{i-1} U'_j - \bigcup_{j=i+1}^{k} U_j)$. The resulting collection $\{U'_i\}$ together with the elements of $\mathfrak{W}$ that do not meet P form the desired collection $\mathfrak{B}$.

Now suppose a simplex $E_i'^r = U'_0U'_1 \cdots U'_r$ of $\mathfrak{B}$ has a nucleus Q_i^r that does not meet F. Then $x \in \overline{Q_i^r} \cap F$ would imply $x \in U_0 \cap U_1 \cap \cdots \cap U_r = N_i^r$, since $\overline{Q_i^r} \subset \overline{U'_0} \cap \cdots \cap \overline{U'_r} \subset U_0 \cap \cdots \cap U_r = N_i^r$. But then $F \cap N_i^r \neq 0$, implying $p_i^r \in U'_0 \cap \cdots \cap U'_r = Q_i^r$; i.e., $Q_i^r \cap F \neq 0$. We must conclude, then, that if a nucleus Q_i^r fails to meet F, then $\overline{Q_i^r} \cap F = 0$. Let Q be any open subset of P which contains F and meets no $\overline{Q_i^r}$ that fails to meet F.

8.8 LEMMA. *If F is a compact subset of a locally compact space S and $\mathfrak{U}$ is a covering of S, then there exists a covering $\mathfrak{W}$ of S such that $\mathfrak{W} > \mathfrak{U}$, and any set of elements of $\mathfrak{W}$ that meet F have a nonempty nucleus only if this nucleus meets F.*

(It will be evident from the proof that only a finite number of elements of $\mathfrak{W}$ meet F — indeed, some open set Q containing F.)

PROOF. We obtain the covering $\mathfrak{B}$ and open set Q as in the proof of Lemma 8.7. As shown there, a set of elements U'_i has nucleus Q_i^r meeting F only if the same held for the nucleus N_i^r in $\mathfrak{U}'$. And by the way Q was selected, a Q_i^r must meet F if it meets Q; i.e., nuclei of simplexes of $\mathfrak{B}$ either meet Q and hence F, or lie in $S - \overline{Q}$. If we replace each U'_i by $U''_i = U'_i \cap Q$, and let $U'_i - F = V''_i$, then the covering $\mathfrak{W}$ consisting of the elements of $\mathfrak{B}$ with each U'_i replaced by U''_i and V''_i is a covering of the desired type.

Let us call a covering such as the covering $\mathfrak{W}$ of Lemma 8.8 *regular with respect to F*; i.e., $\mathfrak{W}$ is regular with respect to F if every simplex of $\mathfrak{W}$ all of whose vertices meet F is *on* F in the sense defined in 7.6.

8.9 LEMMA. *If F is a compact subset of a locally compact space S, then the coverings of S that are regular with respect to F form a complete family of coverings for S; and their intersections with F form a complete family of coverings for F.*

PROOF. The first half of the conclusion follows from Lemma 8.8. As for the second half: If $\mathfrak{U}$ is a covering of F by open subsets of F, then each element U of $\mathfrak{U}$ is the intersection with F of the set $S - (F - U)$ which is open in S. Hence the collection $\{S - (F - U)\}$, $U \in \mathfrak{U}$, is a covering $\mathfrak{V}$ of S, and by Lemma 8.8 there exists a refinement $\mathfrak{W}$ of $\mathfrak{V}$ that is regular with respect to F. For each $W \in \mathfrak{W}$, the set $W \cap F$ is open in F, and the collection of these that are not empty is a refinement of $\mathfrak{U}$.

8.10 COROLLARY. *If F is a compact subset of a locally compact space S, then the homology groups obtained by using cycles and chains of S on F are isomorphic with the groups obtained by using coverings of F.*

8.11 REMARK. As a consequence of Corollary 8.10, we may in the sequel when discussing homology groups, or individual cycles and chains, pass from coverings of a locally compact space to coverings of its compact subsets, and conversely, without explicit mention of its justification above.

The conclusions of 8.7-8.10 continue to hold if S is a normal space, F a closed subset of S, and $\mathfrak{U}$ a fcos.

9. **Vector spaces.** Suppose that the coefficient group used in the definitions of cycles, homologies, etc., is an algebraic field $\mathfrak{F}$. Then if $\{z^n(\mathfrak{U})\}$ is a C-cycle mod L on M, and $f \in \mathfrak{F}$, we define

$$f\{z^n(\mathfrak{U})\} = \{fz^n(\mathfrak{U})\}; \tag{9.a}$$

if $c^n = \sum_{i=1}^{\alpha_n} c^i\sigma_i^n$, then $ac^n = \sum_{i=1}^{\alpha_n} (ac^i)\sigma_i^n$, for all elements a, c^i of $\mathfrak{F}$. With this convention, the additive group of C-cycles mod L on M becomes a *vector space V over* $\mathfrak{F}$; i.e., satisfies, in addition to the property of being an additive abelian group, the following: (1) for every $f \in \mathfrak{F}$, $v \in V$, fv is a unique element of V, (2) for all $a, b \in \mathfrak{F}$, $v \in V$, $a(bv) = (ab)v$; (3) for all $a, b \in \mathfrak{F}$, $v \in V$, $(a + b)v = av + bv$; (4) for all $a \in \mathfrak{F}$, $v_1, v_2 \in V$, $a(v_1 + v_2) = av_1 + av_2$; and (5) $1 \cdot v = v$ for all $v \in V$.

The properties just stated of a vector space V imply that every finite linear combination $f_1v_1 + \cdots + f_kv_k$ (f's $\in \mathfrak{F}$, v's $\in V$) is a unique element of V, and that if 0 is the zero element of $\mathfrak{F}$, then V has a unique element ϕ such that $0 \cdot v = \phi$ for all $v \in V$, and $f\phi = \phi$ for all $f \in \mathfrak{F}$. Inasmuch as no confusion should result, we use 0 instead of ϕ hereafter, while continuing to use the same symbol for the zero element of $\mathfrak{F}$.

9.1 A set $\{v_\nu\}$, finite or infinite, of elements of V is called *linearly independent* if there exists no finite relation of the form

$$f_1v_1 + \cdots + f_kv_k = 0, \qquad f\text{'s} \in \mathfrak{F},\ v\text{'s} \in \{v_\nu\}, \tag{9.1a}$$

unless $f_1 = \cdots = f_k = 0$. If every $v \in V$ is expressible in the form

$$v = f_1 v_1 + \cdots + f_k v_k, \qquad f\text{'s} \in \mathfrak{F},\ v\text{'s} \in \{v_\nu\}, \tag{9.1b}$$

and the set of elements $\{v_\nu\}$ are linearly independent, then the set $\{v_\nu\}$ is called a *base* for V. Although there may be various bases for V, the cardinal number of elements in a base is invariant for fixed V (see the proof by Chevalley in Lefschetz [L, pp. 73-74]), and is called the *dimension* of V. If V and V' are vector spaces over $\mathfrak{F}$ of the same dimension, then there exists a *linear isomorphism* between V and V'; that is, a (1-1)-correspondence $\varphi : V \to V'$ that is not only an isomorphism in the sense of the additive group, but satisfies the condition that $f\varphi(v) = \varphi(fv)$ for all $f \in \mathfrak{F}$, $v \in V$ (it being easily shown that as a consequence $f\varphi^{-1}(v') = \varphi^{-1}(fv')$ for all $f \in \mathfrak{F}$ and $v' \in V'$). Of course such an isomorphism is rather arbitrary, being obtained by proceeding from an arbitrary definition of φ over the bases for V and V' and extending it by linearity to the remaining elements.

9.2 If V_1 is a subset of the vector space V, then V_1 is a *subspace* of V if and only if $v_1, v_2 \in V_1$ imply that $v_1 + v_2 \in V_1$ and $v_1 \in V_1$, $f \in \mathfrak{F}$ imply that $fv_1 \in V_1$. The additive factor group $H = V/V_1$ is made a vector space over $\mathfrak{F}$ in a natural manner by letting, for each coset $\{h\} \in H$, the element $f\{h\}$ be the coset determined by $\{fh\}$ for any element h of $\{h\}$ (easily proved independent of the choice of h in the coset). Then dimension H = dimension V − dimension V_1. In particular, if $V = Z^n(S; M, L; \mathfrak{F})$ and $V_1 = B^n(S; M, L; \mathfrak{F})$, then H is the homology group $H^n(S; M, L; \mathfrak{F})$ and the dimension of H will be called the "Betti number" and be denoted by $p^n(S; M, L; \mathfrak{F})$. These may be abbreviated to $H^n(S; \mathfrak{F})$ and $p^n(S; \mathfrak{F})$ when $M = S$ and $L = 0$; and ultimately to $H^n(S)$, $p^n(S)$ when $\mathfrak{F}$ is fixed (cf. 18.17 below). Cycles that are elements of linearly independent elements of the space $H^n(S; M, L; \mathfrak{F})$ will be called *linearly independent relative to homology* (lirh). A base for $H^n(S; M, L; \mathfrak{F})$ will be called a *homology base* for S.

9.3 A *flat* of a vector space V is a coset modulo some subspace V_1. A necessary and sufficient condition that $F \subset V$ be a flat of V is that there exist a subspace V_1 and $x \in F$ such that $V_1 + x = F$. The element x, in case F is a flat, can be taken to be any element whatsoever of F, and consequently if F_1 and F_2 are flats having a common element x, then $F_1 = V_1 + x$, $F_2 = V_2 + x$, where V_1 and V_2 are subspaces, so that $F_1 \cap F_2 = V_1 \cap V_2 + x$. Since the common part of any number of subspaces is a subspace of V, it follows that the common part of any number of flats, if not empty, is again a flat. Note that, in particular, every $v \in V$ is a flat, and that every subspace is a flat.

Suppose V is a finite-dimensional vector space with given base B. Then a proper subspace V_1 of V, if not the element 0, has a base which, although not necessarily a subset of B, has fewer elements than B. Hence, if $V = V_0 \supset V_1 \supset V_2 \supset \cdots$ is a sequence of subspaces such that, for each i, V_{i+1} is a proper subspace of V_i, the number of terms in the sequence is necessarily finite. Now suppose $\{V_\nu\}$ is any collection of subspaces of V, and let $V' = \bigcap V_\nu$.

Select any $V_1 \in \{V_\nu\}$, and suppose $V_1 - V' \neq 0$. Then if $x_1 \in V_1 - V'$, there must exist $V_2 \in \{V_\nu\}$ such that $x_1 \notin V_2$. Now if $V_1 \cap V_2 - V' \neq 0$, let $x_2 \in V_1 \cap V_2 - V'$. Then there will exist $V_3 \in \{V_\nu\}$ such that $x_2 \notin V_3$. Continuing in this manner, we generate a decreasing sequence of subspaces, $V \supset V_1 \supset V_1 \cap V_2 \supset V_1 \cap V_2 \cap V_3 \supset \cdots$ which, as shown above, must be finite. We have, therefore, that

9.4 *If $\{V_\nu\}$ is a collection of subspaces of the finite-dimensional vector space V, then there exists a finite number of subspaces $V_1, \cdots, V_i, \cdots, V_k \in \{V_\nu\}$ such that $\bigcap V_\nu = \bigcap_{i=1}^{k} V_i$.*

More generally, a decreasing sequence of flats of a finite-dimensional vector space must be finite, and

9.5 *The common part of any set of flats of a finite-dimensional vector space is identical with the common part of a finite number of them.*

9.6 If K is a finite complex, then the chain group $C^n(K; \mathfrak{F})$ is a vector space over $\mathfrak{F}$ whose dimension is the number of n-dimensional cells of K, inasmuch as these form a base. The subspace $Z^n(K; \mathfrak{F})$ is therefore of finite dimension. Suppose F a flat in $V = Z^n(K; \mathfrak{F})$, and let F^* denote the set F augmented by homology; i.e., augmented by the addition of every $z \in V$ which is homologous to an element of F. Being a flat, $F = V_1 + x$, where V_1 is a subspace of V and $x \in F$. We shall show that if V_1^* denotes the set V_1 augmented by homology, then V_1^* is a subspace of V, and that $F^* = V_1^* + x$, and consequently that F^* is a flat.

(1) $F^* \subset V_1^* + x$. For $x^* \in F^*$ implies that there exists $x_1 \in F$ such that $x^* \sim x_1$. But $x_1 \in F$ implies that there exists $v_1 \in V_1$ such that $x_1 = v_1 + x$. Consequently $x^* \sim v_1 + x$, implying that $x^* - x \sim v_1$; hence that $x^* - x \in V_1^*$ and therefore $x^* \in V_1^* + x$.

(2) $F^* \supset V_1^* + x$. For $v^* \in V_1^* + x$ implies that there exists $v_1^* \in V_1^*$ such that $v^* = v_1^* + x$. But $v_1^* \in V_1^*$ implies that there exists $v_1 \in V_1$ such that $v_1^* \sim v_1$. Hence $v^* \sim v_1 + x \in F$, and hence $v^* \in F^*$.

(3) If V_1 is a subspace, then V_1^* is a subspace. (a) Let $x^*, y^* \in V_1^*$. Then $x^* \sim x \in V_1$ and $y^* \sim y \in V_1$. Hence $x^* + y^* \sim x + y \in V_1$, implying that $x^* + y^* \in V_1^*$. (b) $x^* \in V_1^*$ implies that $x^* \sim x \in V_1$, hence $fx^* \sim fx \in V_1$ $(f \in \mathfrak{F})$, implying $fx^* \in V_1^*$.

9.7 *The result of augmenting by homology a flat in the vector space $Z^n(K; \mathfrak{F})$ is again a flat.*

9.8 *Suppose $\mathfrak{U}$ and $\mathfrak{V}$ are finite coverings of a space S such that $\mathfrak{V} > \mathfrak{U}$, and let F be a flat of the vector space $Z^n(\mathfrak{V}; \mathfrak{F})$. Then $\pi_{\mathfrak{U}\mathfrak{V}} F$ is a flat in $Z^n(\mathfrak{U}; \mathfrak{F})$.*

For by definition $F = Z_1 + z$ where Z_1 is a subspace of $Z^n(\mathfrak{V}; \mathfrak{F})$ and $z \in F$. Then $x \in F$ implies that $x = z_1 + z$ where $z_1 \in Z_1$ and $\pi_{\mathfrak{U}\mathfrak{V}} x = \pi_{\mathfrak{U}\mathfrak{V}} z_1 + \pi_{\mathfrak{U}\mathfrak{V}} z$. Thus $\pi_{\mathfrak{U}\mathfrak{V}} F = \pi_{\mathfrak{U}\mathfrak{V}} Z_1 + \pi_{\mathfrak{U}\mathfrak{V}} z$. But $\pi_{\mathfrak{U}\mathfrak{V}} Z_1$ is a subspace of $Z^n(\mathfrak{U}; \mathfrak{F})$. For if

$x, y \in Z_1$, then $\pi_{\mathfrak{U}\mathfrak{B}}x + \pi_{\mathfrak{U}\mathfrak{B}}y = \pi_{\mathfrak{U}\mathfrak{B}}(x + y) \in \pi_{\mathfrak{U}\mathfrak{B}}Z_1$ and $f\pi_{\mathfrak{U}\mathfrak{B}}x = \pi_{\mathfrak{U}\mathfrak{B}}fx \in \pi_{\mathfrak{U}\mathfrak{B}}Z_1(f \in \mathfrak{F})$. Hence $\pi_{\mathfrak{U}\mathfrak{B}}F$ is a flat.

9.9 It will be noted, too, that although for two different projections π_1, π_2 of type $\pi_{\mathfrak{U}\mathfrak{B}}$, the sets π_1F, π_2F may be different, on the other hand, $[\pi_1F]^* = [\pi_2F]^*$. For $x_1^* \in [\pi_1F]^*$ implies that $x_1^* \sim x_1 = \pi_1z_1$ where $z_1 \in F$. But as previously shown (Theorem 7.2), $\pi_1z_1 \sim \pi_2z_1$, hence $x_1^* \sim \pi_2z_1 \in \pi_2F$ and consequently $x_1^* \in [\pi_2F]^*$.

9.10 Definition. If $\mathfrak{B} > \mathfrak{U}$ and $z^n(\mathfrak{U})$, $z^n(\mathfrak{B})$ are cycles mod L on M such that $\pi_{\mathfrak{U}\mathfrak{B}}z^n(\mathfrak{B}) \sim z^n(\mathfrak{U})$ mod L on M, then $z^n(\mathfrak{B})$ is called a *successor* of $z^n(\mathfrak{U})$.

9.11 If $\mathfrak{B} > \mathfrak{U}$, and $z^n(\mathfrak{U})$ is a cycle mod L on M of $\mathfrak{U}$, then the set of all successors of $z^n(\mathfrak{U})$ in $Z^n(\mathfrak{B}; \mathfrak{F})$ is a flat, which, incidentally, is also the set of all successors in $Z^n(\mathfrak{B}; \mathfrak{F})$ of $[z^n(\mathfrak{U})]^*$. The proof of this will be left to the reader.

Remark. Although the last few paragraphs generally use the terminology of absolute cycles, the results stated hold equally well for cycles mod L on M, homology mod L on M, etc.

10. Existence theorems.

In the theorems of this section, Σ will denote the set of all fcos of some space S; and if $\mathfrak{B} > \mathfrak{U}$ and $F(\mathfrak{B})$ is a flat in $Z^n(M, L; \mathfrak{F}, \mathfrak{B})$, then by $\pi_{\mathfrak{U}\mathfrak{B}}F(\mathfrak{B})$ will be understood the set $\{z \mid z = \pi_{\mathfrak{U}\mathfrak{B}}z_1, z_1 \in F(\mathfrak{B})\}^*$.

10.1 Theorem. *Let $\{F(\mathfrak{U})\}$, $\mathfrak{U} \in \Sigma$, be a collection of nonempty flats such that for every $\mathfrak{U} \in \Sigma$, $F(\mathfrak{U}) \subset Z^n(M, L; \mathfrak{F}, \mathfrak{U})$ and $F(\mathfrak{U}) = [F(\mathfrak{U})]^*$; and such that if $\mathfrak{B} > \mathfrak{U}$, then $\pi_{\mathfrak{U}\mathfrak{B}}F(\mathfrak{B}) \subset F(\mathfrak{U})$. Then for each $\mathfrak{U}$, there exists a cycle $z^n(\mathfrak{U}) \in F(\mathfrak{U})$ which has a successor in each $F(\mathfrak{B})$ for which $\mathfrak{B} > \mathfrak{U}$.*

Proof. With $\mathfrak{U}$ fixed, for each $\mathfrak{B} > \mathfrak{U}$, let $G(\mathfrak{B}) = \pi_{\mathfrak{U}\mathfrak{B}}F(\mathfrak{B})$. Then by 9.7 and 9.8, $G(\mathfrak{B})$ is a flat in $F(\mathfrak{U})$. Evidently we wish to show that $\bigcap_{\mathfrak{B}>\mathfrak{U}} G(\mathfrak{B})$ is not empty.

As $Z^n(\mathfrak{U})$ is a finite-dimensional vector space, there exist by 9.5 a finite number of refinements of $\mathfrak{U}$, say $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_k$, such that $\bigcap_{\mathfrak{B}>\mathfrak{U}}G(\mathfrak{B}) = \bigcap_{i=1}^k G(\mathfrak{B}_i)$. Let $\mathfrak{W} > (\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_k)$. For each i, $G(\mathfrak{W}) = \pi_{\mathfrak{U}\mathfrak{W}}F(\mathfrak{W}) = \pi_{\mathfrak{U}\mathfrak{B}_i}\pi_{\mathfrak{B}_i\mathfrak{W}}F(\mathfrak{W}) \subset \pi_{\mathfrak{U}\mathfrak{B}_i}F(\mathfrak{B}_i) = G(\mathfrak{B}_i)$ so that $G(\mathfrak{W}) \subset \bigcap_{i=1}^k G(\mathfrak{B}_i)$. It is trivial that $G(\mathfrak{W}) \supset \bigcap_{\mathfrak{B}>\mathfrak{U}} G(\mathfrak{B}) = \bigcap_{i=1}^k G(\mathfrak{B}_i)$. Hence $G(\mathfrak{W}) = \bigcap_{\mathfrak{B}>\mathfrak{U}} G(\mathfrak{B})$, and as $G(\mathfrak{W})$ is not empty, the theorem is proved.

10.2 Theorem. *Under the hypothesis of Theorem* 10.1, *there exists a C-cycle* $\{z^n(\mathfrak{U})\}$ mod L on M *such that for each* $\mathfrak{U} \in \Sigma$, $z^n(\mathfrak{U}) \in F(\mathfrak{U})$.

Proof. Let the elements of Σ be represented by a well-ordering

$$\mathfrak{U}_1, \mathfrak{U}_2, \cdots, \mathfrak{U}_\alpha, \cdots, \tag{10.2a}$$

of order-type γ. We shall define by transfinite induction, for each $\alpha < \gamma$ and $\beta < \gamma$, a flat $F_\alpha(\mathfrak{U}_\beta)$, such that for fixed β, $F_\alpha(\mathfrak{U}_\beta)$ is a subset of $F(\mathfrak{U}_\beta)$ as well as of all $F_{\alpha'}(\mathfrak{U}_\beta)$ for which $\alpha' < \alpha$, and thereby form another family of flats

$F_\alpha(\mathfrak{U})$ like the family of flats $F(\mathfrak{U})$ of Theorem 10.1. Then the required $z^n(\mathfrak{U}_\alpha)$ will be selected from $F_\alpha(\mathfrak{U}_\alpha)$.

Case $\alpha = 1$. For each $\beta < \gamma$, let $F_1(\mathfrak{U}_\beta) = F(\mathfrak{U}_\beta)$. By Theorem 10.1 there is a cycle $z^n(\mathfrak{U}_1) \in F_1(\mathfrak{U}_1)$ which has a successor in every $F_1(\mathfrak{U}_\beta)$ such that $\mathfrak{U}_\beta > \mathfrak{U}_1$.

Case $1 < \alpha = \tau < \gamma$. If τ is a limiting ordinal, then for each $\beta < \gamma$, $F_\tau(\mathfrak{U}_\beta) = \bigcap_{\alpha<\tau} F_\alpha(\mathfrak{U}_\beta)$. Otherwise, we ask whether $\mathfrak{U}_\beta > \mathfrak{U}_{\tau-1}$; and if not, $F_\tau(\mathfrak{U}_\beta) = F_{\tau-1}(\mathfrak{U}_\beta)$. If, however, $\mathfrak{U}_\beta > \mathfrak{U}_{\tau-1}$, we let $F_\tau(\mathfrak{U}_\beta)$ be the set of all successors of $z^n(\mathfrak{U}_{\tau-1})$ in $F_{\tau-1}(\mathfrak{U}_\beta)$. The flats $F_\tau(\mathfrak{U}_\beta)$ form a system like that of the $F(\mathfrak{U})$ of Theorem 10.1, and $z^n(\mathfrak{U}_\tau) \in F_\tau(\mathfrak{U}_\tau)$ is selected so as to have a successor in every $F_\tau(\mathfrak{U}_\beta)$ such that $\mathfrak{U}_\beta > \mathfrak{U}_\tau$. The justification of these statements follows:

In the case where τ is a limiting ordinal, the set of α's $< \tau$ for which $\mathfrak{U}_\beta > \mathfrak{U}_\alpha$, if any, form a well-ordered subsequence of the subscripts of (10.2a)

$$\text{(10.2b)} \qquad \alpha(1), \alpha(2), \cdots; \alpha(\nu), \cdots, \qquad \alpha(\nu) < \tau.$$

Note that $F_{\alpha(1)}(\mathfrak{U}_\beta) = F(\mathfrak{U}_\beta)$, but that $F_{\alpha(1)+1}(\mathfrak{U}_\beta)$ may be a proper subset of $F_{\alpha(1)}(\mathfrak{U}_\beta)$, being by definition the set of all elements of $F_{\alpha(1)}(\mathfrak{U}_\beta)$ that are successors of $z^n(\mathfrak{U}_{\alpha(1)})$. A like observation holds in regard to each $F_{\alpha(\nu)+1}(\mathfrak{U}_\beta)$: it may or may not be a proper subset of $F_{\alpha(\nu)}(\mathfrak{U}_\beta)$. In any case, however, since each $F_\alpha(\mathfrak{U}_\beta)$ is a flat there can be at most a finite number of $\alpha(\nu)$'s for which $F_{\alpha(\nu)+1}(\mathfrak{U}_\beta)$ is actually a proper subset of $F_{\alpha(\nu)}(\mathfrak{U}_\beta)$, and consequently (10.2b) contains a term $\alpha(\nu')$ such that for all $\alpha(\nu) > \alpha(\nu')$, $F_{\alpha(\nu)}(\mathfrak{U}_\beta) = F_{\alpha(\nu')+1}(\mathfrak{U}_\beta)$. It follows that $F_\tau(\mathfrak{U}_\beta) = \bigcap_{\alpha<\tau} F_\alpha(\mathfrak{U}_\beta) = F_{\alpha(\nu')+1}(\mathfrak{U}_\beta)$. Naturally, if for no $\alpha < \tau$ is $\mathfrak{U}_\beta > \mathfrak{U}_\alpha$, then $F_\tau(\mathfrak{U}_\beta) = F(\mathfrak{U}_\beta)$.

In case τ has an immediate predecessor $\tau - 1$, the set $F_\tau(\mathfrak{U}_\beta)$ is a flat because it is either the flat $F_{\tau-1}(\mathfrak{U}_\beta)$ in case $\mathfrak{U}_\beta \not> \mathfrak{U}_{\tau-1}$, or the intersection of two flats, namely the flat $F_{\tau-1}(\mathfrak{U}_\beta)$ and the flat consisting of all successors of $z^n(\mathfrak{U}_{\tau-1})$ in $Z^n(M, L; \mathfrak{F}, \mathfrak{U}_\beta)$. No such $F_\tau(\mathfrak{U}_\beta)$ is empty since $z^n(\mathfrak{U}_{\tau-1}) \in F_{\tau-1}(\mathfrak{U}_{\tau-1})$ was chosen so as to have a successor in every $F_{\tau-1}(\mathfrak{U}_\beta)$ for which $\mathfrak{U}_\beta > \mathfrak{U}_{\tau-1}$.

In order to justify the statement that the flats $F_\tau(\mathfrak{U}_\beta)$ form a system like that of the flats $F(\mathfrak{U})$ of Theorem 10.1, we have to show that if $\mathfrak{U}_{\beta(2)} > \mathfrak{U}_{\beta(1)}$, then $\pi_{\beta(1),\beta(2)}F_\tau(\mathfrak{U}_{\beta(2)}) \subset F_\tau(\mathfrak{U}_{\beta(1)})$. [For typographical reasons, we shall use the symbol $\pi_{\beta(1)\beta(2)}$ to indicate a projection of $\mathfrak{U}_{\beta(2)}$ into $\mathfrak{U}_{\beta(1)}$.] By the induction assumption $\pi_{\beta(1)\beta(2)}F_{\tau-1}(\mathfrak{U}_{\beta(2)}) \subset F_{\tau-1}(\mathfrak{U}_{\beta(1)})$. If τ is not a limiting ordinal, and $\mathfrak{U}_{\beta(1)} \not> \mathfrak{U}_{\tau-1}$, then $F_\tau(\mathfrak{U}_{\beta(1)}) = F_{\tau-1}(\mathfrak{U}_{\beta(1)})$ and the required inclusion holds no matter whether $\mathfrak{U}_{\beta(2)} > \mathfrak{U}_{\tau-1}$ or not. If $\mathfrak{U}_{\beta(1)} > \mathfrak{U}_{\tau-1}$, then a fortiori $\mathfrak{U}_{\beta(2)} > \mathfrak{U}_{\tau-1}$ and if $z \in F_{\tau-1}(\mathfrak{U}_{\beta(2)})$, $\pi_{(\tau-1)\beta(2)}z \sim \pi_{(\tau-1)\beta(1)}\pi_{\beta(1)\beta(2)}z \bmod L$ on M so that if z is a successor of $z^n(\mathfrak{U}_{\tau-1})$, then $\pi_{\beta(1)\beta(2)}z$ is a successor of $z^n(\mathfrak{U}_{\tau-1})$ in $F_{\tau-1}(\mathfrak{U}_{\beta(1)})$. When τ is a limiting ordinal the required inclusion follows from its validity for the ordinals α such that $\alpha(\nu') < \alpha < \tau$ ($\alpha(\nu')$ as above).

To see that $\{z^n(\mathfrak{U}_\alpha)\}$ is a C-cycle mod L on M, notice that as soon as $z^n(\mathfrak{U}_\alpha)$ is defined, $F_{\alpha+1}(\mathfrak{U}_\alpha)$ becomes $[z^n(\mathfrak{U}_\alpha)]^*$, since $\mathfrak{U}_\alpha > \mathfrak{U}_\alpha$. And all $F_{\alpha'}(\mathfrak{U}_\alpha)$ for which $\alpha' > \alpha$ will be identical with $F_{\alpha+1}(\mathfrak{U}_\alpha)$ since the latter is a single homology class. Thus as soon as $\alpha' > (\alpha, \beta)$, $F_{\alpha'}(\mathfrak{U}_\alpha)$ and $F_{\alpha'}(\mathfrak{U}_\beta)$ are the

homology classes $[z^n(\mathfrak{U}_\alpha)]^*$ and $[z^n(\mathfrak{U}_\beta)]^*$, respectively, and the inclusion $\pi_\beta^\alpha F_{\alpha'}(\mathfrak{U}_\beta) \subset F_{\alpha'}(\mathfrak{U}_\alpha)$ implies that $\pi_{\alpha\beta} z^n(\mathfrak{U}_\beta) \sim z^n(\mathfrak{U}_\alpha)$ mod L on M.

10.3 Definition. A cycle $z^n(\mathfrak{U})$ mod L on M will be called an *essential* cycle mod L on M if it has a successor in $Z^n(M, L; \mathfrak{F}, \mathfrak{V})$ for every refinement $\mathfrak{V}$ of $\mathfrak{U}$.

10.4 Definition. If $\mathfrak{U}, \mathfrak{V} \in \Sigma$ and $\mathfrak{V} > \mathfrak{U}$, then $\mathfrak{V}$ will be called an *n-dimensional normal refinement* of $\mathfrak{U}$ (rel. M, L) if for every cycle $z^n(\mathfrak{V})$ mod L on M, the cycle $\pi_{\mathfrak{U}\mathfrak{V}} z^n(\mathfrak{V})$ is an essential cycle mod L on M. If $\mathfrak{V}$ is an n-dimensional normal refinement mod L on M for every $n \geqq 0$, then $\mathfrak{V}$ is called simply a *normal refinement* of $\mathfrak{U}$ (rel. M, L).

10.5 *For given* $\mathfrak{U}$, M, L, *the cycles* $z^n(\mathfrak{U})$ mod L *on* M *that are essential form a vector space over* $\mathfrak{F}$.

10.6 *If* $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \Sigma$ *and* $\mathfrak{W} > \mathfrak{V} > \mathfrak{U}$, $\mathfrak{V}$ *being an n-dimensional normal refinement of* $\mathfrak{U}$ *(rel.* M, L*), then* $\mathfrak{W}$ *is an n-dimensional normal refinement of* $\mathfrak{U}$ *(rel.* M, L*).*

10.7 Theorem. *Every* $\mathfrak{U} \in \Sigma$ *has a normal refinement (rel.* M, L*).*

Proof. As the complex $\mathfrak{U}$ is of finite dimension, say n, the normality property is trivial for cycles of dimension $>n$, since all such are ~ 0 and the cycle 0 is essential in every dimension. Hence we may prove the validity of the normality property for a particular dimension i $(0 \leqq i \leqq n)$, and the theorem will then follow from 10.6.

For each $\mathfrak{V} > \mathfrak{U}$ let $G(\mathfrak{V})$ denote the set of all cycles $z^i(\mathfrak{U})$ mod L on M which have successors in $Z^i(M, L; \mathfrak{F}, \mathfrak{V})$. Note that if $\mathfrak{W} > \mathfrak{V} > \mathfrak{U}$, then $G(\mathfrak{W}) \subset G(\mathfrak{V})$. Evidently the set of all essential cycles mod L on M of $\mathfrak{U}$ is identical with $\bigcap_{\mathfrak{V}>\mathfrak{U}} G(\mathfrak{V})$. By 9.4 there exist $\mathfrak{V}_1, \mathfrak{V}_2, \cdots, \mathfrak{V}_k > \mathfrak{U}$ such that $\bigcap_{j=1}^k G(\mathfrak{V}_j) = \bigcap_{\mathfrak{V}>\mathfrak{U}} G(\mathfrak{V})$. Let $\mathfrak{W} > (\mathfrak{V}_1, \mathfrak{V}_2, \cdots, \mathfrak{V}_k)$. Then $G(\mathfrak{W}) \subset G(\mathfrak{V}_j)$ for each j, and hence $G(\mathfrak{W}) \subset \bigcap_{j=1}^k G(\mathfrak{V})_j$. It is trivial that $\bigcap_{\mathfrak{V}>\mathfrak{U}} G(\mathfrak{V}) \subset G(\mathfrak{W})$. Hence $G(\mathfrak{W}) = \bigcap_{\mathfrak{V}>\mathfrak{U}} G(\mathfrak{V})$. But if $z^i(\mathfrak{W}) \in Z^i(\mathfrak{W}; \mathfrak{F})$, then $\pi_{\mathfrak{U}\mathfrak{W}} z^i(\mathfrak{W}) \in G(\mathfrak{W})$. Hence $\mathfrak{W}$ is an n-dimensional normal refinement of $\mathfrak{U}$ mod L on M.

10.8 Theorem. *If* $\mathfrak{V}$ *is a normal refinement of* $\mathfrak{U}$ *(rel.* M, L*), and* $z \in Z^n(M, L; \mathfrak{F}, \mathfrak{V})$, *then* $\pi_{\mathfrak{U}\mathfrak{V}} z$ *is a coordinate of a C-cycle* mod L *on* M.

Proof. Because of the isomorphism established in Theorem 7.11, it will only be necessary to prove the theorem for a complete family of coverings. And inasmuch as the family of all refinements of $\mathfrak{U}$ forms a complete family, we need only define the coordinates of the C-cycle required for such refinements.

Let $\mathfrak{W} > \mathfrak{U}$, and let $F(\mathfrak{W})$ be the set of all successors of $\pi_{\mathfrak{U}\mathfrak{V}} z$ in $Z^n(M, L; \mathfrak{F}, \mathfrak{W})$. The set $F(\mathfrak{W})$ is not empty since $\mathfrak{V}$ is a normal refinement of $\mathfrak{U}$ and $\pi_{\mathfrak{U}\mathfrak{V}} z$ is therefore an essential cycle. Of course $F(\mathfrak{U}) = [\pi_{\mathfrak{U}\mathfrak{V}} z]^*$.

Relative to the set Σ' of all refinements of $\mathfrak{U}$, the family of sets $F(\mathfrak{W})$, $\mathfrak{W} \in \Sigma'$, satisfies the hypothesis of Theorem 10.1. For if $\mathfrak{W}' > \mathfrak{W} > \mathfrak{U}$ and $x \in$

$F(\mathfrak{W}')$, then $\pi_{\mathfrak{U}\mathfrak{W}'}x \sim \pi_{\mathfrak{U}\mathfrak{W}}\pi_{\mathfrak{W}\mathfrak{W}'}x \sim \pi_{\mathfrak{U}\mathfrak{B}}z \bmod L$ on M by Theorem 7.2, hence $\pi_{\mathfrak{W}\mathfrak{W}'}x \in F(\mathfrak{W})$. Thus $\pi_{\mathfrak{W}\mathfrak{W}'}F(\mathfrak{W}') \subset F(\mathfrak{W})$, and the theorem follows from Theorem 10.2.

11. Some applications to connectedness and local connectedness. The following theorems will be useful later on, and form interesting applications of the preceding theory.

11.1 THEOREM. *In any space S, if Z^0 is an augmented Čech cycle on a quasi-component of S, then $Z^0 \sim 0$ on S.*

PROOF. If $\mathfrak{U}$ is any covering of S, let $z^0(\mathfrak{U}) = \sum_{i=1}^{k} a^i\sigma_i^0$, $a^i \neq 0$. Inasmuch as $\partial z^0(\mathfrak{U}) = \sum a^i\sigma^{-1} = 0$, it follows that $\sum_{i=1}^{k} a^i = 0$.

Now the σ's correspond to elements of $\mathfrak{U}$, and by use of the simple chain theorem (I 12.3) there can be shown to exist a sequence $\mathfrak{C}^i$ of elements of $\mathfrak{U}$ (not necessarily forming a simple chain!) beginning with the element corresponding to σ_1^0 and ending with the element corresponding to σ_j^0, $j = 2, 3, \cdots, k$; and associated with $\mathfrak{C}^i$ a chain c^i such that $\partial c^i = \sigma_j^0 - \sigma_1^0$. From this we get that $\partial a^i c^i = a^i\sigma_j^0 - a^i\sigma_1^0$ and hence

$$\partial \sum_{j=2}^{k} a^i c^i = \sum_{j=2}^{k} a^i\sigma_j^0 - \sum_{j=2}^{k} a^i\sigma_1^0 . \tag{11.1a}$$

But from $\sum_{i=1}^{k} a^i = 0$ follows that $a^1 = -\sum_{i=2}^{k} a^i$, hence (11.1a) becomes

$$\partial \sum_{j=2}^{k} a^i c^i = \sum_{j=1}^{k} a^i\sigma_j^0 = z^0(\mathfrak{U}).$$

Hence z^0 is a bounding C-cycle.

Actually, a slightly stronger theorem may be stated: Suppose M is an arbitrary subset of a space S. Then if $x, y \in M$ are in the same quasi-component of M, and $\mathfrak{U}$ is a covering of S by open sets, then the set $\{U \cap M | U \in \mathfrak{U}\}$ contains a simple chain from x to y. Hence the argument given above may be modified so as to prove the following theorem, which is of use in the sequel.

11.1a THEOREM. *If M is an arbitrary subset of a space S and Z^0 is an augmented Čech cycle of S on some quasi-component of M, then $Z^0 \sim 0$ on M.*

11.2 THEOREM. *In order that a space S should be connected, it is necessary and sufficient that $p_a^0(S; \mathfrak{F}) = 0$.*

The necessity follows from Theorem 11.1.

Conversely, if $p_a^0(S; \mathfrak{F}) = 0$, S is connected. For if $S = A \cup B$ separated, then the open sets A and B constitute a covering $\mathfrak{U}$ of S. Let $\mathfrak{B}$ be a normal refinement of $\mathfrak{U}$, and let $z^0(\mathfrak{B}) = \sigma_1^0 - \sigma_2^0$ where the elements of $\mathfrak{B}$ corresponding to σ_1^0, σ_2^0 are open subsets of A, B respectively. Then $\pi_{\mathfrak{U}\mathfrak{B}}z^0(\mathfrak{B}) = \sigma_A^0 - \sigma_B^0$ (σ_A^0 corresponding to A, σ_B^0 corresponding to B) is the coordinate on $\mathfrak{U}$ of a C-cycle (Theorem 10.8) which does not bound since its coordinate on $\mathfrak{U}$ does not bound. Hence $p_a^0(S; \mathfrak{F}) \neq 0$.

Since a space which has exactly m components is the union of m separated sets (Corollary I 9.5), or in case the number of components is infinite can be expressed as the union of an arbitrarily large number of separated sets (Theorem I 9.7a), the methods used above can be applied to show the following theorem:

11.3a THEOREM. *The number of components of a space S is exactly* $p_a^0(S; \mathfrak{F}) + 1 = p^0(S; \mathfrak{F})$.

And since by Theorem I 9.3 if the number of components or the number of quasi-components of a space is finite, then components and quasi-components are identical, we have

11.3b THEOREM. *The number of quasi-components of a space is exactly* $p_a^0(S; \mathfrak{F}) + 1 = p^0(S; \mathfrak{F})$.

Of importance later on is the notion of a nontrivial cycle carried by a pair of points. Since there should be no danger of confusion in so doing, we shall frequently use "U's" instead of "σ's" in dealing with 0-chains of a covering $\mathfrak{U}$.

11.4 DEFINITION. Let $x, y \in S$, $x \neq y$. A *nontrivial cycle carried by* $x \cup y$ (or *on* $x \cup y$) is a C-cycle γ^0 such that if $\mathfrak{U}$ is any covering of S, then $\gamma^0(\mathfrak{U}) = U_0 - U_1$, where $U_0, U_1 \in \mathfrak{U}$ and $x \in U_0$, $y \in U_1$, and if $\mathfrak{V} > \mathfrak{U}$, then $\pi_{\mathfrak{U}\mathfrak{V}}\gamma^0(\mathfrak{V}) \sim \gamma^0(\mathfrak{U})$ on $x \cup y$.

11.5 THEOREM. *If x and y are distinct points of a space S, then there exists a nontrivial cycle carried by $x \cup y$.*

PROOF. For each $\mathfrak{U} \in \Sigma$ choose $U_x, U_y \in \mathfrak{U}$ such that $x \in U_x$, $y \in U_y$. Then $\{U_x - U_y\}$ is a C-cycle. For if $\mathfrak{V} > \mathfrak{U}$ and V_x, V_y the elements chosen from $\mathfrak{V}$, then, since $x \in U_x \cap \pi_{\mathfrak{U}\mathfrak{V}}V_x$, the sets U_x and $\pi_{\mathfrak{U}\mathfrak{V}}V_x$ are identical or vertices of a 1-simplex of $\mathfrak{U}$. Consequently $U_x - U_y \sim \pi_{\mathfrak{U}\mathfrak{V}}V_x - \pi_{\mathfrak{U}\mathfrak{V}}V_y$ on $\mathfrak{U}$.

11.6 THEOREM. *In order that two points x, y of a space S should lie in the same quasi-component of S, it is necessary and sufficient that every nontrivial cycle carried by $x \cup y$ bound on S.*

The necessity follows from Theorem 11.1.

PROOF OF SUFFICIENCY. If $S = A \cup B$ separated, where $x \in A$, $y \in B$' then on the covering $\mathfrak{U}$ of S whose elements are A and B, the coordinate of a nontrivial cycle carried by $x \cup y$ must be of the form $A - B$ and does not bound on $\mathfrak{U}$.

And by virtue of Theorem 11.1a we have:

11.7 THEOREM. *If two points x and y lie in the same quasi-component of a subset M of a space S then every nontrivial cycle (of S) carried by $x \cup y$ bounds on M.*

Of great importance in the sequel are the applications to local connectedness. For the 0-dimensional case of local connectedness we establish now the relationship to the lc and lcq of Chapter II 3.

11.8 THEOREM. *In order that a regular space S should be lcq at $x \in S$, it is necessary and sufficient that if P is an open set containing x, then there exists an open set Q such that $x \in Q \subset P$ and such that every augmented Čech 0-cycle on Q bounds on P.*

PROOF. The necessity follows immediately from Theorem 11.1a.

To prove the sufficiency, suppose that a space S satisfies the condition stated in the theorem, but is not lcq at x. Then there exists an open set P containing x such that every open subset of P that contains x meets at least two quasi-components of P.

Let P', Q, Q', R be open sets such that (1) $x \in R \subset Q \Subset Q' \Subset P' \Subset P$; (2) every (augmented) Čech 0-cycle on R bounds on Q. There exist x_1, $x_2 \in R$ and a decomposition $P = P_1 \cup P_2$ separate where $x_1 \in P_1$, $x_2 \in P_2$. Let $\mathfrak{U}$ be the covering of S that consists of the open sets $S - \bar{P}'$, $(P - \bar{Q}) \cap P_1$, $(P - \bar{Q}) \cap P_2$, $Q' \cap P_1$, $Q' \cap P_2$. By definition, the coordinate on $\mathfrak{U}$ of a nontrivial cycle on $x_1 \cup x_2$ must have the form $U_1 - U_2$, where $U_1 = Q' \cap P_1$, $U_2 = Q' \cap P_2$. But $U_1 - U_2 \not\sim 0$ on Q.

REMARK. So far as the sufficiency part of Theorem 11.8 is concerned, the following statement is stronger:

11.8a THEOREM. *Theorem* 11.8 *continues to hold if the words "augmented Čech 0-cycle on Q" are replaced by "nontrivial cycle on a pair of points of Q."*

Since by Theorem II 1.8, a space which is not lc must fail to be lcq at some point, we may state:

11.9 THEOREM. *In order that a regular space S should be lc, it is necessary and sufficient that if $x \in S$ and P an open set containing x, then there exists an open set Q such that $x \in Q \subset P$ and such that every augmented Čech 0-cycle on Q bounds on P.*

[Again "augmented Čech 0-cycle on Q" may be replaced by "nontrivial cycle on a pair of points of Q."]

REMARK. While Theorem 11.8 could be stated *at a point,* Theorem 11.9 had to be stated as *at all points* of the space. The reader may wish to recall Example II 1.4 here. It will also be noted that the regularity condition used in both theorems is needed only for the sufficiency part of the proof.

Of importance in the sequel will be the restriction of homologies to be on certain special subsets of the space; for example, compact subsets. The following theorem, analogous to the "necessary" portion of Theorem 11.1, will be found useful:

11.10 THEOREM. *If K and M are point sets in a locally compact (or normal) space S, $K \subset M$, and only a finite number, m, of nontrivial 0-cycles on K are lirh* (9.2) *on compact subsets of M, then K lies in at most $m + 1$ components of M.*

Proof. Suppose there are points of K in $m + 2$ components M_i of M, $i = 0, 1, \cdots, m + 1$. Let $p_i \in K \cap M_i$, and let γ_i^0 be a nontrivial cycle on $p_0 \cup p_i$, $i = 1, \cdots, m + 1$. By hypothesis, there exists a relation

$$\sum_{i=1}^{m+1} a^i \gamma_i^0 \sim 0, \qquad \text{on } C, \tag{11.10a}$$

where C is some compact subset of M and not all $a^i = 0$.

Consider a covering $\mathfrak{U}$ of S, no element of which contains two of the points p_i ($\mathfrak{U}$ might consist, for example, of the set $S - \bigcup p_i$ and of $m + 2$ disjoint open sets containing the respective points p_i). Then

$$\sum_{i=1}^{m+1} a^i \gamma_i^0(\mathfrak{U}) = \sum_{j=0}^{m+1} b^j U_j , \qquad U_j \in \mathfrak{U}, \tag{11.10b}$$

where $b^0 = \sum_{i=1}^{m+1} a^i$ and $b^j = -a^j$ if $j > 0$. Since $\sum_{j=0}^{m+1} b^j = 0$, at least two of the coefficients b^j are $\neq 0$. It follows that at least two of the points p_i are in C. And since $C \subset M$ and no two of the points p_i lie in the same component of M, the same holds for C. Hence, by Corollary IV 1.4a, $C = \bigcup_{r=1}^{k} C_r$ where (1) k is the number of nonzero coefficients b^j in (11.10b) as well as the number of points p_i in C; (2) each C_r contains one and only one point p_i; and (3) the sets C_r are pairwise separated.

Since by 8.11 we may confine chains and homologies to coverings of C, and the sets C_r form a covering $\mathfrak{U}$ of the type discussed in the preceding paragraph, relation (11.10b) holds. But by (11.10a) there exists a chain $L^1(\mathfrak{U})$ such that $\partial L^1(\mathfrak{U}) = \sum_{j=0}^{m+1} b^j U_j$, where each U_j is a C_r. But the sets C_r are disjoint, and for fixed $j > 0$ this would imply a relation $\partial(-a^j U_j) = b^j U_j$, since the portion of $L^1(\mathfrak{U})$ on C_r is of the form $-a^j U_j$. This is of course impossible.

11.11 Corollary. *If p, q are points of a point set M and γ^0 is a nontrivial cycle on $p \cup q$ which bounds on a compact set C of M, then p and q lie in a continuum of M.*

Proof. We apply Theorem 11.10 by taking $p \cup q$ and C to be the K and M, respectively, of that theorem.

Corollary 11.11 has an interesting application to the theory of local connectedness in locally compact spaces. Analogous to Definition II 1.2 we have:

11.12 Definition. A space S is called *strongly locally connected*—abbreviated lcs—at $x \in S$ if every neighborhood U of x contains a neighborhood V of x which lies in some subcontinuum of $\overline{U}$.

Evidently in locally compact spaces the lcw and lcs properties are equivalent. Examples may be constructed to show that the lcs property is weaker than the lc property, at a point, even in compact spaces.

11.13 Theorem. *In order that a locally compact space S should be lcs at $x \in S$, it is necessary and sufficient that if P is any open set containing x, then there exists an open set Q such that $x \in Q \subset P$ and every augmented Čech cycle on Q bounds on a compact subset of P.*

PROOF OF NECESSITY. Let P' be an open set such that $x \in P' \Subset P$ and $\bar{P}'$ is compact. Then since S is a fortiori lcq at x there exists by Theorem 11.8 an open set Q such that $x \in Q \subset P'$ and every augmented Čech cycle on Q bounds on $\bar{P}'$.

PROOF OF SUFFICIENCY. With x, P and Q as in the statement of the theorem, let p be any point of $Q - x$. Then a nontrivial cycle on $p \cup x$ bounds on a compact subset of P and consequently by Corollary 11.11, $p \cup x$ lies in a subcontinuum of P. Thus Q lies in one component of P, S is lcw at x and hence, since S is locally compact, S is lcs at x.

[As in the case of Theorems 11.8 and 11.9, "augmented Čech 0-cycle on Q" may be replaced by "nontrivial cycle on a pair of points of Q."]

12. **Fundamental systems of cycles for a compact metric space.** In 9.2, homology base for $H^n(S; M, L; \mathfrak{F})$ was defined. Evidently if from each element of such a homology base there is selected a C-cycle Z_ρ^n, then the collection $\{Z_\rho^n\}$ forms a *homology base of n-cycles of S on M* mod L, with the property that if Z^n is a C-cycle on M mod L, then there exists a homology $Z^n \sim \sum_\rho a^\rho Z_\rho^n$ on M mod L, where the a^ρ's are elements of $\mathfrak{F}$, only a finite number being $\neq 0$.

In general, the number of elements in a homology base is not only infinite, but uncountable:

12.1 EXAMPLE. In the cartesian plane, for each natural number k let $M_k = \{(x, y) \mid (x - 3/2^{k+2})^2 + y^2 = 1/2^{2k+4}\}$ and $M = (0, 0) \cup \bigcup M_k$. Then M is a Peano continuum such that $p^1(M; \mathfrak{F}) = \infty$. A homology base of 1-cycles of M would contain an uncountable number of 1-cycles. This is due to the restriction of homologies to finite sets of elements.

We shall see later on, however, that for certain types of compact spaces a countable set of cycles, to be called a fundamental system, will serve as a homology base, through the extension of homologies to infinite sequences of cycles. This must be reserved for later consideration however. For the present, if the space is compact metric, we may show the existence of a countable set of cycles having a special property.

12.2 LEMMA. *If M is a compact metric space, then there exists a countable set of finite coverings $\mathfrak{U}_1, \cdots, \mathfrak{U}_k, \cdots$, of M such that for each k, every $U \in \mathfrak{U}_k$ is of diameter $<1/k$, and $\mathfrak{U}_{k+1}$ is a normal refinement of $\mathfrak{U}_k$.*

PROOF. Let $\mathfrak{U}_1$ be a covering of M by a finite number of open sets of diameter <1, and having defined $\mathfrak{U}_k$ for any integer $k \geqq 1$, define $\mathfrak{U}_{k+1}$ as follows: By Theorem 10.7 there exists a covering $\mathfrak{V}_k$ which is a normal refinement of $\mathfrak{U}_k$. Let $\mathfrak{W}_k$ be a finite covering of M by open sets of diameter $<1/(k + 1)$. Let $\mathfrak{U}_{k+1} = \mathfrak{V}_k \cap \mathfrak{W}_k$.

12.3 DEFINITION. A set of refinements of a compact metric space M such as that whose existence is proved in Lemma 12.2 will be called a *canonical sequence of refinements of M.*

12.4 Theorem. *If M is a compact metric space, then for $r \geqq 0$ there exists a countable set of C-cycles, $Z_1^r, \cdots, Z_{n(1)}^r, \cdots, Z_{n(2)}^r, \cdots, Z_{n(k)}^r, \cdots$, and a canonical sequence of refinements $\mathfrak{U}_1, \cdots, \mathfrak{U}_k, \cdots$, of M such that (1) for $n(k) < i$, $Z_i^r(\mathfrak{U}_h) \sim 0$ on $\mathfrak{U}_h$ for all $h \leqq k$, (2) if Z^r is any cycle of M, then $Z^r(\mathfrak{U}_h) \sim \sum_{i=1}^{n(h)} a^i Z_i^r(\mathfrak{U}_h)$ on $\mathfrak{U}_h$, $a^i \in \mathfrak{F}$.*

Proof. Let $\mathfrak{U}_1, \cdots, \mathfrak{U}_k, \cdots$ be a canonical sequence of refinements of M. For each k, those r-cycles of $\mathfrak{U}_k$ that are coordinates of C-cycles of M form a vector space $Z_C^r(\mathfrak{U}_k)$. Evidently if $x \in Z_C^r(\mathfrak{U}_{k+1})$, $\pi_{k(k+1)}x \in Z_C^r(\mathfrak{U}_k)$. Let $Z_1^r(\mathfrak{U}_1), \cdots, Z_{n(1)}^r(\mathfrak{U}_1)$ be elements of $Z_C^r(\mathfrak{U}_1)$ such that if $Z^r(\mathfrak{U}_1) \in Z_C^r(\mathfrak{U}_1)$, then $Z^r(\mathfrak{U}_1) \sim \sum_{i=1}^{n(1)} a^i Z_i^r(\mathfrak{U}_1)$ on $\mathfrak{U}_1$, $a^i \in \mathfrak{F}$. And for each i, $1 \leqq i \leqq n(1)$, let Z_i^r be a C-cycle of M whose coordinate on $\mathfrak{U}_1$ is $Z_i^r(\mathfrak{U}_1)$.

Similarly, we let $\gamma_{n(1)+1}^r(\mathfrak{U}_2), \cdots, \gamma_{n(2)}^r(\mathfrak{U}_2)$ be elements of $Z_C^r(\mathfrak{U}_2)$ such that if $Z^r(\mathfrak{U}_2) \in Z_C^r(\mathfrak{U}_2)$, then there exists a relation $Z^r(\mathfrak{U}_2) \sim \sum_{h=1}^{n(2)-n(1)} a^h \gamma_{n(1)+h}^r(\mathfrak{U}_2)$, $a^h \in \mathfrak{F}$; and choose C-cycles $\gamma_{n(1)+h}^r$ whose respective coordinates on $\mathfrak{U}_2$ are the cycles $\gamma_{n(1)+h}^r(\mathfrak{U}_2)$.

Now for any h, we have $\pi_{\mathfrak{U}_1\mathfrak{U}_2}\gamma_{n(1)+h}^r(\mathfrak{U}_2) \in Z_C^r(\mathfrak{U}_1)$, and therefore $\pi_{\mathfrak{U}_1\mathfrak{U}_2}\gamma_{n(1)+h}^r(\mathfrak{U}_2) \sim \sum_{i=1}^{n(1)} a_h^i Z_i^r(\mathfrak{U}_1)$ on $\mathfrak{U}_1$; and since $\gamma_{n(1)+h}^r(\mathfrak{U}_1) \sim \pi_{\mathfrak{U}_1\mathfrak{U}_2}\gamma_{n(1)+h}^r(\mathfrak{U}_2)$, we have relations

$$(12.4a) \quad \gamma_{n(1)+h}^r(\mathfrak{U}_1) \sim \sum_{i=1}^{n(1)} a_h^i Z_i^r(\mathfrak{U}_1) \qquad \text{on} \qquad \mathfrak{U}_1, \qquad h = 1, \cdots, n(2) - n(1).$$

Denoting the cycles in the right-hand members of (12.4a) by $\gamma_h^r(\mathfrak{U}_1)$, and correspondingly $\sum_{i=1}^{n(1)} a_h^i Z_i^r$ by γ_h^r, let $Z_{n(1)+h}^r = \gamma_{n(1)+h}^r - \gamma_h^r$. Note that $Z_{n(1)+h}^r(\mathfrak{U}_1) \sim 0$ on $\mathfrak{U}_1$. Now if Z^r is any C-cycle of M, there exists a relation $Z^r(\mathfrak{U}_2) \sim \sum_{h=1}^{n(2)-n(1)} c^h \gamma_{n(1)+h}^r(\mathfrak{U}_2) = \sum_{h=1}^{n(2)-n(1)} c^h(Z_{n(1)+h}^r(\mathfrak{U}_2) + \gamma_h^r(\mathfrak{U}_2)) = \sum_{i=1}^{n(2)} a^i Z_i^r(\mathfrak{U}_2)$. The extension to the general $\mathfrak{U}_k$ is carried out similarly.

12.5 A special, but important case of compact metric space is the complex of II 5.2 of geometric type. If K is such a complex, and S denotes the set of points $\| K \|$ (see II 5.6), then $p^r(K; 2) = p^r(S; \mathfrak{F})$ where $\mathfrak{F}$ is the field of integers mod 2. Moreover, numbers $p^r(K; \mathfrak{F})$ are definable for any field $\mathfrak{F}$—all that is needed is to introduce the orientibility concept in K in suitable fashion—and again $p^r(K; \mathfrak{F}) = p^r(S; \mathfrak{F})$. The proofs of these statements, which may be considered "justification theorems" for the Čech theory of homology, are not included here. They usually proceed via (1) the invariance of $p^r(K; \mathfrak{F})$ under "barycentric subdivisions" of K, and (2) the fact that the "barycentric stars" of a sequence of such subdivisions form a complete family of coverings of S.

13. Alternative definitions. The concept of C-cycle and the corresponding homology group can be introduced in another manner which will be found useful later on, and which we describe now. In the interests of brevity we generally omit the phrase "mod L on M" throughout, but it will be understood that when we speak of cycle, we mean cycle mod L on M, etc., unless explicit exception is made (such exception may be made by the use of the word "absolute" as above).

13.1 Definition. By a *directed system* will be meant a set D and a binary (order) relation $<$ of its elements such that (1), if $x, y, z \in D$ and $x < y$, $y < z$, then $x < z$, and (2) if $x, y \in D$, then there exists $z \in D$ such that $x < z$ and $y < z$.

If we let D be the Σ above, and $\mathfrak{U} < \mathfrak{V}$ mean $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, then Σ and $<$ form a directed system.

13.2 A subset D' of a directed system D is called *cofinal* with D if $x \in D$ implies the existence of $x' \in D'$ such that $x < x'$. Thus, in the above example, any complete family Σ' is cofinal with Σ.

13.3 By the device of "indexing," a directed system may be used to induce a "direction" in another set. Suppose a directed system D is given with order relation $<$, and that E is a set and $\rho : E \to D$ a single-valued mapping of E into D with single-valued inverse. If $e \in E$, $d \in D$ and $\rho(e) = d$, let us denote e by e_d. Then if we let $e_d < e_{d'}$ in the event $d < d'$, the set E clearly becomes a directed system with the new order relation $<$, and we say that E has been *indexed* by D. For example, the cycle groups $Z^n(\mathfrak{U})$ above may be indexed by Σ.

13.4 Let a collection $\{V_x\}$ of vector spaces V_x, which has been given a "direction" through the elements x of some directed system N, be related by a set of linear homomorphisms ρ_{xy} ($x, y \in N$, $x < y$), where ρ_{xy} maps V_y into V_x, and $\rho_{xy}\rho_{yz} = \rho_{xz}$ if $x, y, z \in N$, $x < y < z$. Then the collection $\{V_x, \rho_{xy}\}$ is called an *inverse system* of vector spaces. For such a system we define a limit which we denote by $\lim_{\leftarrow} V_x$, or $\lim_{\leftarrow} \{V_x, \rho_{xy}\}$ if we wish to specify the homomorphisms, as follows: The elements of $\lim_{\leftarrow} V_x$ are the collections $\{v_x\}$ where $v_x \in V_x$, one to each V_x, such that if $x < y$ in N, then $\rho_{xy}(v_y) = v_x$. It will be noted that $\lim_{\leftarrow} V_x$ could conceivably consist of the single collection $\{0\}$, where for each x, 0 is the 0 element of V_x. In any case $\lim_{\leftarrow} V_x$ is nonempty and it forms a vector space according to the following conventions: $\{v_x\} + \{v'_x\} = \{v_x + v'_x\}$, and for any $f \in \mathfrak{F}$, $f\{v_x\} = \{fv_x\}$. It will be this vector space that we denote hereafter by $\lim_{\leftarrow} V_x$.

Consider now the directed systems $\{Z^n(\mathfrak{U}); \mathfrak{F}\}$, $\{B^n(\mathfrak{U}; \mathfrak{F})\}$ and $\{H^n(\mathfrak{U})\}$ where $Z^n(\mathfrak{U}; \mathfrak{F})$ is the vector space of cycles of $\mathfrak{U} \in \Sigma$ over $\mathfrak{F}$, $B^n(\mathfrak{U}; \mathfrak{F})$ that of the cycles of $\mathfrak{U}$ that bound on $\mathfrak{U}$, and $H^n(\mathfrak{U}) = Z^n(\mathfrak{U}; \mathfrak{F})/B^n(\mathfrak{U}; \mathfrak{F})$, the systems being indexed by $\mathfrak{U}$ as described above. The projections $\pi_{\mathfrak{U}\mathfrak{V}}$ induce linear homomorphisms of $H^n(\mathfrak{V})$ into $H^n(\mathfrak{U})$, which we denote by $\rho_{\mathfrak{U}\mathfrak{V}}$ (cf. Lemma 6.4 and its corollary), and the system $\{H^n(\mathfrak{U}), \rho_{\mathfrak{U}\mathfrak{V}}\}$ is an inverse system of vector spaces. The vector space $\lim_{\leftarrow} H^n(\mathfrak{U})$ suggests a new definition of $H^n(S; \mathfrak{F})$. We show, however, that

13.5 Theorem. *There is a natural isomorphism between the vector spaces $H^n(S; \mathfrak{F})$ and $\lim_{\leftarrow} H^n(\mathfrak{U})$.*

Proof. Let $\{F(\mathfrak{U})\}$ be an element of $\lim_{\leftarrow} H^n(\mathfrak{U})$. Then for $\mathfrak{V} > \mathfrak{U}$, $\rho_{\mathfrak{U}\mathfrak{V}}(F(\mathfrak{V})) = F(\mathfrak{U})$. Evidently, then, if $z^n(\mathfrak{V}) \in F(\mathfrak{V})$, and $z^n(\mathfrak{U}) \in F(\mathfrak{U})$, we have $\pi_{\mathfrak{U}\mathfrak{V}}z^n(\mathfrak{V}) \sim z^n(\mathfrak{U})$. Consequently every collection $\{z^n(\mathfrak{U})\}$, such that $z^n(\mathfrak{U}) \in F(\mathfrak{U})$, is a C-cycle. And if $\{z_1^n(\mathfrak{U})\}$, $\{z_2^n(\mathfrak{U})\}$ are two such cycles, then

$\{z_1^n(\mathfrak{U})\} \sim \{z_2^n(\mathfrak{U})\}$ inasmuch as $z_1^n(\mathfrak{U}),\ z_2^n(\mathfrak{U}) \in F(\mathfrak{U})$ implies that $z_1^n(\mathfrak{U}) \sim z_2^n(\mathfrak{U})$ on $\mathfrak{U}$. We have therefore a means of defining a homomorphism of lim$_\leftarrow$ $H^n(\mathfrak{U})$ into $H^n(S; \mathfrak{F})$. This homomorphism is both "onto" and one-to-one, since each element of $H^n(S; \mathfrak{F})$ is a class $\{z^n\}$ of homologous cycles z^n such that $z^n \in \{z^n\}$ implies $z^n = \{z^n(\mathfrak{U})\}$, $\mathfrak{U} \in \Sigma$, and the set of all cycles $z^n(\mathfrak{U})$ corresponding thus to $\{z^n\}$ is identical with an $F(\mathfrak{U})$.

13.6 *Alternative definition of C-cycle.* In view of Theorem 13.5 one can give an alternative definition of C-cycle: For any element $\{F(\mathfrak{U})\}$ of lim$_\leftarrow$ $H^n(\mathfrak{U})$, a collection $\{z^n(\mathfrak{U})\}$ where $z^n(\mathfrak{U}) \in F(\mathfrak{U})$ is a C-cycle. Evidently this definition is equivalent to that given in 7.3.

14. **Dual homomorphisms.** Let G and H be either (1) free abelian groups with finite bases, or (2) vector spaces of finite dimension over a field $\mathfrak{F}$. Let $g_1, \cdots, g_k$ and $h_1, \cdots, h_m$ be bases of G and H respectively. Then if (ρ_i^j) is a matrix of k rows and m columns with integer elements ρ_i^j, we may, in case (1), define a homomorphism $\rho : G \to H$ such that

$$\rho : g_i \to \sum_{j=1}^{m} \rho_i^j h_j \tag{14.1a}$$

and for $g = \sum_{i=1}^{k} a^i g_i$,

$$\rho : g \to \sum_{i=1}^{k} \sum_{j=1}^{m} a^i \rho_i^j h_j . \tag{14.1b}$$

In case (2), if the elements ρ_i^j are in the field $\mathfrak{F}$, then (14.1a), (14.1b) define a homomorphism of G into H which is linear: $a\rho(v) = \rho(av)$ for $a \in \mathfrak{F}$.

Conversely, if $\rho : G \to H$ is a homomorphism, then the matrix (ρ_i^j) defining the homomorphism as above is obtained by merely setting down the relations (14.1a) and selecting the required elements ρ_i^j.

14.1 By using the transpose of the matrix (ρ_i^j) we obtain a homomorphism $\rho^* : H \to G$ by the relations

$$\rho^* : h_j \to \sum_{i=1}^{k} \rho_i^j g_i . \tag{14.1c}$$

We call ρ^* the *dual* of the homomorphism ρ. And ρ is in turn, then, the dual of ρ^*.

Suppose that in addition to G and H we have a third free abelian group (or vector space over $\mathfrak{F}$), K, with finite base $p_1, \cdots, p_r$, and a homomorphism $\psi = (\psi_j^s)$ of H into K. Then the matrix of the resultant homomorphism of G into K is the product $(\rho_i^j)\cdot(\psi_j^s)$, and we write

$$\psi\rho : G \to K. \tag{14.1d}$$

Since the transpose of the product of two matrices is the product of their transposes taken in reverse order, the dual of $\psi\rho$ is $(\psi_j^s)^* \cdot (\rho_i^j)^*$, where the asterisk denotes transposition. Thus,

(14.1e) $$(\psi\rho)^* = \rho^*\psi^*.$$

Linear combinations of homomorphisms are defined in the usual manner: If ρ_1 and ρ_2 are homomorphisms of G into H with matrices (ρ^j_{1i}) and (ρ^j_{2i}) respectively, then for c and d any integers in case (1), or elements of $\mathfrak{F}$ in case (2), the homomorphism $c\rho_1 + d\rho_2$ has as matrix the sum $c(\rho^j_{1i}) + d(\rho^j_{2i})$. For the dual, $(c\rho_1 + d\rho_2)^* = c\rho_1^* + d\rho_2^*$.

EXAMPLES. Consider a finite complex K with cells $\sigma^n_1, \cdots, \sigma^n_k$. The boundary operation ∂ effects a linear homomorphism of the groups $C^n(K; G)$ into the groups $C^{n-1}(K; G)$ with matrix whose ith row contains the coefficients in $\partial\sigma^n_i$. The relations (14.1b) are obtainable in either of the cases where the coefficient group is an arbitrary abelian group or a field, since the elements of the above matrix are all 0, 1 or -1. The dual ∂^* of ∂ will be denoted by δ and will be studied in detail below.

Consider the case of a simplicial mapping π of a complex K into a complex K'. Such a mapping induces (1) a homomorphism of $C^n(K; G)$ into $C^n(K'; G)$, whose dual, π^*, is of importance later, and (2) a homomorphism of $C^n(K; G)$ into $C^{n+1}(\mathfrak{D}K; G)$ (cf. §6). We found previously (Lemma 6.4, 6.6) that for any chain $c^n \in C^n(K; G)$,

$$\partial\pi c^n = \pi\partial c^n, \qquad \partial\mathfrak{D}c^n = \pi c^n - c^n - \mathfrak{D}\partial c^n;$$

in terms of the corresponding homomorphisms (which we denote by the same symbols as for the operators; 1 denotes the identity homomorphism), these relations become

(14.1f) $$\partial\pi = \pi\partial, \qquad \partial\mathfrak{D} = \pi - 1 - \mathfrak{D}\partial.$$

The duals of (14.1f) become, after transposition,

(14.1g) $$\delta\pi^* = \pi^*\delta, \qquad \delta\mathfrak{D}^* = \pi^* - 1 - \mathfrak{D}^*\delta.$$

The P-function used in Theorem 7.2 induces a homomorphism of $C^n(\mathfrak{V}; G)$ into $C^{n+1}(\mathfrak{U}; G)$, and we found in (7.2b) that

$$\partial P = \pi_2 - \pi_1 - P\partial.$$

The dual of this relation, after transposition, is

(14.1h) $$\delta P^* = \pi_2^* - \pi_1^* - P^*\delta.$$

15. **Cocycles; cohomology groups.** Returning to the dual δ of ∂ (cf. 14.1); the operator δ induces a linear homomorphism of $C^n(K; G)$ into $C^{n+1}(K; G)$, K being a complex. We call *cocycle* of K any chain c^n such that $\delta c^n = 0$. That is, the *n-dimensional cocycles* of K are the elements of the kernel of the above homomorphism. Since $\partial^2 = 0$, we have $\delta^2 = 0$, and consequently every chain z_n that satisfies a *coboundary relation*

(15a) $$\delta c^{n-1} = z_n$$

is a cocycle. It will be noticed that we have placed the dimensionality index in the subscript position, a device to distinguish cocycle from cycle, and (below) to distinguish cohomology from homology. A cocycle such as z_n in (15a) will be called a *coboundary*, or *cobounding cocycle*. The coboundaries form a subgroup, $B_n(K; G)$, of the group of cocycles, which is denoted by $Z_n(K; G)$. The factor group $H_n(K; G) = Z_n(K; G)/B_n(K; G)$ is called the *n-dimensional cohomology group of K over the group G*.

As in the case of cycles, we may have both *absolute* and *relative cocycles*. In the case of the relative cycles, we consider chains of the complex in question whose boundaries are chains of a certain subcomplex as cycles mod the latter (7.7). But in the case of relative cocycles, we shall consider chains of the subcomplex whose coboundaries are in its *complement* as relative cocycles. The reason for this is that according to the fundamental conventions (§1), a complex contains the faces of all its simplexes, hence contains the simplexes that correspond to the boundaries of its chains; thus a cycle of a subcomplex is a cycle of the containing complex. But a cocycle of a subcomplex will generally not be a cocycle of the containing complex, since some of the cells involved may correspond to faces of simplexes of the complement of the subcomplex. Consequently, if K_1 is a subcomplex of a complex K, and z_n is a cocycle of K_1, then, from the point of view of K, we call z_n a cocycle of K mod $K - K_1$. On the other hand, a cocycle γ_n of $K - K_1$ is a cocycle of K, since simplexes in $K - K_1$ are faces only of simplexes in $K - K_1$. If we agree to think of a subcomplex K_1 of a complex K as *closed* in K (we could define, for example, a set K_1 of simplexes of K as closed if K_1 contains all faces of its simplexes), and a collection K_2 of simplexes of K as *open* in K if $K - K_2$ is closed, we have an exact analogue of the spacial concepts of closed and open set. By an open subcomplex of a complex K, then, we shall mean such an open set of simplexes as K_2.

The natural habitat of cocycles of a complex is the open subcomplex, just as the natural habitat of the cycles is the closed subcomplex. And we shall get the "dual theory," or "co-theory", by replacing the closed subcomplex and closed point set by the open subcomplex and open point set, respectively. Thus, if K_1 and K_2 are open subcomplexes of a complex K, $K_1 \supset K_2$, then a cocycle of K_1 mod K_2 is a chain of K_1 whose coboundary is a chain of K_2.

15.1 Let us go directly, however, to the general situation of a compact space S and a field $\mathfrak{F}$. Let P be an open subset of S. If σ^n corresponds to a simplex of $\mathfrak{U}$ whose nucleus (7.1) is in P, we say that σ^n is *in* P. And if $c^n = \sum_i g^i\sigma_i^n$ is a chain of $\mathfrak{U}$, whose cells σ_i^n for which $g^i \neq 0$ are all in P, then c^n is called a chain *in* P. It will be noted that the set of all simplexes of $\mathfrak{U}$ whose nuclei are subsets of P forms an open subcomplex of $\mathfrak{U}$. Now suppose Q is an open subset of P. If c_n is a chain of $\mathfrak{U}$ that is in P, but whose coboundary δc_n is in Q, then we call c_n a *cocycle of* $\mathfrak{U}$ mod Q in P. The n-dimensional cocycles of $\mathfrak{U}$ mod Q in P form a vector space which we denote, by analogy with 7.7, by $Z_n(P, Q; \mathfrak{F}, \mathfrak{U})$. Evidently if $\mathfrak{U}(P)$, $\mathfrak{U}(Q)$ are the collections of those simplexes of $\mathfrak{U}$ that lie in P and Q, respectively, then $Z_n(P, Q; \mathfrak{F}, \mathfrak{U})$ is the group of cocycles of $\mathfrak{U}(P)$ mod $\mathfrak{U}(Q)$.

15.2 If z_n is a cocycle of $\mathfrak{U}$ mod Q in P, then we say that z_n *cobounds*, or is *cohomologous to zero* on $\mathfrak{U}$ mod Q in P if there exists a chain c_{n-1} in P such that $\delta c^{n-1} = z_n + z'_n$, where z'_n is in Q; and we express this cohomology by the relation

$$z_n \backsim 0 \qquad \text{on } \mathfrak{U} \text{ mod } Q \text{ in } P, \tag{15.2a}$$

called a *cohomology* or *cohomology relation.* Two such cocycles z_n^1, z_n^2 will be called cohomologous on $\mathfrak{U}$ mod Q in P if $z_n^1 - z_n^2 \backsim 0$ on $\mathfrak{U}$ mod Q in P, and we write

$$z_n^1 \backsim z_n^2 \qquad \text{on } \mathfrak{U} \text{ mod } Q \text{ in } P. \tag{15.2b}$$

The set of cocycles on $\mathfrak{U}$ mod Q in P that are $\backsim 0$ on $\mathfrak{U}$ mod Q in P form the subspace $B_n(P, Q; \mathfrak{F}, \mathfrak{U})$ of $Z_n(P, Q; \mathfrak{F}, \mathfrak{U})$, and we define $H_n(P, Q; \mathfrak{F}, \mathfrak{U})$ as the factor group $Z_n(P, Q; \mathfrak{F}, \mathfrak{U})/B_n(P, Q; \mathfrak{F}, \mathfrak{U})$. We call $H_n(P, Q; \mathfrak{F}, \mathfrak{U})$ the *n-dimensional cohomology group of* $\mathfrak{U}$ *mod Q in P.*

15.3 In order to extend these ideas from the individual covering of the space to the space, let us again consider a directed system of vector spaces, $\{V_x\}$, except that now we suppose there is given a set of linear homomorphisms ρ_{yx} such that if $x < y$, then ρ_{yx} is a homomorphism of V_x into V_y, and if $x < y < z$, then $\rho_{zy}\rho_{yx} = \rho_{zx}$. We call the system $\{V_x, \rho_{yx}\}$ a *direct system* of vector spaces. For such a system we define a limit vector space, $\lim_{\rightarrow} V_x$, or $\lim_{\rightarrow} \{V_x, \rho_{yx}\}$ if we wish to specify the homomorphisms, as follows: If $v_x \in V_x$, $v_y \in V_y$, and if there exists z such that $x, y < z$ and such that $\rho_{zx}v_x = \rho_{zy}v_y$, then we call v_x and v_y equivalent. In this manner, the totality of elements in sets V_x of the collection $\{V_x\}$ is divided into equivalence classes $\{v_x\}$. That this equivalence is transitive is shown as follows: Let v_x and v_y be equivalent as above, and suppose v_y equivalent to v_w by virtue of z' such that $\rho_{z'y}v_y = \rho_{z'w}v_w$. Select $u > z, z'$, and notice that from the definition of direct system the relations $\rho_{uz}(\rho_{zx}v_x) = \rho_{uz}(\rho_{zy}v_y)$, $\rho_{uz'}(\rho_{z'y}v_y) = \rho_{uz'}(\rho_{z'w}v_w)$ imply $\rho_{ux}v_x = \rho_{uy}v_y = \rho_{uw}v_w$.

The elements of $\lim_{\rightarrow} V_x$ are the equivalence classes $\{v_x\}$, and $\lim_{\rightarrow} V_x$ becomes a vector space in natural fashion by virtue of the following conventions: If $g \in \mathfrak{F}$, then $g\{v_x\} = \{gv_x\}$; as the homomorphisms ρ_{yx} are linear, $\rho_{yx}v_x = v_y$ implies that $\rho_{yx}(gv_x) = g\rho_{yx}v_x = gv_y$, and hence $g\{v_x\} \in \lim_{\rightarrow} V_x$. To define a sum $\{v_x\} + \{v'_y\}$, we select $v_x \in \{v_x\}$ and $v'_y \in \{v'_y\}$, and z such that $x, y < z$; then $\{v_x\} + \{v'_y\} = \{\rho_{zx}v_x + \rho_{zy}v'_y\}$. That a unique element of $\lim_{\rightarrow} V_x$ is so determined is left to the reader.

15.4 *Cohomology groups.* Now suppose $\mathfrak{U}, \mathfrak{B} \in \Sigma$, $\mathfrak{B} > \mathfrak{U}$. Since, by (14.1g), $\pi^*_{\mathfrak{U}\mathfrak{B}}\delta c^n(\mathfrak{U}) = \delta\pi^*_{\mathfrak{U}\mathfrak{B}}c^n(\mathfrak{U})$, a dual projection $\pi^*_{\mathfrak{U}\mathfrak{B}}$ maps cocycles of $\mathfrak{U}$ into cocycles of $\mathfrak{B}$, and cobounding cocycles of $\mathfrak{U}$ into cobounding cocycles of $\mathfrak{B}$. And if σ^n is a cell of $\mathfrak{U}$ in an open set P, then $\pi^*_{\mathfrak{U}\mathfrak{B}}\sigma^n$ is also in P. Accordingly $\pi^*_{\mathfrak{U}\mathfrak{B}}$ induces a linear homomorphism of $H_n(P, Q; \mathfrak{F}, \mathfrak{U})$ into $H_n(P, Q; \mathfrak{F}, \mathfrak{B})$, which we also denote by $\pi^*_{\mathfrak{U}\mathfrak{B}}$. That this homomorphism is independent of the particular $\pi^*_{\mathfrak{U}\mathfrak{B}}$ chosen follows from (14.1h). Hence the system $\{H_n(P, Q;$

$\mathfrak{F}$, $\mathfrak{U}$); $\pi^*_{\mathfrak{U}\mathfrak{V}}\}$ is a direct system of vector spaces, and it is the $\lim_{\rightarrow}$ of this system that we call the *n-dimensional cohomology group of S mod Q in P*:

$$H_n(S; P, Q; \mathfrak{F}) = \lim_{\rightarrow} \{H_n(P, Q; \mathfrak{F}, \mathfrak{U}); \pi^*_{\mathfrak{U}\mathfrak{V}}\}.$$

The meanings of $H_n(S; P, 0; \mathfrak{F})$ and $H_n(S; S, 0; \mathfrak{F})$ should be clear from analogy with the corresponding homology groups. The group $H_n(S; S, 0; \mathfrak{F})$ is the cohomology group of S itself, and will hereafter be denoted by $H_n(S)$. And analogously we may denote $H^n(S; \mathfrak{F})$ by $H^n(S)$; and $H_n(S; P, Q, \mathfrak{F})$ by $H_n(S; P, Q)$ or $H_n(P, Q)$ when no confusion would result.

15.5 REMARKS. *Definition of cocycle of a space.* There are some aspects of the cohomology theory which, although not of theoretical importance at first sight, lead to a better intuitive understanding if realized. For example, whereas in the case of the homology groups of a space, a cycle $z^n(\mathfrak{U})$ may or may not give rise to a homology class of $H^n(S)$, the situation in the case of cohomology is quite otherwise. Indeed, a cocycle $z_n(\mathfrak{U})$ determines a unique element of $H_n(S)$. Consequently the necessity for proving such existence theorems as we did for the C-cycles in §10 does not exist in the cohomology theory.

Although in the case of homology we were able to define first the notion of C-cycle and subsequently a homology group, pointing out only as an alternative the possibility of first defining $H^n(S)$ by means of inverse systems, and then defining the individual C-cycle in terms of $H^n(S)$, we have above defined only $H_n(S)$ and have not even considered the question: *What is a cocycle of S?* In order to obtain an answer to this question, note that a single cocycle, $z_n(\mathfrak{U})$, of a single covering $\mathfrak{U}$ of S determines (1) a cohomology class of $H_n(\mathfrak{U}) = H_n(S, 0; \mathfrak{F}, \mathfrak{U})$, and consequently (2) an element of $H_n(S)$, i.e., the set of all $h_n(\mathfrak{V}) \in H_n(\mathfrak{V})$ such that $h_n(\mathfrak{V})$ is equivalent to the cohomology class $[z_n(\mathfrak{U})]^*$ determined by $z_n(\mathfrak{U})$ on $\mathfrak{U}$. It is natural, therefore, to let any $z_n(\mathfrak{U})$ constitute a cocycle of S; and more generally, any $z_n(\mathfrak{U})$ that is a cocycle mod Q in P is a cocycle of S mod Q in P. However, given any such $z_n(\mathfrak{U})$, and $\mathfrak{V} > \mathfrak{U}$, the cocycle $\pi^*_{\mathfrak{U}\mathfrak{V}}z_n(\mathfrak{U})$ determines the same element of $H_n(S)$ as $z_n(\mathfrak{U})$, so that when a cocycle $z_n(\mathfrak{U})$ is given, the elements equivalent to it are immediately obtained on all refinements of $\mathfrak{U}$. Consequently, given a cocycle $Z_n(\mathfrak{U})$ and $\mathfrak{V} > \mathfrak{U}$, we shall sometimes in the sequel use the symbol $Z_n(\mathfrak{V})$ to mean $\pi^*_{\mathfrak{U}\mathfrak{V}}Z_n(\mathfrak{U})$. And as a cocycle of S, $z_n(\mathfrak{U}) \backsim 0$ on S if some $\pi^*_{\mathfrak{U}\mathfrak{V}}z_n(\mathfrak{U}) \backsim 0$ on $\mathfrak{V}$.

15.6 Finally, we emphasize again the dual relation between cycle and cocycle as regards the complexes or point sets in which they are "imbedded." Cycles are absolute if given on closed complexes or closed sets, but relative when given on open complexes or open sets; whereas cocycles are absolute when given in open complexes or sets and relative on closed complexes or sets. Thus, L being a closed subcomplex of K, cocycles of L are really cocycles of K mod $K - L$ when considered as chains of K, since as such they may have coboundaries in $K - L$. And if M is a closed subset of S, there can be defined a corresponding *cohomology group of cocycles of S* mod $S - M$; this is the group

$H_n(S; S, S - M; \mathfrak{F})$, special (and most important) case of $H_n(S; P, Q; \mathfrak{F})$ where $P = S, Q = S - M$.

16. **Chain products for a complex.** In this section we introduce two binary operations for chains, or *chain products* as we shall call them. The first, a "dot" product, is a scalar product defined for chains of equal dimensions as follows:

$$(16a) \qquad (\sum_i a^i \sigma_i^n) \cdot (\sum_i b^i \sigma_i^n) = \sum_i a^i b^i.$$

The second is a noncommutative bilinear product called "cap" product:

$$(16b) \qquad (\sum_i a^i \sigma_i^q) \frown (\sum_j b^j \sigma_j^{p+q}) = \sum_{i,j} a^i b^j (\sigma_i^q \frown \sigma_j^{p+q})$$

for all dimensions p and q; if c^m and c^n are chains such that $m > n$, then $c^m \frown c^n = 0$. Assuming a finite complex K, and for the present using only chains over the additive group of integers, we postulate for each pair of cells σ_i^q, σ_j^{p+q} of K:

I. $\sigma_i^q \frown \sigma_j^{p+q}$ is 0 if the simplex E_i^q to which σ_i^q corresponds is not a face of the simplex E_j^{p+q} to which σ_j^{p+q} corresponds; otherwise, it is a p-chain of the complex $\overline{E}_j^{p+q}$ consisting of E_j^{p+q} and its faces.

II. $\partial(\sigma_i^q \frown \sigma_j^{p+q}) = (-1)^p(\delta\sigma_i^q) \frown \sigma_j^{p+q} + \sigma_i^q \frown (\partial\sigma_j^{p+q})$.

III. There is a fixed integer α, such that for all j and q, $\sum_i \sigma_i^0 \cdot (\sigma_j^q \frown \sigma_j^q) = \alpha$.

A priori, it is possible that any number of different cap products satisfying I—III may be given; even the assigning of a value for α does not necessarily fix the cap product. The question as to whether there exists *any* definition of $\frown$ satisfying the requirements is settled below when we define explicitly the product we shall use throughout. Evidently II will have as a corollary that the cap product of a cocycle and a cycle will be a cycle, and we shall see that the product is extendible to a product between cohomology and homology classes. The ultimate invariance of the latter will depend upon a uniqueness theorem which we dispose of before giving the special theory.

16.1 THEOREM. *Let $\frown$ be any product with $\alpha = 0$. Then there is a bilinear operation $\wedge$ such that (using symbols as in* I, II, III)

I′. *$\sigma_i^q \wedge \sigma_j^{p+q}$ is 0 if E_i^q is not a face of E_j^{p+q}, and otherwise is a $(p + 1)$-chain of $\overline{E}_j^{p+q}$.*

II′. $\sigma_i^q \frown \sigma_j^q = \partial(\sigma_i^q \wedge \sigma_j^q)$.

III′. $\sigma_i^q \frown \sigma_j^{p+q} = \partial(\sigma_i^q \wedge \sigma_j^{p+q}) + (-1)^p(\delta\sigma_i^q) \wedge \sigma_j^{p+q} + \sigma_i^q \wedge (\partial\sigma_j^{p+q})$ *for all $p > 0$.*

PROOF. We can construct $\sigma_i^q \wedge \sigma_j^q$ so that I′ holds for $p = 0$ and II′ holds, as follows:

Define $\sigma_i^q \wedge \sigma_j^q = 0$ if $i \neq j$. For $q = 0$, $\sigma_i^0 \frown \sigma_i^0$ is a 0-chain on E_i^0, hence of form $n\sigma_i^0$; but as $\mathrm{Ki}(\sigma_i^0 \frown \sigma_i^0) = 0$ by III (cf. Definition 2.1), so must $n = 0$ and consequently $\sigma_i^0 \frown \sigma_i^0 = 0$. Therefore we may define $\sigma_i^0 \wedge \sigma_i^0 = 0$. For $q > 0$, $\sigma_i^q \frown \sigma_i^q$ is a 0-cycle of the (augmented) complex $\overline{E}_i^q$ and by Corollary

6.2 there exists a 1-chain C^1 on $\overline{E}_i^q$ such that $\partial C^1 = \sigma_i^q \frown \sigma_i^q$; we define $\sigma_i^q \wedge \sigma_i^q$ to be this C^1.

Suppose that all $\sigma_i^q \wedge \sigma_j^{r+q}$ are constructed for all q as well as for all $r <$ a fixed p. We shall construct $\sigma_i^q \wedge \sigma_j^{p+q}$. We let it be 0 if E_i^q is not a face of E_j^{p+q}. In the contrary case, we first define the chain

$$C^p = \sigma_i^q \frown \sigma_j^{p+q} - (-1)^p \delta\sigma_i^q \wedge \sigma_j^{p+q} - \sigma_i^q \wedge \partial\sigma_j^{p+q}.$$

By II and III′ solved for $\partial(\sigma_i^q \wedge \sigma_j^{p+q})$,

$$\begin{aligned}\partial C^p = (-1)^p \delta\sigma_i^q \frown \sigma_j^{p+q} + \sigma_i^q \frown \partial\sigma_j^{p+q} - (-1)^p[\delta\sigma_i^q \frown \sigma_j^{p+q} - \delta\sigma_i^q \wedge \partial\sigma_j^{p+q}]\\ - [\sigma_i^q \frown \partial\sigma_j^{p+q} - (-1)^{p-1}\delta\sigma_i^q \wedge \partial\sigma_j^{p+q}] = 0.\end{aligned}$$

Hence C^p is a cycle of $\overline{E}_j^{p+q}$, and we may choose $\sigma_i^q \wedge \sigma_j^{p+q}$ as a chain of $\overline{E}_j^{p+q}$ having C^p as boundary.

16.2 Linear combinations of cap products $\frown_1$ and $\frown_2$ may be formed by defining

$$\text{(16.2a)}\qquad C^q(a_1 \frown_1 + a_2 \frown_2)C^{p+q} = a_1(C^q \frown_1 C^{p+q}) + a_2(C^q \frown_2 C^{p+q}).$$

If we denote the α of III by $\alpha(\frown)$, we find by application of III,

$$\text{(16.2b)}\qquad \alpha(a_1 \frown_1 + a_2 \frown_2) = a_1\alpha(\frown_1) + a_2\alpha(\frown_2).$$

16.3 Theorem. *For any two products $\frown$ and $\frown'$,*

$$\text{(16.3a)}\qquad \alpha(\frown)(C^q \frown' C^{p+q}) \sim \alpha(\frown')(C^q \frown C^{p+q}) \text{ if } \delta C^q = \partial C^{p+q} = 0.$$

Proof. Let $\frown'' = \alpha(\frown)\frown' - \alpha(\frown')\frown$. Then $\alpha(\frown'') = 0$. It then follows from Theorem 16.1 that $C^q \frown'' C^{p+q} \sim 0$, and in terms of the given $\frown$ and $\frown'$, this gives (16.3a).

16.4 Theorem. *If L is a closed subcomplex of K, C^q a cocycle in $K - L$, and C^{p+q} a cycle mod L, then for any two products $\frown$, $\frown'$, relation* (16.3a) *holds on $\overline{K - L}$ (the collection of all simplexes not in L, together with their faces).*

Proof. The proof is as for Theorem 16.3, except that we notice that in applying Theorem 16.1 to get $C^q \frown'' C^{p+q} = \partial(C^q \wedge C^{p+q})$, each nonzero chain $\sigma^q \wedge \sigma^{p+q}$ is on $\overline{K - L}$.

16.5 *The special cap product.* As our complexes have been defined in the "simplicial" manner in terms of vertices, we may show the *existence* of a cap product as follows: Let K be the complex as before, and let $v_1, v_2, \cdots, v_k$ denote its vertices (0-cells). In case $\sigma^q = v_{i(p)}v_{i(p+1)} \cdots v_{i(p+q)}$ and $\sigma^{p+q} = v_{i(0)}v_{i(1)} \cdots v_{i(p)} \cdots v_{i(p+q)}$ where $i(0) < i(1) < \cdots < i(p + q)$, then we let $\sigma^q \frown \sigma^{p+q} = \sigma^p = v_{i(0)} \cdots v_{i(p)}$. In every other case, $\sigma^q \frown \sigma^{p+q} = 0$. With $\alpha = 1$, this cap product satisfies I and III. That II holds may be shown thus: Let σ^n, σ^q be any two cells of K. If $q > n$, then $\sigma^q \frown \sigma^n = 0$, and similarly $(\delta\sigma^q) \frown \sigma^n$ and $\sigma^q \frown (\partial\sigma^n)$ both $= 0$. Hence we consider only the case where

$q \leqq n$, and we suppose that $n = p + q$. If E^q is not a face of E^n, then E^q has at least one vertex v not on $\bar{E}^n$; and on the other hand v is also a vertex of the complex corresponding to $\delta\sigma^q$ in case the latter is not 0. Hence in case E^q is not a face of E^n, all terms of II are zero.

The only case, then, where not all terms of II are zero is that where σ^n is of the form $v_0 \cdots v_{p-1}v_p \cdots v_{p+q}$ and σ^q of the form $v_{p-1}v_p \cdots \hat{v}_{p+m} \cdots v_{p+q}$ (by $\hat{v}$ we indicate that the vertex v is to be deleted), $-1 \leqq m \leqq q$. If $-1 < m \leqq q$, the left-hand member of II is zero, and in the right-hand member we get $\sigma^q \frown (\partial\sigma^n) = (-1)^{p+m}v_0 \cdots v_{p-1}$ and $(-1)^p(\delta\sigma^q) \frown \sigma^{p+q} = (-1)^{p+m+1}v_0 \cdots v_{p-1}$. In case $m = -1$, $\partial(\sigma^q \frown \sigma^n) = \partial(v_0 \cdots v_p) = \sum_{i=0}^{p} (-1)^i v_0 \cdots \hat{v}_i \cdots v_p$. And $\delta\sigma^q \frown \sigma^n = v_{p-1}v_p \cdots v_{p+q} \frown \sigma^n = v_0 \cdots v_{p-1}$ so that $(-1)^p\delta\sigma^q \frown \sigma^n = (-1)^p v_0 \cdots v_{p-1}$; whereas $\sigma^q \frown \partial\sigma^n = (v_p \cdots v_{p+q}) \frown \sum_{i=0}^{p-1} (-1)^i v_0 \cdots \hat{v}_i \cdots v_p \cdots v_{p+q} = \sum_{i=0}^{p-1} (-1)^i v_0 \cdots \hat{v}_i \cdots v_p$. Relation II is therefore true in all cases.

16.6 We now extend $\frown$ to an operation between chains with coefficients in $\mathfrak{F}$. We let $\alpha = 1$ and consider this to be the unit in $\mathfrak{F}$. Then III becomes meaningful in $\mathfrak{F}$ for all integers α. And if $a, b \in \mathfrak{F}$, we let $(a\sigma^q) \frown (b\sigma^n) = ab(\sigma^q \frown \sigma^n)$, ab being the product in $\mathfrak{F}$. Extension to arbitrary chains is made in bilinear fashion, and relation II again holds. Theorems 16.1, 16.3 and 16.4 continue to hold. As a consequence, if $\frown$ and $\frown'$ are two cap products with $\alpha = 1$, and z_q, z^n are a cocycle and cycle of K respectively, then $z_q \frown z^n \sim z_q \frown' z^n$. Consequently, so far as homology class is concerned, the result of "multiplying" a cocycle and a cycle is independent of any ordering of the vertices, for instance; and a similar statement holds regarding a cocycle of $K - L$ and a cycle mod L.

We next extend the cap product to an operation on homology and cohomology classes:

16.7 Theorem. *If z_q^1 and z_q^2 are cocycles, $z_q^1 \backsim z_q^2$, and z^n is a cycle, then* $(z_q^1 \frown z^n) \sim (z_q^2 \frown z^n)$.

Proof. Let $\delta c^{q-1} = z_q^1 - z_q^2$. Application of II to $\partial(c^{q-1} \frown z^n)$ gives the latter equal to $(-1)^{n-q+1}(z_q^1 \frown z^n - z_q^2 \frown z^n)$.

If K is a complex, and L a closed subcomplex of K, then Theorem 16.7 continues to hold if the cocycles are cohomologous in $K - L$ and z^n is a cycle mod L. Explicitly,

16.7a Theorem. *If L is a closed subcomplex of a complex K, and $z_q^1 \backsim z_q^2$ in $K - L$, then for any cycle z^n* mod L, $(z_q^1 \frown z^n) \sim (z_q^2 \frown z^n)$ *on* $\overline{K - L}$.

(The proof is a consequence of the fact that a simplex of $K - L$ cannot be a face of a simplex of L.)

In like manner we can prove that

16.7b Theorem. *If L and K are as in Theorem* 16.7a, *and z_q^1, z_q^2 are cocycles mod $K - L$ of K, and z^n is an absolute cycle of L, and $z_q^1 \backsim z_q^2$* mod $K - L$, *then* $(z_q^1 \frown z^n) \sim (z_q^2 \frown z^n)$ *on L.*

PROOF. As a chain of K, z_q^i $(i = 1, 2)$ may have coboundary on $K - L$. As a cycle of a closed subcomplex, L, the chain z^n is also a cycle of K so that $\partial z^n = 0$. Hence by II, $\partial(z_q^i \frown z^n) = (-1)^{n-q}\delta z_q^i \frown z^n = 0$. Thus the chains $z_q^i \frown z^n$ are absolute cycles of K.

By hypothesis, there exists a cobounding relation

$$\delta c^{q-1} = z_q^1 - z_q^2 + z_q \qquad (c^{q-1} \text{ on } L,\ z_q \text{ in } K - L).$$

Applying II,

$$\partial(c^{q-1} \frown z^n) = (-1)^{n-q+1}(z_q^1 - z_q^2 + z_q) \frown z^n$$

$$= (-1)^{n-q+1}[z_q^1 \frown z^n - z_q^2 \frown z^n].$$

16.8 THEOREM. *If z_1^n, z_2^n are cycles of K, z_q a cocycle of K, and $z_1^n \sim z_2^n$, then $z_q \frown z_1^n \sim z_q \frown z_2^n$ on K.*

For the proof, apply II to $\partial(z_q \frown c^{n+1})$, where $\partial c^{n+1} = z_1^n - z_2^n$.

The "relative" forms of Theorem 16.8 are as follows:

16.8a THEOREM. *If L is a closed subcomplex of a complex K, z_q a cocycle in $K - L$, z_1^n, z_2^n cycles* mod L *such that $z_1^n \sim z_2^n$* mod L, *then $(z_q \frown z_1^n) \sim (z_q \frown z_2^n)$ on $\overline{K - L}$.*

Note that the $z_q \frown z_i^n$ $(i = 1, 2)$ are absolute cycles on $\overline{K - L}$, since $z_q \frown \partial z_i^n = 0$ (the chains ∂z_i^n being on L). Like considerations enter into the proof of Theorem 16.8a, which is obtained by applying II to $\partial(z_q \frown c^{n+1})$, where $\partial c^{n+1} = z_1^n - z_2^n + z^n$ (z^n on L).

16.8b THEOREM. *With K and L as before, if z_q is a cocycle* mod $K - L$ *of K, z_1^n, z_2^n cycles of L such that $z_1^n \sim z_2^n$ on L, then $z_q \frown z_1^n \sim z_q \frown z_2^n$ on L.*

We are now in a position to state the main theorem of this section:

16.9 THEOREM. *For any complex K and closed subcomplex L, and $\alpha = 1$, the $\frown$ products exist, and uniquely define $\frown$ products as follows:*

(1) *(cohomology class) $\frown$ (homology class) = (homology class).*

(2) *(cohomology class of $K - L$) $\frown$ (homology class* mod L) = *(homology class of $\overline{K - L}$.*

(3) *(cohomology class of K* mod $K - L$) $\frown$ *(homology class of L) = (homology class of L).*

17. Extension to topological spaces.

In extending the $\frown$ product to C-cycles and cocycles of a space S we shall have need of the following theorem.

17.1 THEOREM. *Let K, K' be complexes, f a simplicial mapping of K into K', and f^* the dual of the chain-mapping f. Then if z'_q, z^n are a cocycle of K' and a cycle of K, respectively,*

$$f(f^*z'_q \frown z^n) \sim z'_q \frown f(z^n). \tag{17.1a}$$

PROOF. Using the special cap product defined above, we may assume that the vertices of K' are first ordered, and then the vertices of K are so ordered that if $f(v_i) = v'_{f(i)}$, then $f(i) < f(j)$ implies that $i < j$. Denoting the special cap product defined in terms of these orderings by $\frown'$, we prove first that

$$f(f^*z'_q \frown' z^n) = z'_q \frown' f(z^n). \tag{17.1b}$$

Consider cells $\sigma'^q = v'_{j(p)} \cdots v'_{j(n)}$, $\sigma^n = v_{i(0)} \cdots v_{i(p)} \cdots v_{i(n)}$, $p = n - q$. Then since $f^*\sigma'^q$ is a sum of cells $v_{m(p)} \cdots v_{m(n)}$, the special product $f^*\sigma'^q \frown' \sigma^n$ will be different from zero, and $= v_{i(0)} \cdots v_{i(p)}$, if and only if for one of the cells $v_{m(p)} \cdots v_{m(n)}$, we have $m(p) = i(p), \cdots, m(n) = i(n)$. In this case, the left-hand member of (17.1b), in terms of individual σ's, is $f(v_{i(0)} \cdots v_{i(p)})$, and the right-hand member is

$$(v'_{j(p)} \cdots v'_{j(n)}) \frown' (f(v_{i(0)}) \cdots f(v_{i(p-1)})v'_{j(p)} \cdots v'_{j(n)})$$
$$= f(v_{i(0)}) \cdots f(v_{i(p)}) = f(v_{i(0)} \cdots v_{i(p)}).$$

Hence (17.1b) is certainly valid, inasmuch as it is valid for arbitrary cells and chains.

By virtue of Theorem 16.3, $f^*z'_q \frown z^n \sim f^*z'_q \frown' z^n$ and $z'_q \frown f(z^n) \sim z'_q \frown' f(z^n)$, and consequently (17.1a) holds.

In the applications of Theorem 17.1, K and K' will be coverings, one of which is a refinement of the other, and f will be a projection. For the most important relative case we shall have need of the following:

17.1a THEOREM. *Let Q be an open subset of S, $z_q(\mathfrak{U}')$ a cocycle in Q and $\{z^n(\mathfrak{U})\}$ a cycle of S mod $S - Q$. Then if $\mathfrak{B} > \mathfrak{U}'$,*

$$\pi_{\mathfrak{U}'\mathfrak{B}}[\pi^*_{\mathfrak{U}'\mathfrak{B}} z_q(\mathfrak{U}') \frown z^n(\mathfrak{B})] \sim z_q(\mathfrak{U}') \frown \pi_{\mathfrak{U}'\mathfrak{B}} z^n(\mathfrak{B}) \qquad \text{on } \mathfrak{U}' \wedge Q, \tag{17.1b'}$$

where $\mathfrak{U}' \wedge Q$ is the subcomplex of $\mathfrak{U}'$ on Q (7.6).

PROOF. In the proof of Theorem 17.1 we showed that for arbitrary chains $z_q(\mathfrak{U}')$, $z^n(\mathfrak{B})$,

$$\pi_{\mathfrak{U}'\mathfrak{B}}[\pi^*_{\mathfrak{U}'\mathfrak{B}} z_q(\mathfrak{U}') \frown' z^n(\mathfrak{B})] = z_q(\mathfrak{U}') \frown' \pi_{\mathfrak{U}'\mathfrak{B}} z^n(\mathfrak{B}). \tag{17.1c}$$

By Theorem 16.4,

$$\pi^*_{\mathfrak{U}'\mathfrak{B}} z_q(\mathfrak{U}') \frown' z^n(\mathfrak{B}) \sim \pi^*_{\mathfrak{U}'\mathfrak{B}} z_q(\mathfrak{U}') \frown z^n(\mathfrak{B}) \qquad \text{on } \mathfrak{B} \wedge Q, \tag{17.1d}$$

and since the projection of a cell on Q is also on Q,

$$\pi_{\mathfrak{U}'\mathfrak{B}}[\pi^*_{\mathfrak{U}'\mathfrak{B}} z_q(\mathfrak{U}') \frown' z^n(\mathfrak{B})] \sim \pi_{\mathfrak{U}'\mathfrak{B}}[\pi^*_{\mathfrak{U}'\mathfrak{B}} z_q(\mathfrak{U}') \frown z^n(\mathfrak{B})] \quad \text{on } \mathfrak{U}' \wedge Q \tag{17.1e}$$

And again by Theorem 16.4

$$z_q(\mathfrak{U}') \frown' \pi_{\mathfrak{U}'\mathfrak{B}} z^n(\mathfrak{B}) \sim z_q(\mathfrak{U}') \frown \pi_{\mathfrak{U}'\mathfrak{B}} z^n(\mathfrak{B}) \qquad \text{on } \mathfrak{U}' \wedge Q. \tag{17.1f}$$

The theorem now follows from relations (17.1c)-(17.1f).

17.2 *Products of cocycles and C-cycles.* Returning to the case of the compact

space S, let $z_q(\mathfrak{U})$ be a cocycle, $\mathfrak{U} \in \Sigma$, and let $\{z^n(\mathfrak{U}')\}$ be a C-cycle. Then, letting $p + q = n$, the chain $z^p(\mathfrak{U}) = z_q(\mathfrak{U}) \frown z^n(\mathfrak{U})$ is a cycle of $\mathfrak{U}$. Let Σ' denote the set of all refinements of $\mathfrak{U}$. Then Σ' is a complete system of coverings of S. And for $\mathfrak{V} \in \Sigma'$, let $z^p(\mathfrak{V}) = \pi^*_{\mathfrak{U}\mathfrak{V}} z_q(\mathfrak{U}) \frown z^n(\mathfrak{V})$. Then the collection $\{z^p(\mathfrak{V})\}$ is a C-cycle relative to Σ'. For let $\mathfrak{W}$, $\mathfrak{V} \in \Sigma'$, where $\mathfrak{W} > \mathfrak{V}$. We must show that $\pi_{\mathfrak{V}\mathfrak{W}} z^p(\mathfrak{W}) \sim z^p(\mathfrak{V})$, where $z^p(\mathfrak{W}) = \pi^*_{\mathfrak{U}\mathfrak{W}} z_q(\mathfrak{U}) \frown z^n(\mathfrak{W})$, etc. This follows from the relations

$$\begin{aligned}
\pi_{\mathfrak{V}\mathfrak{W}} z^p(\mathfrak{W}) &= \pi_{\mathfrak{V}\mathfrak{W}}[\pi^*_{\mathfrak{U}\mathfrak{W}} z_q(\mathfrak{U}) \frown z^n(\mathfrak{W})] \\
&\sim \pi_{\mathfrak{V}\mathfrak{W}}[\pi^*_{\mathfrak{V}\mathfrak{W}}\pi^*_{\mathfrak{U}\mathfrak{V}} z_q(\mathfrak{U}) \frown z^n(\mathfrak{W})] && \text{(Theorem 16.7)} \\
&\sim \pi^*_{\mathfrak{U}\mathfrak{V}} z_q(\mathfrak{U}) \frown \pi_{\mathfrak{V}\mathfrak{W}} z^n(\mathfrak{W}) && \text{(Theorem 17.1)} \\
&\sim \pi^*_{\mathfrak{U}\mathfrak{V}} z_q(\mathfrak{U}) \frown z^n(\mathfrak{V}) && \text{(Theorem 16.8)} \\
&= z^p(\mathfrak{V}).
\end{aligned}$$

Now suppose that $\gamma_q(\mathfrak{U}')$ is a cocycle in the same element of $H_q(S)$ as $z_q(\mathfrak{U})$. Let $\mathfrak{V} > (\mathfrak{U}, \mathfrak{U}')$ such that $\pi^*_{\mathfrak{U}'\mathfrak{V}}\gamma_q(\mathfrak{U}') \backsim \pi^*_{\mathfrak{U}\mathfrak{V}} z_q(\mathfrak{U})$. Then by Theorem 16.7, $\pi^*_{\mathfrak{U}'\mathfrak{V}}\gamma_q(\mathfrak{U}') \frown z^n(\mathfrak{V}) \sim \pi^*_{\mathfrak{U}\mathfrak{V}} z_q(\mathfrak{U}) \frown z^n(\mathfrak{V}) = z^p(\mathfrak{V})$. As the common refinements of $\mathfrak{U}$ and $\mathfrak{U}'$ form a complete family, it follows that cocycles in the same element of $H_q(S)$ will determine the same homology class of $H^p(S)$, since as we have previously shown, a cycle on a complete family of coverings determines a unique homology class. Thus we have shown how, in terms of the $\frown$ product on coverings, to assign to any element of $H_q(S)$ a unique element of $H^p(S)$. And evidently this relationship is reciprocal, since to start above with a $\{\gamma^n(\mathfrak{U}')\} \sim \{z^n(\mathfrak{U}')\}$ is to generate a cycle $\{\gamma^p(\mathfrak{U})\} \sim \{z^p(\mathfrak{U})\}$.

17.3 The above process of obtaining a cap product of absolute cycles and cocycles may be extended to the various relative cases. We mention some of the cases which are most useful in the sequel. For example, suppose M is a closed subset of S, and that $\mathfrak{U}'$ denotes, for $\mathfrak{U} \in \Sigma$, the collection of all simplexes of $\mathfrak{U}$ that are on M, and $\Sigma' = \{\mathfrak{U}'\}$. We found previously (Theorem 16.9) that $\frown$ extends to a product of the cohomology classes of M mod $S - M$ on $\mathfrak{U}$ and homology classes of M on $\mathfrak{U}$ by products that may be defined as above while restricting our complexes to the $\mathfrak{U}'$ coverings. Thus we obtain a product of elements of $H_q(S; S, S - M; \mathfrak{F})$ and $H^n(S; M, 0; \mathfrak{F})$.

Another important case is that of the product of cocycles in an open set Q and the cycles mod $S - Q$; i.e., between the elements of $H_q(S; Q, 0; \mathfrak{F})$ and $H^n(S; S, S - Q; \mathfrak{F})$. This case is the analogue of (2), Theorem 16.9, the resulting products here being elements of $H^p(S; \bar{Q}, 0; \mathfrak{F})$. They are obtained as above for the absolute case $z^p(\mathfrak{U}) = z_q(\mathfrak{U}) \frown z^n(\mathfrak{U})$ where $z_q(\mathfrak{U})$ is in Q and $\{z^n(\mathfrak{U})\}$ a cycle mod $S - Q$. Theorem 17.1a replaces 17.1 in the proof that $\{z^p(\mathfrak{V})\}$ (as above) defines a C-cycle.

Actually, it will be noted, both of the above "relative" cases are only special cases of a product between elements of the groups $H_q(S; P, Q; \mathfrak{F})$ and $H^n(S; CQ, CP; \mathfrak{F})$, where as before P and Q are open subsets of S such that

$P \supset Q$, and where CQ, CP denote the complements in S of the sets Q, P respectively. This general case, where both the cocycles and cycles involved are relative, will not be needed in the sequel, so that we do not give the details of its verification. (Note that the above products yield elements of $H^{n-q}(S; \overline{P} \cap CQ, 0; \mathfrak{F})$.)

18. **Scalar products and dual pairings.** We have already used above (cf. (16a) and 16 III) a scalar product of n-chains in the definition of the $\frown$ product: For any two n-chains, $(\sum_i a^i\sigma_i^n)\cdot(\sum_i b^i\sigma_i^n) = \sum_i a^ib^i$.

If $C^0 = \sum_i a^i\sigma_i^0$ is any 0-chain, then (cf. Definition 2.1) the *Kronecker index* of the chain, $\mathrm{Ki}(C^0)$, is $\sum_i a^i$. With $\alpha = 1$ in the cap product (the value which we assume used throughout from now on), it will be noted that if C_1^n and C_2^n are n-chains of a complex K, then

$$C_1^n\cdot C_2^n = \mathrm{Ki}(C_1^n \frown C_2^n). \tag{18a}$$

The dot product is both commutative and distributive. If L^n, Q^n are any two groups of n-chains, and A is a fixed element of L^n, then for any element B of Q^n we may define $f_A(B) = A\cdot B$. The mapping $f_A : Q^n \to \mathfrak{F}$ is a homomorphism of the additive group Q^n into $\mathfrak{F}$. And similarly if $g_B(A) = A\cdot B$ for fixed $B \in Q^n$, $A \in L^n$, then $g_B : L^n \to \mathfrak{F}$ is a homomorphism. Using the notation introduced by Pontrjagin [a] we let (L^n, Q^n) denote the set of all elements A of L^n such that $f_A : Q^n \to 0$; such elements A are called *annihilators* of Q^n in L^n. *The set* (L^n, Q^n) *is a vector subspace of* L^n.

The following lemma is fundamental:

18.1 Lemma. *If $A^n \in C^n(K, \mathfrak{F})$, $B^{n-1} \in C^{n-1}(K, \mathfrak{F})$, then $A^n\cdot\delta B^{n-1} = (\partial A^n)\cdot B^{n-1}$.*

Proof. If $A^n = \sum_i a^i\sigma_i^n$, $B^{n-1} = \sum_j b^j\sigma_j^{n-1}$, then $A^n\cdot\delta B^{n-1} = \sum_{ij} a^ib^j\eta_j^i$ (where η_j^i is the incidence number: i.e., $\delta\sigma_j^{n-1} = \sum_i \eta_j^i\sigma_i^n$). On the other hand, $(\partial A^n)\cdot B^{n-1} = \sum_{j,i} a^i\eta_j^ib^j$.

In the next few theorems, where no mention is made of the complex under consideration, we assume that we are working with a complex K, and that by $C^n, Z^n, B^n, \cdots$ we denote $C^n(K; \mathfrak{F}), Z^n(K; \mathfrak{F}), B^n(K; \mathfrak{F}), \cdots$.

18.2 Theorem. *For every dimension n,*

$$Z^n = (C^n, B_n) \text{ and } Z_n = (C^n, B^n). \tag{18.2a}$$

Proof. We give the proof of only the first relation, that of the second being similar.

Let $z^n \in Z^n$. Then $\partial z^n = 0$, and hence for any $c^{n-1} \in C^{n-1}$, $z^n\cdot\delta c^{n-1} = (\partial z^n)\cdot c^{n-1} = 0$ by Lemma 18.1. Consequently $Z^n \subset (C^n, B_n)$.

To show that $Z^n \supset (C^n, B_n)$, suppose that $A^n = \sum_i a^i\sigma_i^n$ is an annihilator of B_n. Then for every δc^{n-1}, $A^n\cdot\delta c^{n-1} = 0$, and hence by Lemma 18.1, $(\partial A^n)\cdot c^{n-1} = 0$. In particular, for any chain σ_j^{n-1}, $(\partial A^n)\cdot\sigma_j^{n-1} = \sum_i a^i\eta_j^i = 0$; that is, the coefficient of σ_j^{n-1} in ∂A^n is 0 and A^n is a cycle.

The following are "relative" forms of Theorem 18.2.

18.2a THEOREM. *Relations* (18.2a) *continue to hold if* Z^n *is the group of cycles* mod L *(L a closed subcomplex of K), B^n the elements of Z^n that bound* mod L, *Z_n the group of cocycles of $K - L$, and B_n the elements of Z_n that cobound in $K - L$.*

PROOF. The first inclusion in the proof of Theorem 18.2 holds since $(\partial z^n)\cdot c^{n-1} = 0$ if ∂z^n is a chain of L and c^{n-1} is a chain of $K - L$.

The second part of the proof of Theorem 18.2 may be adapted likewise, keeping in mind that the coboundary of a chain in $K - L$ is in $K - L$.

18.2b THEOREM. *Relations* (18.2a) *continue to hold if Z^n is the group of cycles on L, B^n of cycles that bound on L, Z_n the group of cocycles of K* mod $K - L$ *and B_n the elements of Z_n that cobound* mod $K - L$.

18.3 THEOREM. *For every dimension n,*

$$B^n = (C^n, Z_n) \qquad \textit{and} \qquad B_n = (C^n, Z^n). \tag{18.3a}$$

PROOF OF FIRST EQUATION. That $B^n \subset (C^n, Z_n)$ follows from the fact that the product of a cocycle by a bounding cycle is zero (Lemma 18.1). To show that $B^n \supset (C^n, Z_n)$, let $c^n = \sum_{i=1}^{k(n)} a^i\sigma_i^n$ and $c^n \cdot z_n = 0$ for all cocycles z_n. Denoting z_n generally by $\sum_{i=1}^{k(n)} b^i\sigma_i^n$, this implies that

$$\sum_{i=1}^{k(n)} a^i b^i = 0 \qquad \text{if for } k = 1, 2, \cdots, k(n+1),\ \sum_{i=1}^{k(n)} b^i\eta_i^k = 0, \tag{18.3b}$$

where η_i^k is the incidence number.

We have to show that there exists $c^{n+1} = \sum_{k=1}^{k(n+1)} f^k\sigma_k^{n+1}$ such that $\partial c^{n+1} = c^n$. In other words, we have to find a solution $\{f^k\}$ for the system of equations

$$\sum_{k=1}^{k(n+1)} f^k\eta_i^k = a^i, \qquad i = 1, 2, \cdots, k(n), \tag{18.3c}$$

under the condition that (18.3b) holds. Inasmuch as (18.3b) is precisely the condition which guarantees the consistency of the system (18.3c), the required elements f^k of $\mathfrak{F}$ exist.

18.3a THEOREM. *Relations* (18.3a) *continue to hold for the meanings assigned to Z^n, B^n, etc., in Theorem* 18.2a.

18.3b THEOREM. *Relations* (18.3a) *continue to hold for the meanings assigned to Z^n, B^n, etc., in Theorem* 18.2b.

18.4 THEOREM. *In order that a cycle z^n should bound on K, it is necessary and sufficient that for every cocycle z_n of K, $z_n \cdot z^n = 0$. And in order that a cocycle z_n should cobound on K, it is necessary and sufficient that for every cycle z^n of K, $z_n \cdot z^n = 0$.*

Theorem 18.4 is a corollary of Theorem 18.3.

The relative forms of Theorem 18.4; corollaries of Theorems 18.3a, 18.3b, are:

18.4a THEOREM. *In order that a cycle* z^n mod L *should bound* mod L, *it is necessary and sufficient that for every cocycle* z_n *of* $K - L$, $z_n \cdot z^n = 0$. *And in order that a cocycle* z_n *of* $K - L$ *should cobound in* $K - L$, *it is necessary and sufficient that for every cycle* z^n mod L, $z_n \cdot z^n = 0$.

18.4b THEOREM. *In order that a cycle* z^n *of* L *should bound on* L, *it is necessary and sufficient that for every cocycle of* K mod $K - L$, $z_n \cdot z^n = 0$. *And in order that a cocycle* z_n *of* K mod $K - L$ *should cobound* mod $K - L$, *it is necessary and sufficient that for every cycle* z^n *of* L, $z_n \cdot z^n = 0$.

18.5 COROLLARY. *The dot product between n-chains induces a dot product between homology and cohomology classes of* K; *also between homology classes* mod L *and cohomology classes of* $K - L$, *etc.*

18.6 THEOREM. *Bases* $z_1^{*n}, \cdots, z_k^{*n}$ *for* $H^n(K; \mathfrak{F})$ *and* $z_n^{*1}, \cdots, z_n^{*k}$ *for* $H_n(K; \mathfrak{F})$ *exist such that for* $z_i^n \in z_i^{*n}$, $z_n^i \in z_n^{*i}$, $z_i^n \cdot z_n^i = \delta_i^i$ *(Kronecker delta).*

PROOF. By Corollary 18.5, the dot product may be extended to a dot product between homology and cohomology classes of K. Since bases for $H^n(K; \mathfrak{F})$ and $H_n(K; \mathfrak{F})$ exist, let us suppose that $\gamma_1^{*n}, \cdots, \gamma_k^{*n}$ and $\gamma_n^{*1}, \cdots, \gamma_n^{*r}$ are bases, where $k > r$. Consider the matrix

$$\| \gamma_i^{*n} \cdot \gamma_n^{*j} \| \tag{18.6a}$$

of k columns and r rows, whose element in the ith column and jth row is $\gamma_i^{*n} \cdot \gamma_n^{*j}$. As $k > r$, there exist elements s_i of $\mathfrak{F}$ such that $\sum_{i=1}^k s_i(\gamma_i^{*n} \cdot \gamma_n^{*j}) = 0$ for $j = 1, 2, \cdots, r$. But this implies a relation $\sum_{i=1}^k (s_i \gamma_i^n \cdot \gamma_n^j) = 0$ where $\gamma_i^n \in \gamma_i^{*n}$, $\gamma_n^j \in \gamma_n^{*j}$, which in turn implies that the cycle $z^n = \sum_{i=1}^k s_i \gamma_i^n \sim 0$ by virtue of Theorem 18.4. But then $\sum_{i=1}^k s_i \gamma_i^{*n} = 0$, violating the independence of the γ_i^{*n}. Hence $k \leqq r$. In like manner it may be shown that $k \geqq r$, so that $k = r$. Finally, the elementary transformations on the matrix (18.6a) which produce the identity matrix induce, when carried out on the cycle and cocycle cosets corresponding to the respective rows and columns, the bases desired.

18.6a, 18.6b *Theorems.* (These are the theorems corresponding to Theorem 18.6 when the cohomology and homology classes correspond to the cycles and cocycles in Theorems 18.4a and 18.4b.)

In order to pass to the case of a space S, we need the following lemma:

18.7 LEMMA. *If* z^0 *is a bounding* 0*-chain, then* $\mathrm{Ki}(z^0) = 0$.

For the proof, see §3.

18.8 LEMMA. *Under the hypothesis of Theorem* 17.1, *with* $q = n$,

$$(f^* z_n') \cdot z^n = z_n' \cdot f(z^n). \tag{18.8a}$$

Lemma 18.8 follows from (18a), Theorem 17.1, Lemma 18.7 and the fact that simplicial mappings preserve the Kronecker index.

18.9 DEFINITION. If z_n and z^n are a cocycle and cycle of S, respectively, then by $z_n \cdot z^n$ will be meant that element of $\mathfrak{F}$ obtained from the product $z_n(\mathfrak{U}) \cdot z^n(\mathfrak{U})$, where $\mathfrak{U}$ is a covering of S for which z_n is defined. In the next two lemmas we furnish the basis for extending this product to a product between homology and cohomology classes of S.

18.10 LEMMA. *If $z_n(\mathfrak{U}')$ is a cocycle of S, and $\{z^n(\mathfrak{U})\}$ is a C-cycle, then for all coverings $\mathfrak{U}$ on which there is a cocycle $z_n(\mathfrak{U})$ in the same element of $H_n(S)$ as $z_n(\mathfrak{U}')$, $z_n(\mathfrak{U}) \cdot z^n(\mathfrak{U}) = z_n(\mathfrak{U}') \cdot z^n(\mathfrak{U}')$.*

PROOF. Consider any $\mathfrak{W} > (\mathfrak{U}, \mathfrak{U}')$, such that

$$\pi^*_{\mathfrak{U}\mathfrak{W}} z_n(\mathfrak{U}) \backsim \pi^*_{\mathfrak{U}'\mathfrak{W}} z_n(\mathfrak{U}'); \tag{18.10a}$$

such a covering as $\mathfrak{W}$ exists since $z_n(\mathfrak{U})$ and $z_n(\mathfrak{U}')$ are in the same element of $H_n(S)$. By Lemma 18.8,

$$\begin{aligned} \pi^*_{\mathfrak{U}'\mathfrak{W}} z_n(\mathfrak{U}') \cdot z^n(\mathfrak{W}) &= z_n(\mathfrak{U}') \cdot \pi_{\mathfrak{U}'\mathfrak{W}} z^n(\mathfrak{W}) \\ &= z_n(\mathfrak{U}') \cdot z^n(\mathfrak{U}'). \end{aligned} \tag{18.10b}$$

Similarly,

$$\pi^*_{\mathfrak{U}\mathfrak{W}} z_n(\mathfrak{U}) \cdot z^n(\mathfrak{W}) = z_n(\mathfrak{U}) \cdot z^n(\mathfrak{U}). \tag{18.10c}$$

But (18.10a) implies that

$$\pi^*_{\mathfrak{U}\mathfrak{W}} z_n(\mathfrak{U}) \cdot z^n(\mathfrak{W}) = \pi^*_{\mathfrak{U}'\mathfrak{W}} z_n(\mathfrak{U}') \cdot z^n(\mathfrak{W}). \tag{18.10d}$$

Relations (18.10b-d) imply that $z_n(\mathfrak{U}) \cdot z^n(\mathfrak{U}) = z_n(\mathfrak{U}') \cdot z^n(\mathfrak{U}')$.

18.11 LEMMA. *If $z_n(\mathfrak{U})$ is a cocycle of S and z_1^n, z_2^n are homologous C-cycles of S, then $z_n(\mathfrak{U}) \cdot z_1^n = z_n(\mathfrak{U}) \cdot z_2^n$.*

Since by definition $z_n(\mathfrak{U}) \cdot z_i^n = z_n(\mathfrak{U}) \cdot z_i^n(\mathfrak{U})$, $i = 1, 2$, Lemma 18.11 is an immediate consequence of Theorem 18.2.

18.12 THEOREM. *The dot product between cycles and cocycles induces a product between the elements of $H_n(S)$ and $H^n(S)$.*

18.13 DEFINITION. Two vector spaces V, V' over $\mathfrak{F}$ are called a *dual pair*, and said to be *dually paired to* $\mathfrak{F}$ if there is defined a multiplication $\cdot$ between the elements of V and the elements of V' such that (denoting elements of V and V' by x's and y's, respectively): (1) $x \cdot y \in \mathfrak{F}$ for all x, y; (2) the distributive laws $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ and $(x_1 + x_2) \cdot y = x_1 \cdot y + x_2 \cdot y$ hold; (3) the multiplication is linear in the sense that for a, $b \in \mathfrak{F}$, $(ax) \cdot y = a(x \cdot y)$ and $x \cdot (by) = b(x \cdot y)$. It follows from (1) and (2) that $x \cdot 0 = 0$ and $0 \cdot y = 0$. If $(V, V') = (V', V) = 0$, then the pairing is called *orthogonal.*

An immediate consequence of the theorems on K and L above is:

18.14 THEOREM. *The dot product effects an orthogonal dual pairing to $\mathfrak{F}$ between the vector spaces $H^n(K; \mathfrak{F})$ and $H_n(K; \mathfrak{F})$. And similar statements hold*

for the group pairs constituted by the homology groups mod L *and cohomology groups of* $K - L$, *and the group pairs consisting of the homology groups of* L *and the cohomology groups of* K mod $K - L$.

We now prove the analogous theorem for S:

18.15 Theorem. *The dot product effects an orthogonal dual pairing to* $\mathfrak{F}$ *between the vector spaces* $H^n(S)$ *and* $H_n(S)$.

Proof. Suppose $z^* \in H^n(S)$, $z^* \neq 0$, and let $\{z^n(\mathfrak{U})\} \in z^*$. Then $\{z^n(\mathfrak{U})\} \not\sim 0$, and accordingly there exists a covering $\mathfrak{V}$ of S such that $z^n(\mathfrak{V}) \not\sim 0$ on $\mathfrak{V}$. Then by Theorem 18.4, there exists a cocycle $z_n(\mathfrak{V})$ such that $z_n(\mathfrak{V}) \cdot z^n(\mathfrak{V}) \neq 0$. Then if z_* is the element of $H_n(S)$ determined by $z_n(\mathfrak{V})$, we have $z_* \cdot z^* \neq 0$ by Theorem 18.12.

Suppose $z_* \in H_n(S)$, $z_* \neq 0$. Let $z_n(\mathfrak{U}) \in z_*$. Then $z_n \not\backsim 0$. This means (see 15.5) that for every refinement $\mathfrak{V}$ of $\mathfrak{U}$, $\pi^*_{\mathfrak{U}\mathfrak{V}} z_n(\mathfrak{U}) \not\backsim 0$. In particular, let $\mathfrak{V}$ be a normal refinement (Theorem 10.7) of $\mathfrak{U}$. By Theorem 18.4 there exists a cycle $z^n(\mathfrak{V})$ such that $\pi^*_{\mathfrak{U}\mathfrak{V}} z_n(\mathfrak{U}) \cdot z^n(\mathfrak{V}) \neq 0$. By Lemma 18.8, $z_n(\mathfrak{U}) \cdot \pi_{\mathfrak{U}\mathfrak{V}} z^n(\mathfrak{V}) \neq 0$. Now by Theorem 10.8 the cycle $\gamma^n(\mathfrak{U}) = \pi_{\mathfrak{U}\mathfrak{V}} z^n(\mathfrak{V})$ is the coordinate on $\mathfrak{U}$ of a C-cycle $\{\gamma^n(\mathfrak{U})\}$. By definition, if γ^* is the element of $H^n(S)$ determined by $\{\gamma^n(\mathfrak{U})\}$, $z_* \cdot \gamma^* \neq 0$.

18.16 Lemma. *If the vector spaces* V *and* V' *form an orthogonal dual pair relative to a multiplication* $\cdot$, *and the dimension of either* V *or* V' *is finite, then the spaces* V, V' *have the same dimension.*

Proof. Suppose dimension $V = m <$ dimension V'. Let $x_1, \cdots, x_m$ constitute a base for V, and let $y_1, \cdots, y_n$, where $m < n$, be linearly independent elements of V'. The system of equations

$$\sum_{i=1}^{n} a_i(y_i \cdot x_j) = 0, \qquad j = 1, 2, \cdots, m, \tag{18.16a}$$

has a nontrivial solution $(a_1, \cdots, a_n)$ in $\mathfrak{F}$. Then $y = a_1 y_1 + \cdots + a_n y_n$ is an element of V' which is not zero, and such that $y \cdot x_j = 0$ for $j = 1, 2, \cdots, m$. But then $y \cdot x = 0$ for all $x \in V$ and V and V' cannot be orthogonal.

18.17 Definition. By the *n-dimensional Betti number of* S *over* $\mathfrak{F}$ will hereafter be meant the dimension of $H^n(S)$; it will be denoted by $p^n(S)$. Similarly, the *n-dimensional co-Betti number* $p_n(S)$ is the dimension of $H_n(S)$. (Cf. 9.2, 15.4.)

In view of Theorem 18.15 and Lemma 18.16, we have

18.18 Theorem. *If either the Betti number* $p^n(S)$, *or the co-Betti number* $p_n(S)$ *is finite, then they are equal.*

We remarked in 17.3 that the groups $H^n(S; M, 0; \mathfrak{F})$, $H_n(S; S, S - M; \mathfrak{F})$ are obtainable from the coverings of M which are obtained by selecting from each $\mathfrak{U} \in \Sigma$ the collection of all simplexes of $\mathfrak{U}$ on M. Consequently the dot product may be defined for these groups by the above process, and we have:

18.19 THEOREM. *If M is a closed subset of S, then the dot product effects an orthogonal dual pairing to $\mathfrak{F}$ between the vector spaces $H^n(S;\, M,\, 0;\, \mathfrak{F})$ and $H_n(S;\, S,\, S - M;\, \mathfrak{F})$.*

And if we define numbers as above for these groups, we have,

18.20 THEOREM. *If either of the numbers $p^n(S; M, 0; \mathfrak{F})$, $p_n(S;\, S,\, S - M;\, \mathfrak{F})$ is finite, then they are equal.*

For the case of the cohomology groups of an open subset Q of S and the homology groups of S mod $S - Q$, the reader may verify the following analogue of Lemma 18.8:

18.21 LEMMA. *Under the hypothesis of Theorem* 17.1a, *with* $q = n$, $\pi^*_{\mathfrak{U}'\mathfrak{V}} z_n(\mathfrak{U}') \cdot z^n(\mathfrak{V}) = z_n(\mathfrak{U}') \cdot \pi_{\mathfrak{U}'\mathfrak{V}} z^n(\mathfrak{V})$.

(The proof follows from (18a), Theorem 17.1a and Lemma 18.7.)

18.22 DEFINITION. If $z_n(\mathfrak{U}')$ is a cocycle in Q and $\{z^n(\mathfrak{U})\}$ a cycle mod $S - Q$, then by $z_n(\mathfrak{U}') \cdot \{z^n(\mathfrak{U})\}$ we mean $z_n(\mathfrak{U}') \cdot z^n(\mathfrak{U}')$.

The reader may verify lemmas analogous to Lemmas 18.10 and 18.11 for the case of cocycles in Q and cycles mod $S - Q$. (Relation (18.10a), for example, will be "in Q", and Lemma 18.21 replaces 18.8 in the proof of Lemma 18.10.) And finally we may state the following theorem:

18.23 THEOREM. *The dot product between cocycles in Q and cycles mod $S - Q$ induces a product between the elements of $H_n(S;\, Q,\, 0;\, \mathfrak{F})$ and $H^n(S;\, S,\, S - Q;\, \mathfrak{F})$, relative to which these vector spaces form an orthogonal dual pair. Accordingly, if either of the associated numbers $p_n(S;\, Q,\, 0;\, \mathfrak{F})$, $p^n(S;\, S,\, S - Q;\, \mathfrak{F})$ is finite, then they are equal.*

Among the "working" corollaries of the theorems on orthogonality of cohomology and homology groups, one of the most important is typified by the following:

18.24 COROLLARY. *In order that a C-cycle z^n mod $S - Q$ should bound mod $S - Q$, it is necessary and sufficient that for every cocycle z_n in Q, $z_n \cdot z^n = 0$. Conversely, in order that a cocycle z_n in Q should cobound in Q, it is necessary and sufficient that for every cycle z^n mod $S - Q$, $z_n \cdot z^n = 0$.*

It is necessary, for later purposes, to recognize that in the infinite case, partial "dual bases" may be selected in a manner explained in the next theorem:

18.25 THEOREM. *Let V, V' be vector spaces which form an orthogonal dual pair relative to a multiplication $\cdot$ to a field $\mathfrak{F}$, and let $x_1, \cdots, x_n$ be linearly independent elements of V. Then there exist in V' linearly independent elements $y_1, \cdots, y_n$ such that $x_i \cdot y_j = \delta^j_i$ (Kronecker delta).*

PROOF. Let V_1 be the subspace of V generated by $x_1, \cdots, x_n$, and let $V'_1 = (V', V_1)$. If $V'_* = V'/V'_1$, the multiplication $\cdot$ induces a multiplication

$\times$ between the vector spaces V_1, V'_* as follows: For $x \in V_1$, $y_* \in V'_*$, we let $x \times y_* = x \cdot y$ where $y \in y_*$. Furthermore, V_1 and V'_* form an orthogonal pair relative to $\times$. For if $x \times y_* = 0$ for all $x \in V_1$, then $x \cdot y = 0$ for all $x \in V_1$ and y a fixed element of y_*, implying $y \in V'_1$. And if $x \times y_* = 0$ for all $y_* \in V'_*$, then $x \cdot y = 0$ for all $y \in V'$ so that $x = 0$, inasmuch as V and V' are orthogonal.

By Lemma 18.16, it follows that dimension $V'_* = n$. If $y^1_*, \cdots, y^n_*$ are independent elements of V'_*, then the rank of the matrix $\| x_i \times y^j_* \|$ is n, else a contradiction of the orthogonality will result (see proof of Theorem 18.6). If $y_j \in y^j_*$, then $\| x_i \times y^j_* \| = \| x_i \cdot y_j \|$, and elementary matrix transformations yield the desired result.

The application of Theorem 18.25 to the various sets of cohomology and homology groups over $\mathfrak{F}$ which are dually paired and orthogonal, as shown in the theorems above, should be obvious. For example, if $z^n_1, \cdots, z^n_k$ are C-cycles of S which are linearly independent with respect to homology, then there exist cocycles $z^1_n, \cdots, z^k_n$ such that $z^n_i \cdot z^j_n = \delta^j_i$. Furthermore, since if a z^j_n is given on a covering $\mathfrak{U}$, $\pi^*_{\mathfrak{U}\mathfrak{B}} z^j_n(\mathfrak{U})$ is a cocycle of $\mathfrak{B}$ that is in the same element of $H_n(S)$ as z^j_n if $\mathfrak{B} > \mathfrak{U}$, we may therefore always assume that the cocycles $z^1_n, \cdots, z^k_n$ above are on the same covering of S. Conversely, if the z^i_n above are given linearly independent with respect to cohomology (= lircoh), then cycles z^n_i "dual" to them exist.

Of course similar remarks apply to the various relative cases. In view of Theorem 18.19, for instance, if the z^n_i above are given as linearly independent with respect to homology on a closed set M, then the z^i_n are cocycles mod $S - M$; this is a situation which we frequently encounter later on. Or if the z^i_n are given as cocycles in an open set Q, and are linearly independent with respect to cohomology in Q, then the z^n_i are cycles mod $S - Q$.

Other cases will be handled as they are encountered in the sequel.

Again, partial dual bases may be augmented; this will follow from the next two lemmas.

18.26 Lemma. *If V is a vector space of at least dimension n and $x_1, \cdots, x_m$, $m < n$, are linearly independent elements of V, then there exist $x_{m+1}, \cdots, x_n \in V$ such that $x_1, \cdots, x_m, x_{m+1}, \cdots, x_n$ are linearly independent.*

18.27 Lemma. *Let V' be a vector space which forms, with the V of Lemma 18.26, an orthogonal dual pair relative to a multiplication $\cdot$ to a field $\mathfrak{F}$, and let $y_1, \cdots, y_m$ be linearly independent elements of V' such that $x_i \cdot y_j = \delta^j_i$ for i, $j \leq m$. Then there exist $x_{m+1}, \cdots, x_n \in V$ and $y_{m+1}, \cdots, y_n \in V'$ such that $x_1, \cdots, x_n$* and *$y_1, \cdots, y_n$ are respectively linearly independent sets and, moreover, $x_i \cdot y_j = \delta^j_i$ for all i, j.*

Proof. By Lemma 18.26, there exist $x'_{m+1}, \cdots, x'_n \in V$ such that $x_1, \cdots, x_m, x'_{m+1}, \cdots, x'_n$ are linearly independent. By Lemma 18.25, there exist linearly independent elements $y'_1, \cdots, y'_m, y_{m+1}, \cdots, y_n$ of V' such that

$x_i y_j = \delta_i^j$ for all i, j, neglecting primes. Then the elements $y_1, \cdots, y_m, y_{m+1}, \cdots, y_n$ are linearly independent. For suppose there were a relation

$$\text{(i)} \qquad \sum_{i=1}^{m} a_i y_i + \sum_{j=m+1}^{n} b_j y_j = 0.$$

Then not all the a_i could be zero in (i), since the elements y_j, $j > m$, are linearly independent. Let us suppose, for instance, that $a_1 \neq 0$. But then $x_1 \cdot (\sum_{i=1}^{m} a_i y_i + \sum_{j=m+1}^{n} b_j y_j) = a_1 \neq 0$, since $x_1 \cdot y_j = 0$ for $j > 1$.

Again by Lemma 18.25, there exists a set of linearly independent elements of V, say $z_1, \cdots, z_n$, such that $z_i \cdot y_j = \delta_i^j$. Then $x_1, \cdots, x_m, z_{m+1}, \cdots, z_n$ are linearly independent, by the same sort of argument as used above and if we let $x_{m+1} = z_{m+1}, \cdots, x_n = z_n$, the two sets $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ satisfy the required conditions.

In concluding this section we define two new types of homology and cohomology groups, which include those heretofore given as special cases, and give corresponding duality theorems.

18.28 Definition. By $H^r(S\colon M, L; A, B)$, where M, L, A, B are closed sets such that $M \supset L$, $A \supset B$, will be denoted the vector space of C-cycles of S on M mod L, reduced modulo the subspace of these cycles that bound on A mod B. An important and frequently used case in the sequel is that where $L = B = 0$. The corresponding dimension we denote by $p^r(S\colon M, L; A, B)$.

18.29 Definition. By $H_r(S\colon P, Q; U, V)$, where P, Q, U, V are open sets such that $P \supset Q$, $U \supset V$, will be denoted the vector space of cocycles of S in P mod Q, reduced modulo the subspace of these cocycles that cobound in U mod V. The case usually employed in the sequel will be that where $Q = V = 0$.

18.30 Theorem. *If A, B are compact subsets of a space S such that $A \supset B$, then the vector spaces $H^r(S\colon B, 0; A, 0)$ and $H_r(S\colon S, S - A; S, S - B)$ form an orthogonal dual pair relative to the dot product. Consequently if either is of finite dimension, they are isomorphic.*

Proof. We have already observed in Theorem 18.19 that for any cycle γ^r on B that fails to bound on A, there exists a cocycle Z_r mod $S - A$ such that $Z_r \cdot \gamma^r \neq 0$. Conversely, if a cocycle Z_r mod $S - A$ fails to cobound mod $S - B$, then Z_r is a cocycle mod $S - B$ that fails to cobound mod $S - B$, and there must exist a cycle γ^r on B such that $Z_r \cdot \gamma^r \neq 0$.

18.31 Theorem. *If P and Q are open subsets of a space S such that $\overline{P}$ is compact and $P \supset Q$, then $H_r(S\colon Q, 0; P, 0)$ and $H^r(S\colon S, S - P; S, S - Q)$ form an orthogonal dual pair. Hence if the dimension of either of them is finite, they are isomorphic.*

Remark. The inequality of dimensions of orthogonal dual pairs of infinite-dimensional spaces is exemplified as follows: In the cartesian plane, let $S_k = \{(x, y) \mid (x - 1/k)^2 + y^2 = 1/4k^2(k + 1)^2\}$, $k = 1, 2, 3, \cdots$. With $p = (0, 0)$,

let $S = \bigcup S_k \cup p$. As we shall see later, $p^1(S) = c$, the cardinality of the real numbers, while $p_1(S) = \aleph_0$.

19. **Applications to homology properties of spaces.** The remark following theorem 18.25 concerning the ability to assume of a given finite set of cocycles that they are on the same covering leads to the following important theorems.

19.1 THEOREM. *If $z_n^1, \cdots, z_n^k$ are cocycles that are linearly independent relative to cohomology on S, then there exists a covering $\mathfrak{U}$ such that for all $\mathfrak{V} > \mathfrak{U}$, $z_n^1(\mathfrak{V}), \cdots, z_n^k(\mathfrak{V})$ are linearly independent relative to cohomology on $\mathfrak{V}$.*

Theorem 19.1 is a direct consequence of the definition of cohomology on S.

19.2 THEOREM. *If $Z_1^n, \cdots, Z_k^n$ are C-cycles that are lirh on S, then there exists a covering $\mathfrak{U}$ such that for all $\mathfrak{V} > \mathfrak{U}$, $Z_1^n(\mathfrak{V}), \cdots, Z_k^n(\mathfrak{V})$ are lirh on $\mathfrak{V}$.*

PROOF. As shown above, there exist cocycles $Z_n^1, \cdots, Z_n^k$ such that $Z_i^n \cdot Z_n^j = \delta_i^j$, and we may assume that $\mathfrak{U}$ is such that the Z_n^j are all on $\mathfrak{U}$. Let $\mathfrak{V} > \mathfrak{U}$. Then the $Z_i^n(\mathfrak{V})$ are lirh on $\mathfrak{V}$. For suppose there exists a relation $\sum c^i Z_i^n(\mathfrak{V}) \sim 0$ on $\mathfrak{V}$. Then by Theorem 18.4, $[\sum c^i Z_i^n(\mathfrak{V})] \cdot z_n^j(\mathfrak{V}) = 0$ for all j. Since not all coefficients c^i are zero, we may suppose $c^1 \neq 0$. But then we have $[\sum c^i Z_i^n(\mathfrak{V})] \cdot Z_n^1(\mathfrak{V}) = c^1 Z_1^n(\mathfrak{V}) \cdot Z_n^1(\mathfrak{V}) = c^1 \neq 0$.

19.3 THEOREM. *If $p^n(S; \mathfrak{F})$ is finite, then there exists a covering $\mathfrak{U}$ of S such that if L is a closed subset of $U \in \mathfrak{U}$ and Z^n is an augmented C-cycle on L, then $Z^n \sim 0$ on S. If S is regular, a $\mathfrak{U}$ exists such that every augmented n-dimensional C-cycle on an element of $\mathfrak{U}$ bounds on S.*

PROOF. Since $p^n(S; \mathfrak{F})$ is finite, there exists a finite set of C-cycles, say $Z_1^n, \cdots, Z_k^n$, which are lirh on S and such that every Čech n-cycle is related to a linear combination of these by a homology on S. By Theorem 19.2 there is a covering $\mathfrak{U}$ such that the coordinates of these cycles on $\mathfrak{U}$ are lirh on all refinements of $\mathfrak{U}$. Suppose L and Z^n are as in the statement of the theorem. Let $\mathfrak{V} > \mathfrak{U}$ such that every element of $\mathfrak{V}$ that meets L lies in U.

Now we may choose $\pi_{\mathfrak{U}\mathfrak{V}}$ in such a way that $\pi_{\mathfrak{U}\mathfrak{V}}$ maps all elements of $\mathfrak{V}$ that meet L into U, and consequently $\pi_{\mathfrak{U}\mathfrak{V}} Z^n(\mathfrak{V}) \sim 0$ on $\mathfrak{U}$. But there exists a relation $Z^n \sim \sum c^i Z_i^n$ on S; in particular, $Z^n(\mathfrak{U}) \sim \sum c^i Z_i^n(\mathfrak{U})$. But by definition of C-cycle, $Z^n(\mathfrak{U}) \sim \pi_{\mathfrak{U}\mathfrak{V}} Z^n(\mathfrak{V})$, which as shown above is homologous to zero on $\mathfrak{U}$. It follows that $\sum c^i Z_i^n(\mathfrak{U}) \sim 0$ on $\mathfrak{U}$. But there cannot exist any such relation by virtue of the choice of $\mathfrak{U}$ above unless all $c^i = 0$. Hence $Z^n \sim 0$ on S.

The concluding statement of the theorem follows from Lemma 8.2.

19.4 DEFINITION. A space S is called *semi-n-connected at $x \in S$* if there exists a neighborhood U of x such that all Čech n-cycles on closed subsets of U bound on S; S is called *semi-n-connected* if it is semi-n-connected at every point. (If S is regular, the words "on closed subsets of" may be replaced by "on", of course.) In particular, if S is compact, then S is semi-n-connected if

and only if there exists a fcos $\mathfrak{U}$ of S such that every Čech n-cycle that is on an element of $\mathfrak{U}$ bounds on S. A space S such that $p^n(S) = 0$ is sometimes called *simply n-connected.*

As a corollary of Theorem 19.3 we can state:

19.5 Corollary. *For any space S, $p^n(S; \mathfrak{F})$ finite implies that S is semi-n-connected.*

That a compact space which is semi-0-connected must also have a finite 0-dimensional Betti number is easily shown, since such a space can have only a finite number of components. That semi-n-connectedness does not imply a finite n-dimensional Betti number when $n > 0$, however, may be seen from the following example:

19.6 Example. In the cartesian plane let S consist of (1) all points on the unit square having vertices at (1, 0), (1, 1), (0, 1), but no points inside this square except such as (2) lie on lines $x = 1/n$, n a natural number. Then for any covering $\mathfrak{U}$ of S all of whose elements are of diameter less than 1/2, it will be true that a Čech-1-cycle of S lying in an element of $\mathfrak{U}$ will bound on S, so that S is semi-1-connected. However, the number $p^1(S; \mathfrak{F})$ is infinite.

19.7 Theorem. *If M is a compact subset of a locally compact space S, and γ_i^n, $i = 1, \cdots, k$, are a finite number of C-cycles that are lirh on M, then there exists an open set Q containing M such that the cycles γ_i^n are lirh on Q.*

Proof. By Theorem 19.2, there exists a covering $\mathfrak{U}$ such that for all $\mathfrak{V} > \mathfrak{U}$, the $\gamma_i^n(\mathfrak{V})$ are lirh on M. Let $\mathfrak{V} > \mathfrak{U}$ and Q be as in Lemma 8.7. Then a homology $\sum a^i\gamma_i^n \sim 0$ on Q would imply $\sum a^i\gamma_i^n(\mathfrak{V}) \sim 0$ on Q and hence on M.

20. **Homologies in noncompact spaces.** Up to this point, we have been using only finite coverings of space. This restriction, while of no consequence when the space is compact, may cause difficulty in dealing with noncompact spaces, inasmuch as the fcos do not form a complete family of coverings therein. For this reason another type of homology theory, needed in the applications later on, may logically be inserted at this point. It is especially useful in the case of a space which is the union of a countable collection of open sets that have compact closures, as for instance an open subset of a perfectly normal, compact space:

20.1 Definition. A subset of a space S is called an F_σ in S if it is the union of a countable collection of closed subsets of S. Then a space S is called *perfectly normal* if (1) S is normal, and (2) every open subset of S is an F_σ in S.

Remark. A subset of a space S is called a G_δ in S if it is the intersection of a countable collection of open subsets of S. Condition (2) of 20.1 may be replaced by "every closed subset of S is a G_δ."

20.2 Definition. If S is any space, then by an *unrestricted covering* of S

we shall mean any covering of the space by open sets without restriction as to the number of elements in the covering; we abbreviate the term by the symbol *ucos*. If S is a subset of a space T, then an ucos of S by subsets of S that are open rel. S will be called an *internal* ucos of S.

20.3 Definition. If S is any space, then by the *unrestricted Čech homology theory* of S we shall mean a homology theory set up in the following fashion: If $\mathfrak{U}$ is an ucos of S, then a cycle of $\mathfrak{U}$ is a finite cycle (cf. §2). Such a cycle bounds on $\mathfrak{U}$ if it bounds a finite chain of $\mathfrak{U}$. In terms of finite cycles as coordinates, and with homology meaning the bounding of finite chains by the coordinates, Čech cycles and homology groups are defined as before. Using these Čech cycles and homologies, the theorems of §7 offer no difficulty—their proofs are virtually the same as before. In the unrestricted Čech theory, we indicate that a Čech cycle Z^r bounds by the expression $Z^r \approx 0$, and the numbers analogous to $p^n(S)$ will be denoted by the symbols $p^n(S, \approx)$. And if P is a subset of S, "$Z^r \approx 0$ in P" expresses a homology in terms of internal ucos of P.

We shall prove a theorem of fundamental importance for the later applications. We need the following lemmas:

20.4. Lemma. *If a space S is the union of a countable collection of open sets with compact closures, then $S = \bigcup_{n=1}^{\infty} P_n$, where P_n is an open set with compact closure and for all n, $\overline{P}_n \subset P_{n+1}$.*

Proof. Let $S = \bigcup_{i=1}^{\infty} U_i$, where U_i is open and $\overline{U}_i$ is compact. Let $P_1 = U_1$. Having defined P_n, $n \geqq 1$, let $U_{i(1)}, \cdots, U_{i(m)}$ be a finite collection of elements of $\{U_i\}$ covering $\overline{P}_n$. Let $P_{n+1} = U_{n+1} \cup \bigcup_{j=1}^{m} U_{i(j)}$.

20.5 Lemma. *If S is a perfectly separable, locally compact space, then S has a countable basis* (III 1.8) *of open sets with compact closures.*

Proof. If $\{U_n\}$ is a countable basis for S, let $\{U_{n(i)}\}$ be the collection of all elements of $\{U_n\}$ that have compact closures. Then $\{U_{n(i)}\}$ is a basis of the required type.

20.6 Corollary. *If S is a perfectly separable, locally compact space, then $S = \bigcup_{n=1}^{\infty} P_n$, where P_n is an open set with compact closure and for every n, $\overline{P}_n \subset P_{n+1}$.*

20.7 Theorem. *Let S be a space that is the union of a countable collection of open sets that have compact closures. Then the countable, star-finite coverings of S form a complete family of ucos of S.*

[An ucos is a *countable, star-finite covering* of S if it has only a countable collection of elements and no element meets more than a finite number of other elements of the collection.]

Proof. By Lemma 20.4, $S = \bigcup_{n=1}^{\infty} P_n$, such that for every n, P_n is open and $\overline{P}_n$ is a compact subset of P_{n+1}. Let $\mathfrak{U}$ be an ucos of S.

Since $\bar{P}_n$ is compact, there exists a finite number of elements of $\mathfrak{U}$, say $U_{n,1}, \cdots, U_{n,m(n)}$, which cover $\bar{P}_n$. Let $P_0 = 0$. Let $V_{n,i} = U_{n,i} \cap (P_{n+1} - \bar{P}_{n-1})$, $i = 1, \cdots, m(n)$. Then the collection $\{V_{n,i}\}$ is the required countable star-finite refinement of $\mathfrak{U}$.

(It will be noted that this proof incidentally shows that the elements of the coverings may be so chosen as to have compact closures.)

20.8 Corollary. *If S is a perfectly separable, locally compact space, then the countable, star-finite coverings of S form a complete family of ucos of S.*

20.9 Corollary. *If a space S is homeomorphic with an open, F_σ subset of a compact space, then the countable, star-finite coverings of S form a complete family of ucos of S; hence if S is an open subset of a perfectly normal, compact space, the same conclusion follows.*

20.10 Theorem. *Let M be a topological space that is homeomorphic with a subset of a perfectly normal, compact space S. Then the countable, star-finite coverings of M form a complete family of ucos of M.*

Proof. We may suppose $M \subset S$. Let $\mathfrak{U}$ be an internal ucos of M. For each $U_\nu \in \mathfrak{U}$, let U'_ν be an open subset of S such that $U'_\nu \cap M = U_\nu$. Then $P = \bigcup_\nu U'_\nu$ is an open subset of S and $\mathfrak{U}' = \{U'_\nu\}$ is an ucos of P. By Corollary 20.9, the countable, star-finite coverings of P form a complete family of ucos of P. Hence there exists a countable, star-finite refinement $\mathfrak{V}' = \{V'_i\}$ of $\mathfrak{U}'$ which covers P. For each i, let $V'_i \cap M = V_i$.

Then $\mathfrak{V} = \{V_i\}$ is a star-finite covering of M that is a refinement of $\mathfrak{U}$.

Since every perfectly separable, normal space is homeomorphic with a subset of the Hilbert fundamental parallelopiped, by Theorem III 1.14, we have as a corollary of Theorem 20.10,

20.11 Corollary. *The countable, star-finite coverings form a complete family of ucos for a perfectly separable, normal space.*

We conclude this section with some lemmas useful in later applications of the above:

20.12 Lemma. *Let M be a subset of a completely normal space S and let $U_1, \cdots, U_n$ be subsets of M open rel. M. Then if $\bigcap_{i=1}^n U_i = 0$, there exist sets V_i open in S such that $V_i \supset U_i$, $i = 1, \cdots, n$, and $\bigcap_{i=1}^n V_i = 0$.*

Proof. The lemma being trivial for $k = 1$, we use a mathematical induction argument. Let k denote an integer for which the lemma holds, and let $U_1, \cdots, U_k, U_{k+1}$ be $k + 1$ open subsets of M with empty intersection. Then $U = \bigcap_{i=1}^k U_i$ and U_{k+1} are separated sets (unless $U = 0$, in which case sets V_i, $i = 1, \cdots, k$, exist as in the lemma and we may let $V_{k+1} \supset U_{k+1}$), and since S is completely normal, there exist sets V, V_{k+1} open in S such that $V \supset U$, $V_{k+1} \supset U_{k+1}$, and $V \cap V_{k+1} = 0$. For $i \leq k$, let $U'_i = U_i - U$. By the induction assumption (applied to $M - U$) there exist sets V'_i open in S such that

$V'_i \supset U'_i$ and $\bigcap V'_i = 0$. Let $V_i = V'_i \cup V$. The sets $V_1, \cdots, V_k, V_{k+1}$ satisfy the required conditions.

20.13 Definition. If M is a subset of a space S, and P is an open set containing M, then an internal ucos $\mathfrak{U}$ of P will be called a *neighborhood covering* of M. If $\mathfrak{V}$ is another neighborhood covering of M such that each $V \in \mathfrak{V}$ is contained in some $U \in \mathfrak{U}$, then $\mathfrak{V}$ is called a refinement of $\mathfrak{U}$, symbolized $\mathfrak{V} > \mathfrak{U}$ as before. Note that $\mathfrak{V} > \mathfrak{U}$ implies that the open set covered by $\mathfrak{V}$ is a subset of P.

20.14 Lemma. *Let M be a subset of a completely normal space S, and $\mathfrak{U} = \{U_i\}$ a countable star-finite internal ucos of M. Then there exists a neighborhood covering $\mathfrak{U}' = \{U'_i\}$ of M such that $U'_i \cap M = U_i$ and the complex $\mathfrak{U}$ is isomorphic* (cf. III 4.8, footnote) *with the complex $\mathfrak{U}'$ under the correspondence $U_i \leftrightarrow U'_i$.*

Proof. For each i, U_i and $\bigcup_j U_j$, where j runs through those indices such that $U_i \cap U_j = 0$, are separated sets. Hence there exist disjoint open sets A_i and B_i containing U_i and $\bigcup_j U_j$ respectively; we may select A_i so that $A_i \cap M = U_i$.

Let $\{G_h\}$ be the set of all finite subsets of the set of natural numbers $\{n\}$ such that: If G_h is the set $h(1), \cdots, h(m)$, then (1) each $U_{h(i)}$ meets all the other sets $U_{h(j)}$, $j = 1, \cdots, \hat{\imath}, \cdots, m$; (2) $\bigcap_{i=1}^m U_{h(i)} = 0$. The sets G_h are countable in number and each natural number n occurs in only a finite number of them. And by Lemma 20.12 there exist for each G_h sets $W^h_{h(i)}$, $i = 1, \cdots, m$, open in S, such that $W^h_{h(i)} \supset U_{h(i)}$ and $\bigcap_{i=1}^m W^h_{h(i)} = 0$. Let $U'_i = A_i \cap \bigcap_h W^h_i \cap \bigcap_j B_j$, where h runs through the values of h for which $i \in G_h$, and j runs through the natural numbers $< i$ for which $U_i \cap U_j = 0$. Then $P = \bigcup_{i=1}^\infty U'_i$ is a neighborhood of M and $\mathfrak{U}' = \{U'_i\}$ a neighborhood covering of M having the desired properties.

20.15 Lemma. *If S is perfectly normal, then S is completely normal.*

Proof. Let A, B be separated subsets of S. Since S is perfectly normal, $S - \bar{B} = \bigcup_{i=1}^\infty M_i$, where M_i is closed in S. Let $A \cap M_i = A_i$. Then $\bar{A}_i \subset M_i \subset S - \bar{B}$, and as S is normal, there exists an open set P_i such that $\bar{A}_i \subset P_i \Subset S - \bar{B}$. In a similar manner we obtain sets B_i rel. B and open sets Q_i such that $\bar{B}_i \subset Q_i \Subset S - \bar{A}$.

Now $\bar{A}_i \cap \bar{B}_i = 0$, and there exist disjoint open sets U''_i, V''_i containing $\bar{A}_i$, $\bar{B}_i$ respectively. Let $U'_i = P_i \cap U''_i$, $V'_i = Q_i \cap V''_i$. Then U'_i, V'_i are disjoint open sets containing $\bar{A}_i$, $\bar{B}_i$ respectively, and such that $\bar{U}'_i \subset S - \bar{B}$, $\bar{V}'_i \subset S - \bar{A}$.

Define $U_0 = V_0 = 0$, and for $i > 0$ let $U_i = U'_i - \bigcup_{j=0}^{i-1} \bar{V}_j$, $V_i = \bar{V}'_i - \bigcup_{j=0}^{i-1} \bar{U}_j$. Finally, let $U = \bigcup_{i=1}^\infty U_i$, $V = \bigcup_{i=1}^\infty V_i$. Then U and V are disjoint open sets containing A and B respectively.

20.16 COROLLARY. *Every subset of a perfectly normal space is itself a perfectly normal space.*

21. **Approximate homologies.** A notion sometimes useful is that of a C-cycle approximately on a given set. It arises from the fact that a C-cycle may be on each element of a decreasing sequence of compact point sets and yet not be on their intersection—a circumstance which rules out the possibility of a "smallest" carrier of the cycle, for instance (see VII 2, however). We assume throughout this section that all coverings are finite.

21.1 DEFINITION. A C-cycle γ^r will be said to be *approximately on* a set M if for arbitrary open set $P \supset M$ there exists a covering $\mathfrak{U}$ such that if $\mathfrak{W} > \mathfrak{V} > \mathfrak{U}$, then $\pi_{\mathfrak{V}\mathfrak{W}}\gamma^r(\mathfrak{W}) \sim \gamma^r(\mathfrak{V})$ on P. Such a covering $\mathfrak{U}$ will be said to *govern* γ^r on P. ($\mathfrak{U}$ governs γ^r on P, then, if γ^r, in terms of the complete family of all refinements of $\mathfrak{U}$, is on P.)

21.2 THEOREM. *If $M \subset S$, then the Čech r-cycles approximately on M form a vector space $Z^r_P(M, \mathfrak{F})$.*

21.3 DEFINITION. If γ^r is a C-cycle approximately on M, then γ^r will be said to *bound approximately on M*, or to be *homologous to zero approximately on M*, if for arbitrary open set $P \supset M$ there exists a covering $\mathfrak{U}$ such that if $\mathfrak{V} > \mathfrak{U}$, then $\gamma^r(\mathfrak{V}) \sim 0$ on P. Such a covering $\mathfrak{U}$ will be said to *govern bounding of*, or *homologies of*, γ^r *on* P.

21.4 THEOREM. *If $M \subset S$, then the Čech r-cycles approximately on M that bound approximately on M form a vector space $B^r_P(M, \mathfrak{F})$.*

21.5 The space $Z^r_P(M, \mathfrak{F})/B^r_P(M, \mathfrak{F})$ will be called the *r-dimensional homology group approximately on M*, and will be denoted by the symbol $H^r_P(M, \mathfrak{F})$.

[It would be preferable to include the "S" in these new symbols—$H^r_P(S; M, \mathfrak{F})$, for instance—since the new cycles make little sense without S; but we omit the S for brevity.]

In the case where M is a compact subset of a locally compact space S (instead of local compactness, we could impose normality on S; it is the former hypothesis that is needed in later applications), we shall establish an isomorphism between $H^r(M; \mathfrak{F})$ and $H^r_P(M, \mathfrak{F})$. Since, as is easily seen, the group $H^r_P(M, \mathfrak{F})$ is not changed by restriction to a complete family of coverings, we shall assume throughout the proof that *all coverings employed have the property that if all vertices of a cell meet M, then the cell is on M*; under the given conditions, the set of all such coverings forms a complete family (Lemma 8.8).

21.6 DEFINITION. Let M be a subset of a space S, and $\mathfrak{U}$ a covering of S. Let $G = \mathrm{St}(M, \mathfrak{U})$ and let P be an open set such that $M \subset P \Subset G$. Then the covering $\mathfrak{V}$ which consists of the elements of $\mathfrak{U} \cap M$ (7.6) and of the sets $U - \bar{P}$ for all $U \in \mathfrak{U} - \mathfrak{U} \cap M$ will be called a *refinement of* $\mathfrak{U}$ *adjusted to* M.

If we wish to fix $\mathfrak{V}$ more exactly, we call it a *refinement of* $\mathfrak{U}$ *adjusted to* M *by means of* P.

21.7 Lemma. *If* M *is a closed subset of a normal space* S *and* $\mathfrak{U}$ *is a covering of* S, *then there exists a refinement of* $\mathfrak{U}$ *adjusted to* M.

21.8 Lemma. *If* $\mathfrak{V}$ *is a refinement of* $\mathfrak{U}$ *adjusted to* M *by means of* P, *then* $\mathfrak{V} \cap P = \mathfrak{U} \cap M$.

Now suppose γ^r is a C-cycle approximately on $M \subset S$, and let $\mathfrak{U}$ be any covering of S. Let $G = \mathrm{St}(M, \mathfrak{U})$, and let P be an open set such that $M \subset P \Subset G$. Let $\mathfrak{V}' = \mathfrak{V}'(\mathfrak{U}, P)$ be a refinement of $\mathfrak{U}$ adjusted to M by means of P, and let $\mathfrak{U}' = \mathfrak{U}'(P)$ govern γ^r on P. Finally, let $\mathfrak{U}'' = \mathfrak{U}''(\mathfrak{U}, P) > (\mathfrak{U}', \mathfrak{V}')$. Let $Z^r(\mathfrak{U}) = \pi_{\mathfrak{U}\mathfrak{V}'}\pi_{\mathfrak{V}'\mathfrak{U}''}\gamma^r(\mathfrak{U}'') = \pi_{\mathfrak{U}\mathfrak{U}''}\gamma^r(\mathfrak{U}'')$ (cf. Lemma 21.8). Although $Z^r(\mathfrak{U})$ is not unique, this is of no importance for our purposes.

21.9 Remark. If γ^r is actually on M, then we may take $Z^r(\mathfrak{U}) = \gamma^r(\mathfrak{U})$. For if a cycle is on M, it is approximately on M, and any covering governs it on P. Hence we may let $\mathfrak{U}'(P) = \mathfrak{U}$ and take $\mathfrak{U}'' = \mathfrak{V}' > (\mathfrak{U}, \mathfrak{V}')$. Then $Z^r(\mathfrak{U}) = \pi_{\mathfrak{U}\mathfrak{U}''}\gamma^r(\mathfrak{U}'') = \pi_{\mathfrak{U}\mathfrak{V}'}\gamma^r(\mathfrak{V}') = \pi_{\mathfrak{U}\mathfrak{U}}\gamma^r(\mathfrak{U}) = \gamma^r(\mathfrak{U})$.

21.10 Lemma. *The collection* $\{Z^r(\mathfrak{U})\}$ *is a* C*-cycle on* M.

Proof. Let $\mathfrak{V} > \mathfrak{U}$. Let $Z^r(\mathfrak{U})$ and $Z^r(\mathfrak{V})$ be defined as above. We denote the sets of type P used in the definition of $Z^r(\mathfrak{U})$ and $Z^r(\mathfrak{V})$ by P and P', respectively. When we wish to distinguish, for instance, the sets $\mathfrak{U}''$ used in the definitions, we attach the respective arguments—thus, $\mathfrak{U}''(\mathfrak{U}, P)$, $\mathfrak{U}''(\mathfrak{V}, P')$, although we shall omit the arguments unless needed. Let $\mathfrak{W} > (\mathfrak{U}''(\mathfrak{U}, P), \mathfrak{U}''(\mathfrak{V}, P'))$. Then the following relations hold:

$$\pi_{\mathfrak{U}''\mathfrak{W}}\gamma^r(\mathfrak{W}) \sim \gamma^r(\mathfrak{U}'') \qquad \text{on } P,$$

$$\pi_{\mathfrak{U}\mathfrak{U}''}\pi_{\mathfrak{U}''\mathfrak{W}}\gamma^r(\mathfrak{W}) \sim \pi_{\mathfrak{U}\mathfrak{U}''}\gamma^r(\mathfrak{U}'') = Z^r(\mathfrak{U}) \qquad \text{on } M. \tag{21.10a}$$

And similarly we have the relations:

$$\pi_{\mathfrak{U}''\mathfrak{W}}\gamma^r(\mathfrak{W}) \sim \gamma^r(\mathfrak{U}'') \qquad \text{on } P',$$

$$\pi_{\mathfrak{V}\mathfrak{U}''}\pi_{\mathfrak{U}''\mathfrak{W}}\gamma^r(\mathfrak{W}) \sim \pi_{\mathfrak{V}\mathfrak{U}''}\gamma^r(\mathfrak{U}'') = Z^r(\mathfrak{V}) \qquad \text{on } M,$$

$$\pi_{\mathfrak{U}\mathfrak{V}}\pi_{\mathfrak{V}\mathfrak{U}''}\pi_{\mathfrak{U}''\mathfrak{W}}\gamma^r(\mathfrak{W}) \sim \pi_{\mathfrak{U}\mathfrak{V}}Z^r(\mathfrak{V}) \qquad \text{on } M. \tag{21.10b}$$

Now we consider the two cycles on the extreme left of relations (21.10a) and (21.10b). The projections involved in the above relations may be considered as projections (1) from $\mathfrak{W}$ to $\mathfrak{U}''(P) \cup \mathfrak{U}''(P')$, (2) from $\mathfrak{U}''(P) \cup \mathfrak{U}''(P')$ to $(\mathfrak{U} \cap P) \cup (\mathfrak{V} \cap P') = (\mathfrak{U} \cap M) \cup (\mathfrak{V} \cap M)$, and (3) from $(\mathfrak{U} \cap M) \cup (\mathfrak{V} \cap M)$ to $\mathfrak{U} \cap M$. Thus, the cycle on the left in (21.10a) may be expressed as $\pi_{\mathfrak{U}\,\mathfrak{U}\cup\mathfrak{V}}\pi_{\mathfrak{U}\cup\mathfrak{V}\,\mathfrak{U}''(P)\cup\mathfrak{U}''(P')}\pi_{\mathfrak{U}''(P)\cup\mathfrak{U}''(P')\,\mathfrak{W}}\gamma^r(\mathfrak{W})$. Now in the proof which we gave earlier to show that different projections map a cycle into the same homology class of a covering (Theorem 7.2), we employed a chain-mapping "P"

such that if $\sigma^r = w_0 \cdots w_r$, then $P(w_0 \cdots w_r) = \sum (-1)^i u_0 \cdots u_i v_i \cdots v_r$, where $u_i \supset w_i$, $v_i \supset w_i$, etc. If a σ^r is on a set F, then $P(\sigma^r)$ is on F. Evidently $\gamma^r(\mathfrak{W})$, applied to the two cycles of (21.10a) and (21.10b) in question, is on $\mathfrak{U} \wedge M$ (7.6), and since as a consequence these cycles are homologous on M, we have $Z^r(\mathfrak{U}) \sim \pi_{\mathfrak{U}\mathfrak{B}} Z^r(\mathfrak{B})$ on M.

21.11 Lemma. *If $\gamma^r \sim 0$ approximately on M, then $Z^r = \{Z^r(\mathfrak{U})\} \sim 0$ on M.*

Proof. We are to show that given a covering $\mathfrak{U}$, $Z^r(\mathfrak{U}) \sim 0$ on $\mathfrak{U} \wedge M$. Let P, $\mathfrak{U}' = \mathfrak{U}'(P)$, $\mathfrak{B}' = \mathfrak{B}'(P)$, $\mathfrak{U}'' = \mathfrak{U}''(\mathfrak{U}, P)$ have the same meanings as above. In addition, let $\mathfrak{E} = \mathfrak{E}(P)$ be a covering governing homologies of γ^r on P. Let $\mathfrak{D} = \mathfrak{D}(P) > (\mathfrak{U}'', \mathfrak{E})$. Then $\gamma^r(\mathfrak{D}) \sim 0$ on P. And since from $\mathfrak{D} > \mathfrak{U}''$ we have $\pi_{\mathfrak{U}''\mathfrak{D}}\gamma^r(\mathfrak{D}) \sim \gamma^r(\mathfrak{U}'')$ on P, it follows that $\gamma^r(\mathfrak{U}'') \sim 0$ on P. Then the projection $\pi_{\mathfrak{U}\mathfrak{B}'}\pi_{\mathfrak{B}'\mathfrak{U}''} = \pi_{\mathfrak{U}\mathfrak{U}''}$ from $\mathfrak{U}''$ to $\mathfrak{U} \wedge M$ gives $\pi_{\mathfrak{U}\mathfrak{U}''}\gamma^r(\mathfrak{U}'') = Z^r(\mathfrak{U}) \sim 0$ on M.

In view of 21.9 and Lemma 21.11 we have:

21.12 Corollary. *If M is a compact set and γ^r is a C-cycle on M such that for arbitrary open set P containing M, there exists $\mathfrak{U}$ such that for $\mathfrak{B} > \mathfrak{U}$, $\gamma^r(\mathfrak{B}) \sim 0$ on P, then $\gamma^r \sim 0$ on M.*

Remark. One can now state Theorem 19.7 for any normal space:

Theorem. *If M is a closed set in a normal space S and γ_i^r, $i = 1, \cdots, k$, are C-cycles lirh on M, then there exists an open set U containing M such that the cycles γ_i^r are lirh on U.*

Proof. Suppose no such U exists. Let U_1 be an open set containing M, and L a linear form in the γ's such that $L \sim 0$ on U_1. Since all coefficients lie in $\mathfrak{F}$, there exist only a finite number, say $m(U_1)$, of such linear forms as L that are linearly independent in the algebraic sense. Let m denote the minimum value of $m(U_1)$ for all open sets U_1 containing M.

Let U be an open set such that $m(U) = m$, and let L be a linear form in the γ's that bounds on U. Since $L \not\sim 0$ on M, by Corollary 21.12 there is an open set V containing M such that $L \not\sim 0$ on V. Let $W = U \cap V$. Then $L \not\sim 0$ on W, and $m(W) = m$. Let $L_1, \cdots, L_m$ be independent linear forms in the γ's that bound on W (hence on U). There exists a relation $L = \sum a^i L_i$. But each $L_i \sim 0$ on W, hence $L \sim 0$ on W.

As a consequence of the two lemmas just proved, and the remark preceding them, the mapping $\varphi : \gamma^r \to Z^r$ induces a homomorphism $\Phi : H_P^r(M, \mathfrak{F}) \to H^r(M; \mathfrak{F})$. We show Φ is an isomorphism.

21.13 Lemma. *Let P_1 be an open set containing M. Then there exists $\mathfrak{U}$ such that if $\mathfrak{B} > \mathfrak{U}$, then $\gamma^r(\mathfrak{B}) \sim \varphi\gamma^r(\mathfrak{B})$ on P_1.*

Proof. Let $\mathfrak{U}'(P_1)$ be a covering governing γ^r on P_1, and let $\mathfrak{U} > \mathfrak{U}'(P_1)$ such that $\mathrm{St}(M, \mathfrak{U}) \subset P_1$. By definition, $Z^r(\mathfrak{U}) = \pi_{\mathfrak{U}\mathfrak{U}''(P)}\gamma^r(\mathfrak{U}''(P))$ where

$\overline{P} \subset \mathrm{St}(M, \mathfrak{U})$. Since $\mathfrak{U}'' > \mathfrak{U}'(P_1)$, $\pi_{\mathfrak{U}\mathfrak{U}''}\gamma^r(\mathfrak{U}'') \sim \gamma^r(\mathfrak{U})$ on P_1. Hence $Z^r(\mathfrak{U}) \sim \gamma^r(\mathfrak{U})$ on P_1.

If $\mathfrak{V} > \mathfrak{U}$, then $\mathfrak{V} > \mathfrak{U}'(P_1)$ and $\mathrm{St}(M, \mathfrak{V}) \subset \mathrm{St}(M, \mathfrak{U}) \subset P_1$. Hence as just shown, $Z^r(\mathfrak{V}) \sim \gamma^r(\mathfrak{V})$ on P_1.

21.14 Corollary. *The cycles γ^r and $Z^r = \varphi\gamma^r$ determine the same element of $H^r_P(M; \mathfrak{F})$.*

Obviously if $Z^r(\mathfrak{U}) \sim 0$ on M, then $Z^r(\mathfrak{U}) \sim 0$ on any open set P_1 which contains M.

We have, then,

21.15 Theorem. *The groups $H^r_P(M, \mathfrak{F})$ and $H^r(M; \mathfrak{F})$ are isomorphic.*

Bibliographical Comment

§6. The proof of Lemma 6.1 and its application to 6.2 were indicated verbally to the author by S. Eilenberg. See Eilenberg [a].

§7. The Čech cycles, etc., were originally given in Čech [a].

§8. Lemma 8.7 (attributed to A. D. Wallace by Lefschetz [L; 263]) was Lemma V of Čech [g].

§10. The notion of *normal refinement* is due to Čech [a; 160]. For an essentially different treatment of the theorems of this section, see Lefschetz [L].

§§14, 16. Compare H. Whitney [a].

§17. Theorem 17.1 is to be found in Whitney [a; Th. 6].

§20. See S. Kaplan [a], especially with reference to Corollary 20.11 and Lemma 20.14. Lemma 20.12 is to be found in Čech [a; 170].

CHAPTER VI

LOCAL CONNECTEDNESS AND LOCAL CO-CONNECTEDNESS

As a preliminary to defining manifolds and discussing their properties, we establish in this chapter some properties of local connectedness as defined in terms of C-cycles and cocycles. *Throughout only augmented cycles and chains are employed.*

1. **Local connectedness in n dimensions.** If $\mathfrak{U}$ is any covering and P a point set, then (V 7.6) by $\mathfrak{U} \wedge P$ is denoted the subcomplex of $\mathfrak{U}$ consisting of simplexes of $\mathfrak{U}$ on P; hence for a chain $C^n(\mathfrak{U})$ to lie on P is equivalent to being a chain of $\mathfrak{U} \wedge P$. By $\mathfrak{U} \cap P$ is denoted the collection of all elements of $\mathfrak{U}$ that meet P.

1.1 Definition. S is n-lc (= *locally connected in dimension* n) at $x \in S$ if given an open set P containing x there must exist an open set Q such that $x \in Q \subset P$ and such that every n-dimensional C-cycle on Q bounds on P.

This form of the definition, although having the advantage of simplicity, is not the best for some purposes, inasmuch as we shall sometimes be concerned with an individual covering, and Definition 1.1, phrased as it is in terms of C-cycles, involves at least a complete family of coverings.

1.2 Definition. S is n-lc at $x \in S$ if given an open set P containing x and a covering $\mathfrak{U}$ of S, there must exist an open set Q (dependent only on P) such that $x \in Q \subset P$ as well as a covering $\mathfrak{V} > \mathfrak{U}$ such that if $z^n(\mathfrak{V})$ is a cycle of $\mathfrak{V} \wedge Q$, then $\pi_{\mathfrak{U}\mathfrak{V}} z^n(\mathfrak{V}) \sim 0$ on $\mathfrak{U} \wedge P$.

An alternative form of Definition 1.2 is as follows:

1.2′ Definition (Čech). S is n-lc at $x \in S$ if, given an open set P containing x and a covering $\mathfrak{U}$ of S, there exists an open set Q such that $x \in Q \subset P$ as well as a covering $\mathfrak{V} > \mathfrak{U}$, such that if $z^n(\mathfrak{V})$ is a cycle of $\mathfrak{V} \wedge Q$, then $z^n(\mathfrak{V}) \sim 0$ on $(\mathfrak{U} \cup \mathfrak{V}) \wedge P$.

(By $\mathfrak{U} \cup \mathfrak{V}$ we indicate the covering of S that is obtained from the elements of $\mathfrak{U}$ and $\mathfrak{V}$ combined; specifically, $\{W \mid (W \in \mathfrak{U}) \vee (W \in \mathfrak{V})\}$.)

To see, first, that Definitions 1.2 and 1.2′ are equivalent, note that if Definition 1.2 is satisfied, there exists for given $\mathfrak{U}$ a chain $c^{n+1}(\mathfrak{U})$ of $\mathfrak{U} \wedge P$ such that $\partial c^{n+1}(\mathfrak{U}) = \pi_{\mathfrak{U}\mathfrak{V}} z^n(\mathfrak{V})$. As shown previously [Lemma V 6.5], there exists on $\mathfrak{U} \cup \mathfrak{V}$ a chain $\mathfrak{D}z^n(\mathfrak{V})$ such that $\partial \mathfrak{D}z^n(\mathfrak{V}) = \pi_{\mathfrak{U}\mathfrak{V}} z^n(\mathfrak{V}) - z^n(\mathfrak{V})$; and from the definition, it is clear that $\mathfrak{D}z^n(\mathfrak{V})$ is on Q since $z^n(\mathfrak{V})$ is on Q. Hence $\partial[c^{n+1}(\mathfrak{U}) - \mathfrak{D}z^n(\mathfrak{V})] = z^n(\mathfrak{V})$. Thus $z^n(\mathfrak{V}) \sim 0$ on $(\mathfrak{U} \cup \mathfrak{V}) \wedge P$. And conversely, Definition 1.2′ implies Definition 1.2. For $\mathfrak{U} \cup \mathfrak{V}$ is a refinement of $\mathfrak{U}$, and hence

$z^n(\mathfrak{V}) \sim 0$ on $(\mathfrak{U} \cup \mathfrak{V}) \wedge P$ implies that $\pi_{\mathfrak{U}\,\mathfrak{U}\cup\mathfrak{V}} z^n(\mathfrak{V}) \sim 0$ on $\mathfrak{U} \wedge P$, which in turn implies that $\pi_{\mathfrak{U}\mathfrak{V}} z^n(\mathfrak{V}) \sim 0$ on $\mathfrak{U} \wedge P$.

We shall show that Definitions 1.1 and 1.2 are also equivalent. First, 1.1 implies 1.2. Given P and $\mathfrak{U}$ as in Definition 1.2, we select Q as in Definition 1.1, and let $\mathfrak{V}$ be a normal refinement of $\mathfrak{U}$ (rel. $\bar{Q}$) [Theorem V 10.7].

Suppose z^n a cycle of $\mathfrak{V} \wedge Q$. Then $\pi_{\mathfrak{U}\mathfrak{V}} z^n$ is a cycle of $\mathfrak{U} \wedge Q$ and as such is a coordinate of a C-cycle γ^n of $\bar{Q}$. Hence $\gamma^n \sim 0$ on P. In particular, then, $\pi_{\mathfrak{U}\mathfrak{V}} z^n \sim 0$ on $\mathfrak{U} \wedge P$.

Definition 1.2 implies Definition 1.1. Given P as in Definition 1.1, select Q as in Definition 1.2, and suppose z^n is a C-cycle on Q. Then if $\mathfrak{U} \in \Sigma$, and $\mathfrak{V} > \mathfrak{U}$ as in Definition 1.2, we have that $\pi_{\mathfrak{U}\mathfrak{V}} z^n(\mathfrak{V}) \sim 0$ on $\mathfrak{U} \wedge P$. By hypothesis z^n is a C-cycle on Q, so that $\pi_{\mathfrak{U}\mathfrak{V}} z^n(\mathfrak{V}) \sim z^n(\mathfrak{U})$ on Q. Hence $z^n(\mathfrak{U}) \sim 0$ on $\mathfrak{U} \wedge P$. Thus every coordinate of z^n bounds on P and Definition 1.1 is satisfied.

2. **Chain-realizations.** As we see later, spaces with sufficient local connectedness properties resemble a finite complex homologically. Of fundamental importance in the demonstration of this fact is the notion of "chain-realization".

2.1 DEFINITION. Let K and L be complexes. Then by a *chain-realization of K on L* we mean a function, or chain-mapping τ, which assigns to each $c^n \in C^n(K; \mathfrak{F})$, $n = 0, 1, 2, \cdots$, an n-chain τc^n of L such that

(1) $\tau(ac_1^n + bc_2^n) = a\tau c_1^n + b\tau c_2^n$, $a, b \in \mathfrak{F}$.

(2) $\tau\partial c^n = \partial\tau c^n$.

(3) $\mathrm{Ki}(\tau c^0) = \mathrm{Ki}(c^0)$ *for every 0-chain* c^0.

For example, a simplicial mapping of K into L induces a chain-mapping such as τ, hence a chain-realization of K on L.

2.2 DEFINITION. If K' is a subcomplex of K that contains all the vertices of K, then a chain-realization of K' on L is called a *partial chain-realization of K on L*. Thus, when we defined for each $\mathfrak{V} > \mathfrak{U}$ that the projection of a $V \in \mathfrak{V}$ should be a $U \in \mathfrak{U}$ such that $U \supset V$, we also set up a partial chain-realization of $\mathfrak{V}$, as a complex, on $\mathfrak{U}$ as a complex, where the set analogous to K' was the set of all vertices V of $\mathfrak{V}$.

2.3 DEFINITION. If τ' is a partial chain-realization of K on L, then a chain-realization τ of K on L is called an *extension* of τ' if τ agrees with τ' wherever the latter is defined. We also say that "τ' can be extended to the realization τ."

In the application of the notions just defined we shall encounter two types of theorems. One of these is concerned with what we might call *realizations in the large*, meaning thereby that the whole space S is involved; the other is concerned with *realizations in the small*, in that only a portion of the space is involved. We consider the former type first, and as a preliminary we introduce a type of uniform local connectedness which we denote by the symbol Culc, in analogy with lc.

2.4 Definition. If $\mathfrak{E}$ is an arbitrary covering, and $c^n(\mathfrak{U})$ is a chain of some covering $\mathfrak{U}$, then $c^n(\mathfrak{U})$ will be said to be *of diameter* $< \mathfrak{E}$—symbolically, diameter $c^n(\mathfrak{U}) < \mathfrak{E}$—if there exists $E \in \mathfrak{E}$ such that $c^n(\mathfrak{U})$ is on E.

2.5 Definition. A space S is called *n-Culc* (*n-dimensionally uniformly locally connected in the sense of Čech*) if given any $\mathfrak{U}, \mathfrak{V} \in \Sigma$, there exist[1] $\mathfrak{U}'_n = \mathfrak{U}'_n(\mathfrak{U})$ and $\mathfrak{V}'_n = \mathfrak{V}'_n(\mathfrak{U}, \mathfrak{V})$ such that if $z^n(\mathfrak{V}'_n)$ is of diameter $< \mathfrak{U}'_n$, then $\pi_{\mathfrak{V}\mathfrak{V}'_n} z^n(\mathfrak{V}'_n)$ bounds a chain of $\mathfrak{V}$ of diameter $< \mathfrak{U}$.

2.6 By lc^n we denote the property of being r-lc for all $r \leqq n$; and similarly the symbol Culc^n denotes the property of being r-Culc for all $r \leqq n$.

2.7 As in the case of Definition 1.2, there is an alternative form of Definition 2.5 in which the phrase "then $\cdots$" is replaced by "then $z^n(\mathfrak{V}'_n)$ bounds a chain on $\mathfrak{V} \cup \mathfrak{V}'_n$ of diameter $< \mathfrak{U}$."

By way of justification of the phrase "uniformly locally connected" above, we prove:

2.8 Theorem. *If the compact space S is n-lc at every point, then S is n-Culc.*

Proof. Let $\mathfrak{U}, \mathfrak{V} \in \Sigma$ be given. For each $x \in S$ select $U(x) \in \mathfrak{U}$ such that $x \in U(x)$. Then with $U(x)$ as the P of Definition 1.2, and $\mathfrak{V}$ in place of its $\mathfrak{U}$, there exists by the n-lc assumption an open set $Q(x)$ such that $x \in Q(x) \subset U(x)$ and a covering $\mathfrak{V}(x)$ such that if z^n is a cycle of $\mathfrak{V}(x)$ on $Q(x)$, then $\pi_{\mathfrak{V}\mathfrak{V}(x)} z^n \sim 0$ on $\mathfrak{V} \wedge U(x)$.

Let $x_1, \cdots, x_i, \cdots, x_m \in S$ such that the set $\mathfrak{U}'$ consisting of the sets $Q(x_i)$ forms a covering of S. Let $\mathfrak{V}' > [\mathfrak{U}, \mathfrak{V}(x_1), \cdots, \mathfrak{V}(x_i), \cdots, \mathfrak{V}(x_m)]$. Then consider any $z^n(\mathfrak{V}')$ of diameter $< \mathfrak{U}'$. The cycle $z^n(\mathfrak{V}')$ is on some Q, say $Q(x_i)$. Then $\pi_{\mathfrak{V}(x_i)\mathfrak{V}'} z^n(\mathfrak{V}')$ is also on $Q(x_i)$ and therefore $\pi_{\mathfrak{V}\mathfrak{V}(x_i)} \pi_{\mathfrak{V}(x_i)\mathfrak{V}'} z^n(\mathfrak{V}') \sim 0$ on $\mathfrak{V} \wedge U(x_i)$. Hence $\pi_{\mathfrak{V}\mathfrak{V}'} z^n(\mathfrak{V})$ bounds a chain on $\mathfrak{V}$ of diameter $< \mathfrak{U}$.

Remark. It should be noticed that as a consequence of the above proof, if $z^n(\mathfrak{V}')$ is any cycle on $U' = Q(x_i)$, then there exists $U = U(x_i) \in \mathfrak{U}$ such that (1) $U \supset U'$ and (2) $z^n(\mathfrak{V}') \sim 0$ on $(\mathfrak{V}' \cup \mathfrak{V}) \wedge U$. The reader may prove the converse of Theorem 2.8.

2.9 Definition. If $\mathfrak{E}$ is a covering and τ is a partial chain-realization of a complex K on some covering $\mathfrak{U}$, then τ is said to be of *norm* $< \mathfrak{E}$ if for each simplex E^n of K there exists $E \in \mathfrak{E}$ such that for every chain c^k of E^n, for which τc^k is defined, the chain τc^k is on E.

2.10 Theorem. *A necessary and sufficient condition for a compact space S to be lc^n is that for every pair $\mathfrak{U}, \mathfrak{V} \in \Sigma$ there exist coverings $\mathfrak{U}^*_n = \mathfrak{U}^*_n(\mathfrak{U})$ and $\mathfrak{V}^*_n = \mathfrak{V}^*_n(\mathfrak{U}, \mathfrak{V})$ such that if K is a complex of dimension $\leqq n + 1$ and τ' is a*

[1]Whenever a symbol $\mathfrak{U}(\mathfrak{U}_1, \cdots, \mathfrak{U}_k)$ is employed, it will be understood that $\mathfrak{U} > \mathfrak{U}_1, \cdots, \mathfrak{U}_k$.

partial chain-realization of K on $\mathfrak{B}_n^*$, or a refinement thereof, of norm $< \mathfrak{U}_n^*$, then τ' can be extended to a chain-realization of K on $\mathfrak{B} \cup \mathfrak{B}_n^*$ of norm $< \mathfrak{U}$.

PROOF OF THE NECESSITY. Given coverings $\mathfrak{U}$ and $\mathfrak{B}$, define consecutively the coverings (cf. Lemma V 8.6):

$$\mathfrak{U}_n >^* \mathfrak{U}_n'(\mathfrak{U}), \qquad \mathfrak{B}_n > \{\mathfrak{B}_n'(\mathfrak{U}, \mathfrak{B}), \mathfrak{U}_n\},$$

$$\mathfrak{U}_{n-1} >^* \mathfrak{U}_{n-1}'(\mathfrak{U}_n), \qquad \mathfrak{B}_{n-1} > \{\mathfrak{B}_{n-1}'(\mathfrak{U}_n, \mathfrak{B}_n), \mathfrak{U}_{n-1}\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathfrak{U}_0 >^* \mathfrak{U}_0'(\mathfrak{U}_1), \qquad \mathfrak{B}_0 > \{\mathfrak{B}_0'(\mathfrak{U}_1, \mathfrak{B}_1), \mathfrak{U}_0\},$$

where the $\mathfrak{U}'$'s and $\mathfrak{B}'$'s are as in the proof of Theorem 2.8. Let $\mathfrak{U}_n^*(\mathfrak{U}) = \mathfrak{U}_0$ and $\mathfrak{B}_n^*(\mathfrak{U}, \mathfrak{B}) = \mathfrak{B}_0$.

For any $U_0 \in \mathfrak{U}_0$, there exists a sequence

(2.10a) $\quad U_0 \subset \mathrm{St}\,(U_0, \mathfrak{U}_0) \subset U_0' \subset U_1 \subset \cdots \subset U_{k-1}' \subset U_k \subset \cdots \subset U,$

where $U_i \in \mathfrak{U}_i$, $U_i' \in \mathfrak{U}_i'$, for $i = 0, 1, \cdots, n$; $U \in \mathfrak{U}$; and such that any i-cycle of a refinement of $\mathfrak{B}_i'$ on U_i' bounds a chain on U_{i+1} if $i < n$, and any n-cycle of a refinement of $\mathfrak{B}_n'$ on U_n' bounds a chain on U (see Remark following proof of Theorem 2.8).

Now let τ' be a partial chain-realization of a complex K of dimension $\leqq n + 1$ on $\mathfrak{B}_n^*$ of norm $< \mathfrak{U}_n^*$. If K' is the subcomplex of K of which τ' is a chain-realization (Definition 2.2), then for any σ^k of K' we let $\tau(\sigma^k) = \tau'(\sigma^k)$. The definition of τ on $K - K'$ will be given inductively.

Suppose σ^1 is a cell of K but not of K'. There exists U_0 such that $\tau'\partial\sigma^1$ is on U_0. Hence $\tau'\partial\sigma^1$ is on the U_0' of (2.10a) and there exists a chain C^1 of $\mathfrak{B}_0 \cup \mathfrak{B}_1$ on U_1 of (2.10a) such that $\partial C^1 = \tau'\partial\sigma^1$. We let $\tau\sigma^1 = C^1$. Note that 2.1 (2) is satisfied.

Suppose τ has been defined on all i-cells of K not in K' for $i \leqq k - 1$ and let σ^k be a cell corresponding to an E^k of K not in K'. There exists a U_0 such that for each σ^0 corresponding to a vertex of E^k, $\tau\sigma^0$ is on U_0, as is also each given $\tau'\sigma^i$ for σ^i corresponding to a face of E^k. Suppose σ^{k-1} corresponds to a face E^{k-1} of E^k and the $\tau'\sigma^{k-1}$ was not given. Then $\tau\sigma^{k-1}$ was defined by the induction by first selecting a U_0 which we denote by $U_0^{(1)}$, subsequent to which $\tau\sigma^{k-1}$ was defined as a chain of $\mathfrak{B}_0 \cup \mathfrak{B}_1 \cup \cdots \cup \mathfrak{B}_{k-1}$ on $U_{k-1}^{(1)}$, element of the sequence of type (2.10a) associated with $U_0^{(1)}$. We now have two sequences, (2.10a) and the sequence of type (2.10a) in which $U_i^{(1)}$ replaces U_i.

Now we took $\mathfrak{B}_0 > \mathfrak{U}_0$, and if all $\tau\sigma^0$'s corresponding to vertices of E^{k-1} lie on both U_0 and $U_0^{(1)}$, then U_1 and $U_1^{(1)}$ meet and consequently U_{k-1} and $U_{k-1}^{(1)}$ meet. It follows that all $\tau\sigma^{k-1}$ lie on elements of $\mathfrak{U}_{k-1}$ that meet U_{k-1}, and hence $\tau\partial\sigma^k$ is a cycle of $\mathfrak{B}_0 \cup \cdots \cup \mathfrak{B}_{k-1}$ on U_{k-1}'. Hence there exists a chain C^k of $\mathfrak{B}_0 \cup \cdots \cup \mathfrak{B}_{k-1} \cup \mathfrak{B}_k$ on U_k such that $\partial C^k = \tau\partial\sigma^k$. We let $\tau\sigma^k = C^k$. If we denote U by U_{n+1}, $\mathfrak{B}$ by $\mathfrak{B}_{n+1}$, etc., the induction is now complete.

Finally, having defined τ for all cells of K not in K', all chains may be "projected" onto $\mathfrak{V} \cup \mathfrak{V}_0 = \mathfrak{V} \cup \mathfrak{V}_n^*$ so as to obtain the desired chain-realization of K on $\mathfrak{V} \cup \mathfrak{V}_n^*$.

Proof of the Sufficiency. Let $x \in S$, open set P containing x, and $\mathfrak{U} \in \Sigma$ be given. Let Q_1 be an open set such that $x \in \overline{Q}_1 \subset P$, and U, V be disjoint open sets containing $F(P)$, $F(Q_1)$ respectively. Then $S - \overline{P}$, U, $P - \overline{Q}_1$, V, Q_1 constitute a covering $\mathfrak{U}'$ of S. Then there exist $\mathfrak{U}_n^*(\mathfrak{U}')$, $\mathfrak{V}_n^*(\mathfrak{U}', \mathfrak{U})$, etc. Let Q be an open set containing x and common to Q_1 and the elements of $\mathfrak{U}_n^*$ that contain x. Suppose z^r, $r \leqq n$, is a cycle of $\mathfrak{V}_n^*$ on Q. If $| z^r |$ denotes the smallest complex of $\mathfrak{V}_n^*$ of which z^r is a cycle, let Y be a vertex of $| z^r |$, and let $\hat{z}^r$ denote the cone-complex (V 6) formed by the join of $| z^r |$ and Y. Then on the complex $| z^r |$ can be defined an obvious partial chain-realization τ' of $\hat{z}^r$ on $\mathfrak{V}_n^*$ of norm $< \mathfrak{U}_n^*$, and this is extendible to a chain-realization of $\hat{z}^r$ on $\mathfrak{U} \cup \mathfrak{V}_n^*$ of norm $< \mathfrak{U}'$. Evidently, since $\tau' z^r$ is on Q, the extension τ must be on P, and since, by Lemma V 6.1, $z^r \sim 0$ on $\hat{z}^r$, we have $z^r \sim 0$ on $(\mathfrak{U} \cup \mathfrak{V}_n^*) P$.

3. Complex-like character of compact lc^n spaces.

3.1 Theorem. *If the compact space S is lc^n, then there exists a complex K such that the homology groups $H_a^r(S; \mathfrak{F})$, $r \leqq n$, are isomorphic with subgroups of the corresponding groups $H_a^r(K; \mathfrak{F})$.* (Compare Theorem 4.5 below.)

Proof. Let $\mathfrak{U}$ be a covering of S, and let $\mathfrak{U}_0 >^* \mathfrak{U}_n^*(\mathfrak{U})$ ($\mathfrak{U}_n^*$ as defined in the statement of Theorem 2.10). For each C-cycle z^r, let $\varphi(z^r) = z^r(\mathfrak{U}_0)$. Then φ induces a homomorphism, which we also denote by φ, of $H_a^r(S; \mathfrak{F})$ into $H_a^r(\mathfrak{U}_0 ; \mathfrak{F})$. We shall show that if the cycle $z^r(\mathfrak{U}_0) \sim 0$ on $\mathfrak{U}_0$, then $z^r(\mathfrak{V}) \sim 0$ for every refinement $\mathfrak{V}$ of $\mathfrak{U}_0$; and since the refinements of $\mathfrak{U}_0$ form a complete system of coverings, it will follow that $z^r \sim 0$, and that φ is an isomorphism into a subgroup of $H_a^r(\mathfrak{U}_0 ; \mathfrak{F})$.

Suppose, then, $z^r(\mathfrak{U}_0) \sim 0$ on $\mathfrak{U}_0$, z^r being a C-cycle. For $\mathfrak{V} > \mathfrak{U}_0$, let $\mathfrak{V}_n^*(\mathfrak{U}, \mathfrak{V})$ be the covering defined in the statement of Theorem 2.10. Associated with the projections $\pi_{\mathfrak{U}_0\mathfrak{V}_n*}$ are the homomorphisms $\mathfrak{D}$ (see V 6) into $\mathfrak{D}\mathfrak{V}_n^*$, such that $\partial \mathfrak{D} z^r(\mathfrak{V}_n^*) = \pi_{\mathfrak{U}_0\mathfrak{V}_n*} z^r(\mathfrak{V}_n^*) - z^r(\mathfrak{V}_n^*)$ (Lemma V 6.6). Let $c_1^{r+1}(\mathfrak{U}_0)$, $c_2^{r+1}(\mathfrak{U}_0)$ be chains such that $\partial c_1^{r+1}(\mathfrak{U}_0) = z^r(\mathfrak{U}_0)$, $\partial c_2^{r+1}(\mathfrak{U}_0) = z^r(\mathfrak{U}_0) - \pi_{\mathfrak{U}_0\mathfrak{V}_n*} z^r(\mathfrak{V}_n^*)$. Then $c^{r+1} = c_1^{r+1}(\mathfrak{U}_0) - c_2^{r+1}(\mathfrak{U}_0) - \mathfrak{D} z^r(\mathfrak{V}_n^*)$ is a chain of $\mathfrak{V}_n^* \cup \mathfrak{U}_0$ such that $\partial c^{r+1} = z^r(\mathfrak{V}_n^*)$.

Let $| c^{r+1} |$ denote the minimal subcomplex of $\mathfrak{V}_n^* \cup \mathfrak{U}_0$ of which c^{r+1} is a chain. Let v_i, $i = 1, 2, \cdots, m$, be the vertices of $| c^{r+1} |$ on $\mathfrak{U}_0$, and for each i let v_i^* be a vertex of $\mathfrak{V}_n^*$ such that $v_i \supset v_i^*$. Define a partial chain-realization τ' of $| c^{r+1} |$ on $\mathfrak{V}_n^*$ as follows: For each cell σ of $| z^r(\mathfrak{V}_n^*) |$, $\tau'\sigma = \sigma$; and $\tau' v_i = v_i^*$. Then norm $\tau' < \mathrm{St}(\mathfrak{U}_0, \mathfrak{U}_0) > \mathfrak{U}_n^*(\mathfrak{U})$. Hence by the definition of $\mathfrak{V}_n^*(\mathfrak{U}, \mathfrak{V})$, τ' can be extended to a chain-realization τ of $| c^{r+1} |$ on $\mathfrak{V}_n^* \cup \mathfrak{V}$ (of norm $< \mathfrak{U}$). Then $\partial \tau c^{r+1} = \tau \partial c^{r+1} = \tau z^r(\mathfrak{V}_n^*) = z^r(\mathfrak{V}_n^*)$. Consequently the chain $\pi_{\mathfrak{V}\mathfrak{V}_n*} \tau c^{r+1}$ is bounded by $\pi_{\mathfrak{V}\mathfrak{V}_n*} z^r(\mathfrak{V}_n^*)$, and since the latter is homologous to $z^r(\mathfrak{V})$ on $\mathfrak{V}$, we have that $z^r(\mathfrak{V}) \sim 0$ on $\mathfrak{V}$.

3.2 Corollary. *If a compact space S is lc^n, then the numbers $p_a^r(S; \mathfrak{F})$, $r = 0, 1, \cdots, n$, are all finite.*

3.3 Lemma. *Let c_0, c_1, $\cdots$, c_n be any set of $n + 1$ nonnegative integers. Then there exists an at most n-dimensional (geometric as in* II 5.2*) complex K such that $p_a^r(K; \mathfrak{F}) = c_r$, $r = 0, 1, \cdots, n$.*

Proof. The lemma is true for $n = 0$; since the number of components in K is to be $c_0 + 1$ (Theorem V 11.3), we let K consist of $c_0 + 1$ vertices.

Suppose the lemma is true for $n - 1$. Then there exists an at most $(n - 1)$-dimensional (geometric) complex K' such that $p_a^r(K') = c_r$ for $r = 0, 1, \cdots, n - 1$. Now let L be a complex consisting of c_n n-spheres, S_1^n, $\cdots$, S_k^n, where $k = c_n$, such that S_i^n, S_j^n, $i < j$, have exactly one vertex in common if $j = i + 1$, and otherwise have nothing in common. Let K' be augmented by the addition of the complex L in such a manner that K' and L have just one vertex in common, say a vertex of S_1^n not on S_2^n. The resulting complex K is at most n-dimensional, and since its n-dimensional simplexes lie entirely in L, satisfies the condition $p_a^n(K) = c_n$ since $p_a^n(L) = c_n$.

That $p_a^r(K) = p_a^r(K') = c_r$ for $1 < r < n$ follows readily from the fact that (1) $p_a^r(S^n) = 0$ (see V 12.5 and Corollary II 5.11) and (2) every r-cycle z^r of K can be expressed in the form

$$z^r = \bar{z}^r + z_1^r + \cdots + z_k^r, \tag{3.2a}$$

where z_j^r, $j = 1, 2, \cdots, c_n$, is an absolute cycle of S_j^n. And since the number of components of K is the same as that of K', $p_a^0(K) = c_0$.

In order to see that $p^1(K)$, in case $1 < n$, is the same as $p^1(K')$, we can again get a decomposition like (3.2a), with $r = 1$, except that here it is not so obvious that the individual terms on the right represent cycles. However, consider z_1^1, for example. Since

$$z^1 = \bar{z}^1 + z_1^1 + \cdots + z_k^1, \tag{3.2b}$$

and $\partial z^1 = 0$, it follows that $\partial z_1^1 = -\partial \bar{z}^1 - \partial z_2^1 - \cdots - \partial z_k^1$. Now if $\partial z_1^1 \neq 0$, then must $\partial z_1^1 = a(\sigma_1^0 - \sigma_2^0)$, $a \in \mathfrak{F}$, where σ_1^0 corresponds to $K' \cap L$, σ_2^0 corresponds to $S_1^n \cap S_2^n$ and $a \neq 0$. For the presence of any other 0-cell of S_1^1 in ∂z_1^1 would imply its presence in ∂z^1; and in addition, ∂z_1^1 being a bounding cycle it must satisfy the condition $\mathrm{Ki}(\partial z_1^1) = 0$ (Lemma V 18.7). But then

$$-\partial \bar{z}^1 - \partial z_2^1 - \cdots - \partial z_k^1 = a(\sigma_1^0 - \sigma_2^0),$$

implying that $-\partial \bar{z}^1 = a\sigma_1^0$, which is impossible since $\mathrm{Ki}(a\sigma_1^0) = a \neq 0$. We must conclude then that $\partial z_1^1 = 0$. Similar arguments apply in the case of the other terms of the right-hand member of (3.2b), with a slight variation in the case of $\bar{z}^1$ and z_k^1.

In view of Corollary 3.2 and the fact that two vector spaces of the same finite dimension are isomorphic, we have

3.4 THEOREM. *If the compact space S is lc^n, then there exists an at most n-dimensional (geometric) complex K such that $H_a^r(S; \mathfrak{F})$ and $H_a^r(K; \mathfrak{F})$ are isomorphic, $r = 0, 1, \cdots, n$.*

The reader will recognize that Theorem 3.4 is the extension of the theorem that a compact space which is lc in the sense of Chapter I has an at most finite number of components.

For the situation "in the small", we give below analogues of Theorems 2.10 and 3.1. It will be useful, for later purposes, to state the theorems with only local compactness assumed.

3.5 THEOREM. *Let S be locally compact and lc^n, M a compact subset of S and P an open set containing M. Then for any coverings $\mathfrak{U}$ and $\mathfrak{V}$, there exist coverings $\mathfrak{U}_n' = \mathfrak{U}_n'(\mathfrak{U}; M, P)$ and $\mathfrak{V}_n' = \mathfrak{V}_n'(\mathfrak{U}, \mathfrak{V}; M, P)$ such that if $z^r(\mathfrak{V}_n')$, $r \leqq n$, is a cycle of $\mathfrak{V}_n' \wedge M$ of diameter $< \mathfrak{U}_n'$, then $z^r(\mathfrak{V}_n')$ bounds a chain of $(\mathfrak{V}_n' \cup \mathfrak{V}) \wedge P$ of diameter $< \mathfrak{U}$.*

The proof is analogous to that of Theorem 2.8; the differences in proof are due only to the fact that we have here a uniformity with regard to M rather than to S.

REMARK. Any refinement of $\mathfrak{V}_n'$ will serve as well as $\mathfrak{V}_n'$. In particular, $\mathfrak{V}_n'$ may be assumed to be a refinement of $\mathfrak{U}_n'$.

3.6 THEOREM. *Under the hypothesis of Theorem 3.5, if $\mathfrak{U}$ and $\mathfrak{V}$ are coverings, there exist coverings $\mathfrak{U}_n^* = \mathfrak{U}_n^*(\mathfrak{U}; M, P)$ and $\mathfrak{V}_n^* = \mathfrak{V}_n^*(\mathfrak{U}, \mathfrak{V}; M, P)$ such that if K is a complex of dimension $\leqq n + 1$ and τ' is a partial chain-realization of K on $\mathfrak{V}_n^* \wedge M$ of norm $< \mathfrak{U}_n^*$, then τ' can be extended to a chain-realization τ of K on $(\mathfrak{V}_n^* \cup \mathfrak{V}) \wedge P$ of norm $< \mathfrak{U}$.*

PROOF. Let $P_1, \cdots, P_n$ be open sets such that (1) $M \subset P_1 \Subset \cdots \Subset P_k \Subset \cdots \Subset P_n \Subset P$, and (2) $\bar{P}_n$ is compact. Let $\mathfrak{W}_n > \mathfrak{U}$ be a covering such that $\mathrm{St}(\bar{P}_n, \mathfrak{W}_n) \subset P$, and select $\mathfrak{U}_n >^* \mathfrak{U}_n'(\mathfrak{W}_n; \bar{P}_n, P)$, $\mathfrak{V}_n > \{\mathfrak{V}_n'(\mathfrak{W}_n, \mathfrak{V}; \bar{P}_n, P), \mathfrak{U}_n\}$ (Theorem 3.5).

Next, let $\mathfrak{W}_{n-1} > \mathfrak{U}_n$ be a covering such that $\mathrm{St}(\bar{P}_{n-1}, \mathfrak{W}_{n-1}) \subset P_n$, and select $\mathfrak{U}_{n-1} >^* \mathfrak{U}_{n-1}'(\mathfrak{W}_{n-1}; \bar{P}_{n-1}, P_n)$, $\mathfrak{V}_{n-1} > \{\mathfrak{V}_{n-1}'(\mathfrak{W}_{n-1}, \mathfrak{V}_n; \bar{P}_{n-1}, P_n), \mathfrak{U}_{n-1}\}$. Continue in this manner until, finally, we let $\mathfrak{W}_1 > \mathfrak{U}_2$ be a covering such that $\mathrm{St}(\bar{P}_1, \mathfrak{W}_1) \subset P_2$, and select $\mathfrak{U}_1 >^* \mathfrak{U}_1'(\mathfrak{W}_1; \bar{P}_1, P_2)$, $\mathfrak{V}_1 > \{\mathfrak{V}_1'(\mathfrak{W}_1, \mathfrak{V}_2; \bar{P}_1, P_2), \mathfrak{U}_1\}$. We then let $\mathfrak{W}_0 > \mathfrak{U}_1$ be a covering such that $\mathrm{St}(M, \mathfrak{W}_0) \subset P_1$, and select $\mathfrak{U}_0 >^* \mathfrak{U}_0'(\mathfrak{W}_0; M, P_1)$ and $\mathfrak{V}_0 > \{\mathfrak{V}_0'(\mathfrak{W}_0, \mathfrak{V}_1; M, P_1), \mathfrak{U}_0\}$.

We then proceed as in the proof of Theorem 2.10. Let τ' be a partial chain-realization of a complex K of dimension $\leqq n + 1$ on $\mathfrak{V}_0 \wedge M$ of norm $< \mathfrak{U}_0$. With K' defined as before, suppose σ^1 is a cell of K not in K'. There exists U_0 such that $\tau'\partial\sigma^1$ is on U_0 and hence on a $U_0' \supset U_0$ as before; hence by definition of $\mathfrak{U}_0'$, there exists a chain C^1 of $\mathfrak{V}_0 \cup \mathfrak{V}_1$ on a $W_0 \in \mathfrak{W}_0$ such that $\partial C^1 = \tau'\partial\sigma^1$. By the choice of $\mathfrak{W}_0$, $W_0 \subset P_1$. Let $U_1 \in \mathfrak{U}_1$ such that $W_0 \subset U_1$; then C^1 is a fortiori on U_1. Let $\tau\sigma^1 = C^1$ as before.

Having τ defined on all i-cells of K for $i \leqq k - 1$, let σ^k be as before. This time $\tau\partial\sigma^k$ is a cycle of $\mathfrak{V}_0 \cup \cdots \cup \mathfrak{V}_{k-1}$, on $P_{k-1} \cap U'_{k-1}$ where $U'_{k-1} \in \mathfrak{U}'_{k-1}$ and accordingly bounds a chain C^k of $\mathfrak{V}_0 \cup \cdots \cup \mathfrak{V}_k$ on a $W_{k-1} \in \mathfrak{W}_{k-1}$. Since $W_{k-1} \subset P_k$, C^k is on P_k.

If we denote $\mathfrak{U}$ by $\mathfrak{U}_{n+1}$, $\mathfrak{V}$ by $\mathfrak{V}_{n+1}$, etc., we ultimately arrive at definitions of all $\tau\sigma^{n+1}$ on P, etc.

3.7 Corollary. *If the locally compact space S is lc^n, M is a compact subset of S and P is an open set containing M, then there is a covering $\mathfrak{U}_0$ of S such that if z^r, γ^r, $r \leqq n$, are cycles on M such that $z^r(\mathfrak{U}_0) \sim \gamma^r(\mathfrak{U}_0)$ on $\mathfrak{U}_0 \wedge M$, then $z^r \sim \gamma^r$ on P.*

The proof is similar to that of Theorem 3.1.

3.8 Corollary. *If the locally compact space S is lc^n, M is a compact subset of S and P is an open set containing M, then at most a finite number of n-cycles on M are independent with respect to homology on P.*

3.9 Corollary. *If M is a compact subset of a locally compact, lc^n space S, and P is an open set containing M such that $\bar{P}$ is compact, then $H^n(S: M, 0; \bar{P}, 0)$ and $H_n(S: S, S - \bar{P}; S, S - M)$ have the same, finite, dimension. And they have respective sets of generators $\{\Gamma^n_j\}$, $\{\Gamma^i_n\}$ such that $\Gamma^i_n \cdot \Gamma^n_j = \delta^j_i$.*

Proof. The proof is a consequence of Corollary 3.8 and Theorem V 18.30.

4. **Noncompact cases.** We shall give in this section certain theorems that are of fundamental importance in the applications later on, especially in the duality theory. We begin with a theorem which establishes an analogue of the ulc property. First we note the following lemma:

4.1 Lemma. *If $\mathfrak{U}$ is a star-finite ucos of a space S, and M is a compact subset of S, then only a finite number of elements of $\mathfrak{U}$ meet M.*

The proof is left to the reader.

4.2 Theorem. *Let P be an open subset of a perfectly normal, compact, lc^n space S, and $\mathfrak{U}$, $\mathfrak{V}$ star-finite internal ucos of P.*[2] *Then there exist countable, star-finite internal ucos $\mathfrak{U}'_n = \mathfrak{U}'_n(\mathfrak{U}; P)$ and $\mathfrak{V}'_n = \mathfrak{V}'_n(\mathfrak{U}, \mathfrak{V}; P)$ of P such that if $Z^r(\mathfrak{V}'_n)$, $r \leqq n$, is a finite cycle of diameter $< \mathfrak{U}'_n$, then $Z^r(\mathfrak{V}'_n)$ bounds a finite chain on $\mathfrak{V}'_n \cup \mathfrak{V}$ of diameter $< \mathfrak{U}$.*

Proof. Let $\{Q_i\}$ be a collection of open sets such that $Q_{i+1} \supset Q_i$ and $P = \bigcup Q_i$, each $\bar{Q}_i$ being compact (Lemma V 20.4). Let $Q_0 = Q_{-1} = 0$.

By Lemma 4.1, so far as any particular set $\bar{Q}_i$ is concerned, $\mathfrak{U}$ and $\mathfrak{V}$ may be considered as fcos. Hence by Theorem 3.5, there exist, for each i, fcos $\mathfrak{U}'_{n,i} =$

[2]Evidently the hypothesis in this form is more general that if it were assumed that $\mathfrak{U}$ and $\mathfrak{V}$ are countable, star-finite ucos of S.

$\mathfrak{U}'_{n,i}(\mathfrak{U}; \overline{Q}_{i+1} - Q_{i-1}, Q_{i+2} - \overline{Q}_{i-2})$ and $\mathfrak{B}'_{n,i} = \mathfrak{B}'_{n,i}(\mathfrak{U}, \mathfrak{B}; \overline{Q}_{i+1} - Q_{i-1}, Q_{i+2} - \overline{Q}_{i-2})$ of S such that $\mathfrak{B}'_{n,i} > \mathfrak{U}'_{n,i}$, and if $Z^r(\mathfrak{B}'_{n,i})$, $r \leqq n$, is a cycle of $\mathfrak{B}'_{n,i} \wedge (\overline{Q}_{i+1} - Q_{i-1})$ of diameter $< \mathfrak{U}'_{n,i}$, then $Z^r(\mathfrak{B}'_{n,i})$ bounds a chain of $(\mathfrak{B}'_{n,i} \cup \mathfrak{B}) \wedge (Q_{i+2} - \overline{Q}_{i-2})$ of diameter $< \mathfrak{U}$. We may assume that for each i, $\mathfrak{B}'_{n,i+1} > \mathfrak{B}'_{n,i}$.

For each i, let $\mathfrak{S}_{n,i} = \{U \cap (Q_{i+1} - \overline{Q}_{i-1}) \mid U \in \mathfrak{U}'_{n,i}\}$, and $\mathfrak{J}_{n,i} = \{V \cap (Q_i - \overline{Q}_{i-2}) \mid V \in \mathfrak{B}'_{n,i}\}$. Let $\mathfrak{U}'_n = \bigcup \mathfrak{S}_{n,i}$, $\mathfrak{B}'_n = \bigcup \mathfrak{J}_{n,i}$.

Consider a cycle $Z^r(\mathfrak{B}'_n)$, $r \leqq n$, of diameter $< \mathfrak{U}'_n$. Then $| Z^r(\mathfrak{B}'_n) |$ lies on at least one element of $\mathfrak{U}'_n$; let i be the smallest integer such that $| Z^n(\mathfrak{B}'_n) |$ lies on a $U \in \mathfrak{S}_{n,i}$. Then U meets no vertices of $\mathfrak{J}_{n,i-1}$, or of any $\mathfrak{J}_{n,j}$ such that $j < i$; indeed, $| Z^n(\mathfrak{B}'_n) |$ lies on $\mathfrak{B}'_{n,i} \cup \mathfrak{B}'_{n,i+1} \cup \mathfrak{B}'_{n,i+2}$, and is on $\overline{Q}_{i+1} - Q_{i-1}$ as well as of diameter $< \mathfrak{U}'_{n,i}$. Consequently $Z^r(\mathfrak{B}'_n)$ bounds a chain of $\mathfrak{B}'_n \cup \mathfrak{B}$ of diameter $< \mathfrak{U}$.

4.3 THEOREM. *Under the hypothesis of Theorem 4.2, there exist countable, star-finite internal ucos* $\mathfrak{U}^*_n = \mathfrak{U}^*_n(\mathfrak{U}; P)$ *and* $\mathfrak{B}^*_n = \mathfrak{B}^*_n(\mathfrak{U}, \mathfrak{B}; P)$ *of* P *such that if* K *is a complex (finite or infinite) of dimension* $\leqq n + 1$ *and* τ' *is a partial realization of* K *on* $\mathfrak{B}^*_n$ *of norm* $< \mathfrak{U}^*_n$, *then* τ' *can be extended to a realization of* K *on* $\mathfrak{B} \cup \mathfrak{B}^*_n$ *of norm* $< \mathfrak{U}$.

The proof is practically a repetition of the proof of Theorem 2.10, of which Theorem 4.3 is the analogue, except that Theorem 4.2 is used instead of Theorem 2.8.

4.4 THEOREM. *Let* P *be an open subset of a perfectly normal, compact,* lc^n *space* S, *and let* M *be a compact subset of* P *and* Z^r, $r \leqq n$, *a C-cycle on* M *such that* $Z^r \approx 0$ *in* P.[3] *Then* $Z^r \sim 0$ *on a compact subset of* P.

PROOF. The set P may be represented as $\bigcup_{i=1}^{\infty} Q_i$ where the Q_i are open sets such that $Q_1 \supset M$, $Q_{i+1} \supseteq \overline{Q}_i$. Let $\mathfrak{U}$ be the covering of P whose elements are the set $U_1 = Q_2$ and the sets $U_{i+1} = Q_{i+2} - \overline{Q}_i$, $i = 1, 2, 3, \cdots$. Then let $\mathfrak{U}_0 >^* \mathfrak{U}^*_n(\mathfrak{U})$, where $\mathfrak{U}^*_n$ is as defined in Theorem 4.3.

By hypothesis, $Z^r(\mathfrak{U}_0) \sim 0$ on $\mathfrak{U}_0$. For any $\mathfrak{B} > \mathfrak{U}_0$, where $\mathfrak{B}$ is a star-finite, internal ucos of P, let $\mathfrak{B}^*_n(\mathfrak{U}, \mathfrak{B})$ be as defined in Theorem 4.3. Referring to the proof of Theorem 3.1, there exist chains c^{r+1}, c_1^{r+1}, c_2^{r+1}, $\mathfrak{D}Z^r(\mathfrak{B}^*_n)$ as defined therein, where the last two are on M. Let U_k be the element of $\mathfrak{U}$ with greatest subscript k such that $U_k \cap | c^{r+1} | \neq 0$, and let $U = \bigcup_{i=1}^{k} U_i$. Then, following the proof as given in the latter part of the proof of Theorem 3.1, it is shown that $Z^r(\mathfrak{B}) \sim 0$ on $\mathfrak{B} \wedge \overline{U}$. And since, by Corollary V 20.9, coverings such as $\mathfrak{B}$ form a complete family for P, we conclude that $Z^r \sim 0$ on $\overline{U}$.

Note that by the same type of argument, we have:

4.5 THEOREM. *Let* P *be an open subset of a perfectly normal, compact,* lc^n *space* S. *Then there exists a countable, star-finite complex* K *such that the homology*

[3]Cf. Definition V 20.3.

groups $h_a^r(P; \mathfrak{F})$, $r \leqq n$, determined by C-cycles and homologies on compact subsets of P, are isomorphic with subgroups of the corresponding homology groups of K as determined by finite chains of K.[4]

We shall conclude this section with a theorem that is of fundamental importance for the duality theory of Chapter VIII.

4.6 THEOREM. *If M is an arbitrary subset of a perfectly normal, compact, lc^n space S, and Z^r, $r \leqq n$, is a C-cycle on a compact subset of M, then a necessary and sufficient condition that $Z^r \approx 0$ in M is that for every open subset P of S containing M, $Z^r \sim 0$ on a compact subset of P.*

PROOF OF NECESSITY. Every internal ucos $\mathfrak{U}$ of P gives, by the intersections of its elements with M, an internal ucos $\mathfrak{U}'$ of M, and by hypothesis $Z^r \sim 0$ on $\mathfrak{U}'$ and hence on $\mathfrak{U}$. Therefore $Z^r \approx 0$ in P, and by Theorem 4.4 must therefore bound on a compact subset of P.

PROOF OF SUFFICIENCY. By Theorem V 20.10, it is sufficient to show that Z^r bounds on all countable, star-finite coverings of M. Let $\mathfrak{U} = \{U_i\}$ be any such covering. By Lemmas V 20.14, 20.15, there exists a neighborhood covering $\mathfrak{U}' = \{U_i'\}$ of M such that $U_i' \cap M = U_i$ and the complex $\mathfrak{U}$ is isomorphic with the complex $\mathfrak{U}'$ under the correspondence $U_i \leftrightarrow U_i'$. Let $P = \bigcup_i U_i'$. Then P is an open set and by hypothesis there exists a compact subset F of P such that $Z^r \sim 0$ on F.

Now by Lemma 4.1, only a finite number of elements of $\mathfrak{U}'$ meet F, and these form a fcos $\mathfrak{B}'$ of F. Hence $Z^r(\mathfrak{B}') \sim 0$ on $\mathfrak{B}' \cap F$. But this implies that $Z^r(\mathfrak{U}')$ bounds a finite chain on $\mathfrak{U}'$. This in turn implies, because of the isomorphism between $\mathfrak{U}$ and $\mathfrak{U}'$, that $Z^r(\mathfrak{U}) \sim 0$ on $\mathfrak{U}$.

For purposes of an application in Chapter XII, we note here that Theorem 4.4 can be generalized as follows (the proof being not essentially different from that of Theorem 4.4):

4.4a THEOREM. *Let P be an open subset of a perfectly normal, compact, lc^n space S, and let Z^r, $r \leqq n$, be a C-cycle on a compact subset M of P such that $Z^r \approx 0 \bmod K$ in P, where K is also a compact subset of P. Then $Z^r \sim 0 \bmod K$ on a compact subset of P.*

And Theorem 4.6 generalizes as follows:

4.6a THEOREM. *If M is an arbitrary subset of a perfectly normal, compact, lc^n space S, and Z^r, $r \leqq n$, is a C-cycle on a compact subset of M, then a necessary and sufficient condition that $Z^r \approx 0 \bmod K$ in M, where K is a compact subset of M, is that for every open subset P of S that contains M, $Z^r \sim 0 \bmod K$ on a compact subset of P.*

5. **Fundamental systems of cycles.** In this section we shall show that for compact G_δ subsets of an lc^n space there exist countable sets of cycles analogous

[4]Compare Theorem 3.1 above.

to homology bases. The reader may wish to compare the material herein with that of §V 12.

5.1 Theorem. *Suppose that S is lc^r for some nonnegative integer r, and that $U_1, \cdots, U_k, \cdots$ is a sequence of open sets such that (1) $\overline{U}_k$ is compact, (2) $U_k \supset \overline{U}_{k+1}$; let $M = \bigcap U_k$. Then there exists a sequence of C-cycles $\gamma_1^r, \cdots, \gamma_{n(1)}^r, \cdots, \gamma_{n(k)}^r, \cdots$, on M such that for each k, $\gamma_1^r, \cdots, \gamma_{n(k)}^r$ form a base for r-cycles on M relative to homologies on $\overline{U}_k$, and $\gamma_i^r \sim 0$ on $\overline{U}_k$ for $i > n(k)$.*

Proof. By Corollary 3.8 only a finite number of Čech r-cycles on M are lirh on $\overline{U}_1$. Let $\gamma_1^r, \cdots, \gamma_{n(1)}^r$ be a base for such cycles. Then by Lemma V 18.26 there exists a base for r-cycles of M relative to homologies on $\overline{U}_2$ of the form $\gamma_1^r, \cdots, \gamma_{n(1)}^r, Z_{n(1)+1}^r, \cdots, Z_{n(2)}^r$ (if the γ's alone form such a base we would of course proceed to a U_k where the Z's form a nonempty system). Now we have

$$Z_j^r \sim \sum_{i=1}^{n(1)} a_j^i \gamma_i^r = \Gamma_j^r \qquad \text{on} \qquad \overline{U}_1, \qquad j = n(1) + 1, \cdots, n(2).$$

Let $\gamma_j^r = \Gamma_j^r - Z_j^r$. Then $\gamma_1^r, \cdots, \gamma_{n(1)}^r, \gamma_{n(1)+1}^r, \cdots, \gamma_{n(2)}^r$ form a base for cycles of M relative to homology on $\overline{U}_2$. For suppose there exists a relation $\sum_{i=1}^{n(2)} b^i \gamma_i^r \sim 0$ on $\overline{U}_2$. Then substituting for the γ_i^r for $i > n(1)$, the cycles $\Gamma_i^r - Z_i^r$, we get

$$\sum_{i=1}^{n(1)} \left[b^i + \sum_{j=n(1)+1}^{n(2)} b^j a_j^i \right] \gamma_i^r - \sum_{j=n(1)+1}^{n(2)} b^j Z_j^r \sim 0 \qquad \text{on} \qquad \overline{U}_2 .$$

If some b^j is not zero, a contradiction of the lirh of the original set of cycles γ_i^r, Z_j^r on $\overline{U}_2$ results. And if all b^j are zero, then a relation $\sum_{i=1}^{n(1)} b^i \gamma_i^r \sim 0$ on $\overline{U}_2$ results. As the γ_i^r, $i = 1, \cdots, n(2)$, then, are lirh on $\overline{U}_2$ and $n(2)$ in number, they form a base of the required sort.

5.2 Corollary. *If S is a locally compact and lc^r space, and M is a compact G_δ in S, then there exist open sets U_k as in the hypothesis of Theorem 5.1, and consequently C-cycles of type $\gamma_1^r, \cdots, \gamma_{n(k)}^r, \cdots$. In particular, if γ^r is any C-cycle on M and P is an open set containing M, then γ^r is homologous on $\overline{P}$ to a finite linear combination of the cycles γ_i^r.*

5.3 Definition. A set of cycles such as the collection γ_i^r whose existence is proved above will be called a *fundamental system of r-cycles* of M. The justification for this terminology will appear in the sequel. In the meantime we record the following obvious corollaries:

5.4 Corollary. *Every compact subset of euclidean space has a countable fundamental system of Čech r-cycles for every dimension r.*

For instance, reverting to the Peano continuum M of V 12.1, it may be shown that M has a fundamental system $\{\gamma_k^1\}$ such that γ_k^1 is a cycle on M_k.

5.5 Corollary. *Every compact subset of a locally compact, metric, lc^r space has a countable fundamental system of Čech r-cycles.*

The r-dimensional homology group of a compact set M having a countable fundamental system of Čech r-cycles as above will bear further analysis. Let the cycles $\gamma_1^r, \cdots, \gamma_{n(k)}^r, \cdots$ be determined as before (Theorem 5.1) relative to the open sets $U_1, \cdots, U_k, \cdots$. Let $\{Z_1^r\}$, $\{Z_2^r\}$ be elements of $H^r(M; \mathfrak{F})$, with $Z_1^r \in \{Z_1^r\}$, $Z_2^r \in \{Z_2^r\}$. Then we define a distance function $\rho(\{Z_1^r\}, \{Z_2^r\}) = 1/k$, where k is the largest positive integer such that $Z_1^r \sim Z_2^r$ on $\overline{U}_k$, if such a k exists. If $Z_1^r \nsim Z_2^r$ on $\overline{U}_1$, we let $\rho(\{Z_1^r\}, \{Z_2^r\}) = 1$, and if $Z_1^r \sim Z_2^r$ on all $\overline{U}_k$, we let $\rho(\{Z_1^r\}, \{Z_2^r\}) = 0$. By Corollary V 21.12, $\rho(\{Z_1^r\}, \{Z_2^r\}) = 0$ implies that $Z_1^r \sim Z_2^r$ on M, hence $\{Z_1^r\} = \{Z_2^r\}$. We leave to the reader the rest of the details of the proof that in this manner $H^r(M; \mathfrak{F})$ becomes a metric space.

The metric space so defined is complete. Consider a Cauchy sequence (I 3.2) of homology classes $\{Z_i^r\}$, $i = 1, 2, 3, \cdots$, and $Z_i^r \in \{Z_i^r\}$. By Lemma V 8.7 there exists a complete family Σ' of coverings such that if $\mathfrak{U} \in \Sigma'$, there exists an open set P containing M such that if $U \in \mathfrak{U}$ meets P, then U meets M. For each such $\mathfrak{U}$ let us choose the U_k with smallest subscript $k(\mathfrak{U}) > 1$ such that $U_{k(\mathfrak{U})}$ can serve as a set "P" for $\mathfrak{U}$. Let $i(\mathfrak{U})$ be the smallest natural number such that $\rho(\{Z_{i(\mathfrak{U})}^r\}, \{Z_{i(\mathfrak{U})+s}^r\}) < 1/k(\mathfrak{U})$ for all s: that is, $Z_{i(\mathfrak{U})}^r \sim Z_{i(\mathfrak{U})+s}^r$ on $U_{k(\mathfrak{U})}$ for all s. Let $Z^r(\mathfrak{U}) = Z_{i(\mathfrak{U})}^r(\mathfrak{U})$. Then Z^r is a C-cycle on M. For suppose $\mathfrak{U}, \mathfrak{V} \in \Sigma'$, $\mathfrak{V} > \mathfrak{U}$. We must show that $\pi_{\mathfrak{U}\mathfrak{V}} Z_{i(\mathfrak{V})}^r(\mathfrak{V}) \sim Z_{i(\mathfrak{U})}^r(\mathfrak{U})$ on M.

There are two cases: (1) If $k(\mathfrak{V}) \geqq k(\mathfrak{U})$, then $i(\mathfrak{V}) \geqq i(\mathfrak{U})$ and $Z_{i(\mathfrak{U})}^r(\mathfrak{U}) \sim Z_{i(\mathfrak{V})}^r(\mathfrak{U})$ on $\overline{U}_{k(\mathfrak{U})}$ and hence on M. Since $\pi_{\mathfrak{U}\mathfrak{V}} Z_{i(\mathfrak{V})}^r(\mathfrak{V}) \sim Z_{i(\mathfrak{V})}^r(\mathfrak{U})$ on M, the required homology follows. (2) If $k(\mathfrak{V}) < k(\mathfrak{U})$, then $i(\mathfrak{V}) \leqq i(\mathfrak{U})$ and $Z_{i(\mathfrak{V})}^r(\mathfrak{V}) \sim Z_{i(\mathfrak{U})}^r(\mathfrak{V})$ on $\overline{U}_{k(\mathfrak{V})}$ and hence on M. Consequently $\pi_{\mathfrak{U}\mathfrak{V}} Z_{i(\mathfrak{V})}^r(\mathfrak{V}) \sim \pi_{\mathfrak{U}\mathfrak{V}} Z_{i(\mathfrak{U})}^r(\mathfrak{V})$ on M, and from $\pi_{\mathfrak{U}\mathfrak{V}} Z_{i(\mathfrak{U})}^r(\mathfrak{V}) \sim Z_{i(\mathfrak{U})}^r(\mathfrak{U})$ on M the required relation again follows.

To see that $\lim_{i\to\infty}\{Z_i^r\} = \{Z^r\}$, select any U_k and let N_k be the smallest natural number such that $Z_i^r \sim Z_j^r$ on $\overline{U}_k$ for all $i, j \geqq N_k$. Let Σ'' be the set of all elements of Σ' that are refinements of the covering constituted by the open sets U_k, $S - M$. If $\mathfrak{U} \in \Sigma''$, then $k(\mathfrak{U}) \geqq k$, hence $i(\mathfrak{U}) \geqq N_k$. Consequently $Z_{i(\mathfrak{U})}^r(\mathfrak{U}) \sim Z_i^r(\mathfrak{U})$ on $\overline{U}_k$ for all $i \geqq N_k$.

If, now, we define $\sum_{i=1}^{\infty} a^i\gamma_i^r = \lim_{m\to\infty} \sum_{i=1}^{m} a^i\gamma_i^r$, where the γ_i^r are the elements of the fundamental system of r-cycles of M, we see that every such sum exists since the partial sums $\sum_{i=1}^{m} a^i\gamma_i^r$ form a Cauchy sequence. Conversely, every cycle γ^r of M satisfies a unique homology of the form $\gamma^r \sim \sum_{i=1}^{\infty} a^i\gamma_i^r$. The latter may be obtained as follows: Let $\gamma^r \sim \sum_{i=1}^{n(1)} a^i\gamma_i^r$ on $\overline{U}_1$ and $\gamma^r \sim \sum_{i=1}^{n(2)} b^i\gamma_i^r$ on $\overline{U}_2$. Then $a^i = b^i$ for $i = 1, \cdots, n(1)$, since otherwise there would exist a homology $\sum_{i=1}^{n(1)} (a^i - b^i)\gamma_i^r \sim 0$ on $\overline{U}_1$. In other words, the coefficient of γ_i^r, for $n(k) < i \leqq n(k+1)$, is the same as in the homology $\gamma^r \sim \sum_{i=1}^{n(k+1)} a^i\gamma_i^r$. Then $\gamma^r \sim Z^r = \sum_{i=1}^{\infty} a^i\gamma_i^r$, by application of Corollary V 21.12.

Viewed from the standpoint of the Cauchy-Cantor-Meray-Hausdorff construction for completing a metric space, the space $H^r(M; \mathfrak{F})$ is the "completion" of the space $H^r_f(M; \mathfrak{F})$ which consists of all finite linear forms $\sum_{i=1}^m a^i \gamma^r_i$, $a^i \in \mathfrak{F}$, and whose metric is determined as above. We have proved, then:

5.6 THEOREM. *If S is a locally compact, lc^r space and M is a compact G_δ in S, then the group $H^r(M; \mathfrak{F})$ forms a complete metric space having a countable subset $\Gamma^r_1, \cdots, \Gamma^r_i, \cdots$ such that $\lim_{i\to\infty} \Gamma^r_i = 0$ and every $\Gamma^r \in H^r(M; \mathfrak{F})$ is uniquely expressible in the form $\Gamma^r = \sum_{i=1}^\infty a^i \Gamma^r_i$, $a^i \in \mathfrak{F}$. In particular, the vector subspace $H^r_f(M; \mathfrak{F})$ generated by the base elements Γ^r_i is dense in $H^r(M; \mathfrak{F})$ and uniquely determines the latter.*

Finally, we observe that the space $H^r_f(M; \mathfrak{F})$ above—and hence $H^r(M; \mathfrak{F})$—is of a unique type. This is trivial for the finite-dimensional case, of course. If M_1, M_2 are compact spaces of type M—i.e., each is a G_δ in some locally compact lc^r space—and γ^r_i, $i = 1, 2, \cdots$, Z^r_i, $i = 1, 2, \cdots$ are respective fundamental systems of r-cycles, then for each $\sum_{i=1}^m a^i\{\gamma^r_i\} \in H^r_f(M_1; \mathfrak{F})$ let correspond the element $\sum_{i=1}^m a^i\{Z^r_i\}$ of $H^r_f(M_2; \mathfrak{F})$. This is not only an isomorphism between these spaces as vector spaces, but a homeomorphism between the complete metric spaces. This is a direct result of the fact that a relation $\sum_{i=1}^h c^i\{\gamma^r_i\} \sim 0$ on $\overline{U}_k$ implies that $c^i = 0$ for $i \leqq n(k)$. Hence if $\lim_{\nu\to\infty} \sum_{i=1}^{h_\nu} a^i_\nu\{\gamma^r_i\} = \sum_{i=1}^h a^i\{\gamma^r_i\}$, then ultimately $a^i_\nu = a^i$ for $i = 1, 2, \cdots, h$ and a^i_ν for $i > h$ is nonzero only for greater and greater values of i.

Another way of looking at the matter is to observe that the topology of $H^r(M; \mathfrak{F})$ as set up above turns out to be the same as that of the product space (I 12) formed by a countable collection of sets $\mathfrak{F}_i$, where each $\mathfrak{F}_i$ is isomorphic with the field $\mathfrak{F}$. We leave the details of the proof of this fact to the reader.

5.7 THEOREM. *The groups $H^r(M; \mathfrak{F})$ of compact G_δ subsets of locally compact, lc^r spaces are completely determined as to topological structure by the field $\mathfrak{F}$ and the Betti numbers $p^r(M; \mathfrak{F})$; for $p^r(M; \mathfrak{F})$ infinite, this structure is always that of the product space of a denumerable collection of sets $\mathfrak{F}_i$ each of which is isomorphic with the field $\mathfrak{F}$.*

REMARK. Theorem 5.6 is a generalization of a theorem of Vietoris [a] to the effect that the r-dimensional Betti group of any compact metric space has a fundamental system of cycles Γ^r_i such as that of Theorem 5.6. For since (1) in a compact metric space the homology theory in terms of the Vietoris cycles and C-cycles are equivalent, and (2) every compact metric space can be imbedded in the fundamental parallelopiped of Hilbert space (Theorem III 1.14), it follows that every compact metric space satisfies the hypothesis and conclusion of Theorem 5.6.

The significance of the above theorems for the cohomology groups of compact spaces imbeddable as G_δ subsets of locally connected spaces is brought out by the following theorem:

5.8 THEOREM. *If the compact space M is imbeddable as a G_δ in a locally compact, lc^r space S, then there exist a fundamental system $\{\gamma_i^r\}$ of r-cycles of M and a countable base $\{\gamma_r^i\}$ for cocycles of M relative to cohomologies on M such that $\gamma_i^r \cdot \gamma_r^i = \delta_i^i$. In particular, $H_f^r(M) = H_r(M)$.*

PROOF. Identify M with the M of Theorem 5.1. Having selected $\gamma_1^r, \cdots, \gamma_{n(1)}^r$ as in the proof of that theorem, let $\gamma_r^1, \cdots, \gamma_r^{n(1)}$ be cocycles mod $S - \overline{U}_1$ such that $\gamma_i^r \cdot \gamma_r^j = \delta_i^j$; $i, j \leqq n(1)$. This is possible by Theorem V 18.30. The cocycles γ_r^j form a base for cocycles mod $S - \overline{U}_1$ relative to cohomologies mod $S - M$.

By Lemma V 18.27, a base for cycles of M relative to homologies on $\overline{U}_2$ may be formed by the addition of cycles γ_i^r, $i = n(1) + 1, \cdots, n(2)$ to the set $\gamma_1^r, \cdots, \gamma_{n(1)}^r$, and a base for cocycles mod $S - \overline{U}_2$ relative to cohomologies mod $S - M$ can be formed by the addition of cocycles γ_r^j, $j = n(1) + 1, \cdots, n(2)$, mod $S - \overline{U}_2$, to the set $\gamma_r^1, \cdots, \gamma_r^{n(1)}$, in such a way that $\gamma_i^r \cdot \gamma_r^j = \delta_i^j$ for all $i, j \leqq n(2)$.

Continuing in this manner, there are generated two (finite or infinite) sequences $\{\gamma_i^r\}$, $\{\gamma_r^j\}$ such that (1) for each $\overline{U}_m$, the γ_i^r for $i \leqq n(m)$ form a base for cycles of M relative to homologies on $\overline{U}_m$, (2) the γ_r^j for $j \leqq n(m)$ form a base for cocycles mod $S - \overline{U}_m$ relative to cohomologies mod $S - M$, and (3) $\gamma_i^r \cdot \gamma_r^j = \delta_i^j$ for all i, j.

Now we assert that the cocycles γ_r^j, as cocycles mod $S - M$, form a base for cohomologies mod $S - M$. In the first place, they are lircoh mod $S - M$. For a relation $\sum_{j=1}^h b_j\gamma_r^j \backsim 0$ mod $S - M$ would imply for some j (such that $b_j \neq 0$) that $\gamma_j^r \cdot \gamma_r^j = 0$ (the cycles γ_i^r being on M). And in the second place, if γ_r is any cocycle mod $S - M$, given on some covering $\mathfrak{U}$ of S, let $\mathfrak{V} > \mathfrak{U}$ such that if the closure of a simplex of $\mathfrak{V}$ meets M, then the simplex is on M (Lemma V 8.7). Then $\delta\pi^*_{\mathfrak{U}\mathfrak{V}}\gamma_r$ is in $S - \overline{U}_m$ for some m, and hence there exists $\mathfrak{W} > \mathfrak{V}$ such that $\pi^*_{\mathfrak{V}\mathfrak{W}}\pi^*_{\mathfrak{U}\mathfrak{V}}\gamma_r \backsim \sum_{j=1}^{n(m)} a_j\gamma_r^j$ mod $S - M$.

That the set $\{\gamma_r^j\}$ forms a base for cocycles of M relative to cohomology on M follows from V 8.11.

6. Local co-connectedness; local connectivity numbers and local dualities. As in the case of local connectedness, the concept of local-co-connectedness may be formulated in various ways. The strict analogue of Definition 1.1 above is as follows:

6.1 DEFINITION. The space S is *n-colc (locally co-connected in dimension n)* at the point x if for each open set P containing x there exists an open set Q such that $x \in Q \subset P$ and such that if z_n is a cocycle *in* Q, then $z_n \backsim 0$ *in* P.

It will be recalled that if z_n is defined as $z_n(\mathfrak{U})$, then for any $\mathfrak{V} > \mathfrak{U}$, $\pi^*_{\mathfrak{U}\mathfrak{V}}z_n(\mathfrak{U})$ is in Q if $z_n(\mathfrak{U})$ is in Q. And by $z_n \backsim 0$ in P is meant that for some $\mathfrak{V} > \mathfrak{U}$, there is a chain $c^{n-1}(\mathfrak{V})$ in P such that $\delta c^{n-1}(\mathfrak{V}) = \pi^*_{\mathfrak{U}\mathfrak{V}}z_n(\mathfrak{U})$. Consequently the above definition may be stated in the following equivalent form:

6.2 DEFINITION. S is n-colc at $x \in S$ if for each open set P containing x there exists an open set Q such that $x \in Q \subset P$, and for each covering $\mathfrak{U}$ there exists a covering $\mathfrak{V} > \mathfrak{U}$ such that if $z_n(\mathfrak{U})$ is a cocycle in Q, then $\pi^*_{\mathfrak{U}\mathfrak{V}} z_n(\mathfrak{U}) \backsim 0$ in P.

To see that Definitions 6.1 and 6.2 are equivalent, suppose x, P, Q are given as in Definition 6.1. Then, given $\mathfrak{U}$, there exists a finite base of cocycles of $\mathfrak{U}$ in Q, say $z_n^i(\mathfrak{U})$, $i = 1, \cdots, k$, relative to cohomologies on $\mathfrak{U}$ in P. For each $z_n^i(\mathfrak{U})$ there exists a $\mathfrak{V}_i > \mathfrak{U}$ such that $\pi^*_{\mathfrak{U}\mathfrak{V}_i} z_n^i(\mathfrak{U}) \backsim 0$ on $\mathfrak{V}_i$ in P. Let $\mathfrak{V} > (\mathfrak{V}_1, \cdots, \mathfrak{V}_k)$. Then

(a) $$\pi^*_{\mathfrak{U}\mathfrak{V}} z_n^i(\mathfrak{U}) \backsim 0 \qquad \text{on } \mathfrak{V} \text{ in } P, \qquad i = 1, \cdots, k.$$

Then if $z_n(\mathfrak{U})$ is any cocycle of $\mathfrak{U}$ in Q, we have

(b) $$z_n(\mathfrak{U}) \backsim \sum_{i=1}^{k} c^i z_n^i(\mathfrak{U}), \qquad c^i \in \mathfrak{F}, \text{ on } \mathfrak{U} \text{ in } P.$$

Relation (b) implies that

$$\pi^*_{\mathfrak{U}\mathfrak{V}} z_n(\mathfrak{U}) \backsim \sum_{i=1}^{k} c^i \pi^*_{\mathfrak{U}\mathfrak{V}} z_n^i(\mathfrak{U}) \qquad \text{on } \mathfrak{V} \text{ in } P,$$

and relations (a) and (b) imply that $\pi^*_{\mathfrak{U}\mathfrak{V}} z_n(\mathfrak{U}) \backsim 0$ on $\mathfrak{V}$ in P.

The converse, namely that if the conditions of Definition 6.2 are satisfied, then those of Definition 6.1 follow, is immediate.

In Definition 6.2, given Q and $\mathfrak{V}$ satisfying the stated requirements, evidently any open set Q' such that $x \in Q' \subset Q$, and covering $\mathfrak{W} > \mathfrak{V}$, will serve in place of Q and $\mathfrak{V}$ respectively.

The definition of *uniform n-colc* (= *n-coulc*) may be given as follows:

6.3 DEFINITION. S is *n-coulc* if given $\mathfrak{U}$, $\mathfrak{V} \in \Sigma$, there exist $\mathfrak{U}_1^n = \mathfrak{U}_1^n(\mathfrak{U})$ and $\mathfrak{V}_1^n = \mathfrak{V}_1^n(\mathfrak{U}, \mathfrak{V})$ such that if $z_n(\mathfrak{V})$ is in an element of $\mathfrak{U}_1^n$, then $\pi^*_{\mathfrak{V}\mathfrak{V}_1^n} z_n(\mathfrak{V})$ cobounds a chain of $\mathfrak{V}_1^n$ which lies in some element of $\mathfrak{U}$.

6.4 THEOREM. *If the compact space S is n-colc at every point, then it is n-coulc.*

The proof is virtually a paraphrase of the proof of Theorem 2.8, with Definition 6.2 replacing Definition 1.2, π^* replacing π, etc.

6.5 DEFINITION. If a space is r-colc for $r = k, k + 1, \cdots, m$, we may indicate this by the symbol colc$_k^m$; colc$_0^m$ is abbreviated to colc^m.

6.6 *Local connectivity numbers.* Let $x \in S$, and let P and Q be open sets such that $x \in Q \subset P$. Let $Z^n(x; P, Q)$ be the vector space formed by the Čech n-cycles of S mod $S - P$ (that is, given P as defined above, $Z^n(x; P, Q)$ is the group $Z^n(S; S, S - P; \mathfrak{F})$ of V 7.10), and $B^n(x; P, Q)$ the subspace formed by those cycles of $Z^n(x; P, Q)$ that bound mod $S - Q$. Let $H^n(x; P, Q) = Z^n(x; P, Q)/B^n(x; P, Q)$, made into a vector space in the usual way, and let $p^n(x; P, Q)$ be the dimension of $H^n(x; P, Q)$. Evidently $p^n(x; P, Q)$ is the maxi-

mum number of Čech n-cycles of S mod $S - P$ that are linearly independent with respect to homologies mod $S - Q$.

If Q' is an open set such that $Q' \subset Q$, then $p^n(x; P, Q') \leqq p^n(x; P, Q)$, and consequently there is a greatest cardinal number $p^n(x; P) \leqq p^n(x; P, Q)$ for all open sets Q which lie in P and contain x. For an open set P' such that $P \supset P' \supset Q$, $p^n(x; P, Q) \leqq p^n(x; P', Q)$, so there is a least cardinal number $p^n(x) \geqq p^n(x; P)$ for all open sets P that contain x. The number $p^n(x)$ may be called the *n-dimensional local Betti number* (*over* $\mathfrak{F}$) *of S at the point x.* (In case it is necessary in order to avoid confusion, the space S will be indicated in the symbol for $p^n(x)$—thus, $p^n(S, x)$. This will be particularly convenient in case S is a subspace of another space.) We shall not attempt to distinguish between the cardinalities of infinite values of $p^n(x)$, as a rule. However, in case $p^n(x; P)$ is finite for all P, but has no finite upper bound, we write $p^n(x) = \omega$. And in case $p^n(x; P)$ is infinite for some P, we shall write simply $p^n(x) = \infty$. We shall find these distinctions of value later on.

Denote by $Z_n(x; P, Q)$ the vector space formed by the set of all n-cocycles in Q and by $B_n(x; P, Q)$ the subspace formed by those elements of $Z_n(x; P, Q)$ that are cohomologous to zero in P. Let $H_n(x; P, Q) = Z_n(x; P, Q)/B_n(x; P, Q)$, made into a vector space, and denote the dimension of $H_n(x; P, Q)$ by $p_n(x; P, Q)$. In analogy to the above, we may define a number $p_n(x; P) \leqq p_n(x; P, Q)$ and a number $p_n(x) \geqq p_n(x; P)$. The number $p_n(x)$ may be called the *n-dimensional local co-Betti number of S at x.* (The symbol $p_n(S, x)$ will be used when it is advisable to indicate the space determining the number $p_n(x)$.) Regarding its infinite values we make the same conventions as in the case of $p^n(x)$.

6.7 THEOREM. *If either of the numbers $p_n(x; P, Q)$, $p^n(x; P, Q)$ is finite, then they are equal.*

PROOF. It is only necessary to notice that the two vector spaces $H_n(x; P, Q)$, $H^n(x; P, Q)$ form an orthogonal dual pair relative to the usual multiplication between cocycles and relative cycles as defined in Chapter V. If z^n is a C-cycle mod $S - P$ which fails to bound mod $S - Q$, then a fortiori z^n is a cycle mod $S - Q$ which fails to bound mod $S - Q$. Hence by Corollary V 18.24, there exists a cocycle z_n in Q such that $z_n \cdot z^n \neq 0$. Conversely, if a cocycle z_n in Q fails to cobound in P, again by Corollary V 18.24 there exists a C-cycle z^n mod $S - P$ such that $z_n \cdot z^n \neq 0$. Hence $H_n(x; P, Q)$ and $H^n(x; P, Q)$ form an orthogonal dual pair.

6.8 COROLLARY. *For each $x \in S$, $p_n(x) = p^n(x)$.*

The reader may prove the following theorem:

6.9. THEOREM. *In any topological space S, if $x \in S$, then $p^0(x)$ is always* 0, 1 *or* ∞. *The value* 1 *occurs if and only if x is not a limit point of $S - x$. And if for some neighborhood P of x, every quasi-component of S meets $S - P$, then* $p^0(x) = 0$.

It has probably already occurred to the reader that the case $p_n(x) = 0$ corresponds precisely to the case where S is n-colc at x. And in combination with Corollary 6.8 we have,

6.10 THEOREM. *In order that S should be n-colc at $x \in S$, it is necessary and sufficient that either $p_n(x) = 0$ or $p^n(x) = 0$.*

6.11 In arriving at the conclusion just made, it was necessary to observe that if $p^n(x)$ is finite, then any open set which contains x will also contain open sets P and Q such that $x \in Q \subset P$ and $p^n(x; P, Q) = p^n(x) = p_n(x) = p_n(x; P, Q)$. Such a pair of neighborhoods P and Q will be called a *canonical pair of neighborhoods of x* (relative to n and the local Betti number).

6.12 COROLLARY. *If the space S has only a finite number of components, then it is 0-colc at all points.*

Exactly as the numbers $p_n(x)$ and $p^n(x)$ were defined, there may be defined numbers $g^n(x)$ and $g_n(x)$ by interchanging the roles of cycle and cocycle. That is, $g^n(x)$ is defined in terms of absolute cycles and $g_n(x)$ in terms of relative cocycles. Starting with x, P and Q as before, $g^n(x; P, Q)$ will be the number of C-cycles on Q that are linearly independent with respect to homologies on P. For $Q \supset Q'$, $g^n(x; P, Q) \geqq g^n(x; P, Q')$, so that $g^n(x; P)$ can be defined as the greatest cardinal number such that $g^n(x; P) \leqq g^n(x; P, Q)$ for all Q. If $P \supset P' \supset Q$, $g^n(x; P, Q) \leqq g^n(x; P', Q)$, so that $g^n(x)$ can be defined as the least cardinal number $\geqq g^n(x; P)$ for all P. As in the case of $p^n(x)$, if $g^n(x; P)$ is infinite for some P, then we write $g^n(x) = \infty$. It will be unnecessary to make any further conventions for the case of $g^n(x)$, however, because of the following theorem.

6.13 THEOREM. *The only two possible values of $g^n(x)$ are 0 and ∞.*

PROOF. Suppose that $g^n(x; P)$ is finite for all P. Then, given P, there exists Q such that $g^n(x; P, Q) = g^n(x; P)$; denote this common value by k and suppose $k > 0$. On Q there are exactly k C-cycles, say $z_1^n, \cdots, z_k^n$, that are linearly independent with respect to homologies on P. There exists $\mathfrak{U} \in \Sigma$ such that for all $\mathfrak{B} > \mathfrak{U}$, $z_1^n(\mathfrak{B}), \cdots, z_k^n(\mathfrak{B})$ are linearly independent with respect to homologies on $\mathfrak{B} \wedge P$ (Theorem V 19.2).

Denote those elements of $\mathfrak{U}$ that meet Q by $U_1, \cdots, U_m$. Suppose $x \in U_1$. Let V be an open set such that $x \in V$ and $\overline{V} \subset U_1 \cap Q$. For $i > 1$, let $V_i = U_i - \overline{V}$. Then the covering $\mathfrak{B}$ obtained by replacing U_i by V_i, $i > 1$, in $\mathfrak{U}$ is a refinement of $\mathfrak{U}$ in which only one element, U_1, meets V.

Let γ_1^n be a C-cycle on V that is $\nsim 0$ on P. Then $\gamma_1^n \sim \sum_{i=1}^k c^i z_i^n$ on P, where each $c^i \in \mathfrak{F}$ and not all c^i's are zero. In particular,

$$\gamma_1^n(\mathfrak{B}) \sim \sum_{i=1}^{k} c^i z_i^n(\mathfrak{B}) \qquad \text{on } \mathfrak{B} \wedge P. \tag{6.13a}$$

But $\gamma_1^n(\mathfrak{B}) = 0$.

Hence $\sum_{i=1}^{k} c^i z_i^n(\mathfrak{B}) \sim 0$ on $\mathfrak{B} \wedge P$, contradicting the fact that the cycles $z_i(\mathfrak{B})$ are linearly independent with respect to homologies on $\mathfrak{B} \wedge P$. It must be concluded, then, that $k = 0$; i.e., $g^n(x) = 0$.

The proof of 6.13 also establishes the following theorem which will be found very useful later on.

6.14 THEOREM. *In order that S should be n-lc at $x \in S$, it is necessary and sufficient that each open set which contains x also contain a pair of open sets P, Q such that $x \in Q \subset P$ and such that $g^n(x; P, Q)$ is finite.*

6.15 COROLLARY. *In order that S should be n-lc at $x \in S$, it is necessary and sufficient that either $g^n(x) = 0$ or $g_n(x) = 0$.*

(The dualities $g^n(x; P, Q) = g_n(x; P, Q)$, and hence $g^n(x) = g_n(x)$, are proved as in the case of the numbers "p".)

As a corollary of 3.8 and Theorem 6.14 we may state:

6.16 THEOREM. *If S is a locally compact space, then a necessary and sufficient condition that S be lc^n is that for every compact subset M and open set P containing M, the number of r-cycles on M that are lirh on P is finite, $r \leqq n$.*

7. **Property $(P, Q)_n$.** The various types of local connectivity defined in the previous section may be extended so as to be special cases of what we might call *connectivity numbers about a set.* The sole change, to accomplish this, is to replace the point x by a subset of S which is not necessarily a single point. Although we shall not have occasion to pursue this idea further in the present chapter, we shall find it useful later on (see VII 8). For the present we propose to investigate a related notion.

7.1 DEFINITION. A locally compact space S will be said to have *property $(P, Q)_n$*, n a non-negative integer, if for every pair of open sets P, Q such that $\overline{Q}$ is compact and $\overline{Q} \subset P$, the vector space $H_n(S\colon Q, 0; P, 0)$ is of finite dimension.

We shall prove the following theorem:

7.2 THEOREM. *If the locally compact space S has property $(P, Q)_{n+1}$, and $p_n(x)$ is finite or ω for all $x \in S$, then S has property $(P, Q)_n$.*

PROOF. Evidently if $P \supset P' \supseteq Q' \supset Q$, and the number $p_n(S\colon Q', 0; P', 0)$ is finite, then $p_n(S\colon Q, 0; P, 0)$ is finite. For each $x \in S$ and open set O containing x, there exist $P(x)$ and $Q(x)$ such that $O \supset P(x) \supset Q(x) \supset x$ and $p_n(S\colon Q(x), 0; P(x), 0)$ is finite. Then any open set $Q'(x)$ such that $x \in Q'(x)$ and $Q'(x) \Subset Q(x)$ yields a pair of sets $P(x)$, $Q'(x)$ which we call a *special canonical pair for x.*

Suppose now that P and Q are given. Since $\overline{Q}$ is compact, we may replace P by a smaller open set whose closure is compact, so that we may as well assume that $\overline{P}$ is compact. If $x \in \overline{Q}$, we may select a special canonical pair $P(x)$, $Q'(x)$ such that $P(x) \subset P$, and since $\overline{Q}$ is compact we may cover $\overline{Q}$ by a finite number

$Q'(x_1), \cdots, Q'(x_k)$ of the sets $Q'(x)$. Then if $p_n(S: \bigcup_{i=1}^k Q'(x_i), 0; \bigcup_{i=1}^k P(x_i), 0)$ is finite, the proof is complete in view of the opening remarks of the proof. Hence we may confine our attention to the case where Q is a set of type $\bigcup_{i=1}^k Q'(x_i)$.

Evidently if $k = 1$ above, the theorem is proved. Suppose it has been proved for the case where Q is the union of at most m sets $Q'(x_i)$, $m \geqq 1$, and let $P(x_i)$, $Q'(x_i)$, $i = 1, \cdots, m + 1$, be a set of special canonical pairs. Let $P_k = \bigcup_{i=1}^k P(x_i)$, $Q'_k = \bigcup_{i=1}^k Q'(x_i)$. If $Q'_m \supset Q'_{m+1}$, the proof is complete. Otherwise, let $Q''(x_i)$ be chosen such that $P(x_i) \supset Q(x_i) \Supset Q''(x_i) \Supset Q'(x_i)$, for each i; the pair $P(x_i)$, $Q''(x_i)$ is again a special canonical pair and the induction assumption applies. We may therefore assume that the set $Q''_m = \bigcup_{i=1}^m Q''(x_i)$ does not contain $Q''(x_{m+1})$. Let $R = Q''_m \cap Q''(x_{m+1})$. Then

$$\overline{(Q''(x_{m+1}) - R)} \cap (\overline{Q}'_m - Q''(x_{m+1})) = 0. \tag{7.2a}$$

For if this set contained a point p, then since $Q''_m \supset \overline{Q}'_m$, p would be a point of the boundary of $Q''(x_{m+1})$ in Q''_m and $p \in \overline{Q}'(x_i) \subset Q''(x_i)$, $i \leqq m$; but all points of $Q''(x_{m+1})$ in $Q''(x_i)$ are in R by definition and have therefore been removed from $Q''(x_{m+1})$ when R has been deleted from this set. Hence $p \notin \overline{Q''(x_{m+1}) - R}$. Let $Y = \overline{Q''(x_{m+1}) - R}$, $T = \overline{Q}'_m - Q''(x_{m+1})$; Y and T are disjoint compact sets by virtue of (7.2a).

Since Y is compact, there exists an open set $O \supset Y$ such that $\overline{O} \cap T = 0$. Let $F = \overline{Q}'_{m+1} \cap (\overline{O} - O)$. As $\overline{O}$ does not meet $\overline{Q}'_m - Q''(x_{m+1})$, a point of F must be a point of $\overline{Q}'_m$ in $Q''(x_{m+1})$, or of $\overline{Q}'(x_{m+1})$ in Q''_m, hence a point of R. That is, $F \subset R$. Also, $Q'_{m+1} - F = A \cup B$ separate, where $B = Q'_{m+1} \cap O$. Let G be an open set such that $R \Supset G \supset F$.

Since S has property $(P, Q)_{n+1}$, there exists an integer ρ such that every ρ $(n + 1)$-cocycles of G satisfy a cohomology relation in R. Also, by the induction assumption, there exists an integer α such that every α n-cocycles of Q''_m satisfy a cohomology relation in P_m. And there exists an integer β such that every β n-cocycles of $Q''(x_{m+1})$ satisfy a cohomology relation in $P(x_{m+1})$.

Now suppose there exist $\alpha \cdot \beta \cdot \rho$ cocycles $z_n^1, \cdots, z_n^{\alpha \cdot \beta \cdot \rho}$ of Q'_{m+1} that are independent with respect to cohomology in P. We may select a covering $\mathfrak{U}$ of S such that to each z_n^i, $i = 1, \cdots, \alpha \cdot \beta \cdot \rho$, corresponds a cocycle of $\mathfrak{U}$ in the same cohomology class with z_n^i; with no loss of generality we may suppose these to be the z_n^i themselves. And we may suppose that $\mathfrak{U}$ is selected so that St $(F, \mathfrak{U}) \subset G$. We split each z_n^i into chains z_{n1}^i, z_{n2}^i, where z_{n1}^i consists of the portion of z_n^i that is in A. Evidently each cocycle $z_{n+1}^i = \delta z_{n1}^i$ is in G. Hence each ρ of them satisfy a cohomology relation

$$\sum_{i=(\mu-1)\rho+1}^{i=\mu\rho} c_i z_{n+1}^i \backsim 0 \quad \text{in} \quad R; \qquad \mu = 1, 2, \cdots, \alpha \cdot \beta. \tag{7.2b}$$

Consider the cocycles

$$\Gamma_n^\mu = \sum_{i=(\mu-1)\rho+1}^{i=\mu\rho} c_i z_n^i, \qquad \mu = 1, 2, \cdots, \alpha\beta. \tag{7.2c}$$

As these are linear combinations of the z_n^i, they must be lircoh in P. However, the cohomologies (7.2b) imply that for each μ there exists a chain C_n^μ in R such that, if

$$\Gamma_{n1}^\mu = \sum_{i=(\mu-1)\rho+1}^{\mu\rho} c_i z_{n1}^i, \qquad \Gamma_{n2}^\mu = -\sum_{i=(\mu-1)\rho+1}^{\mu\rho} c_i z_{n2}^i,$$

then

$$\Gamma_n^\mu = \Gamma_{n1}^\mu - \Gamma_{n2}^\mu = (\Gamma_{n1}^\mu - C_n^\mu) + (C_n^\mu - \Gamma_{n2}^\mu), \tag{7.2d}$$

where the chains in the parentheses are cocycles of Q_m'' and $Q''(x_{m+1})$ respectively.

Letting $\gamma_{n1}^\mu = \Gamma_{n1}^\mu - C_n^\mu$, $\gamma_{n2}^\mu = C_n^\mu - \Gamma_{n2}^\mu$, and recalling that every α cocycles of Q_m'' satisfy a cohomology relation in P_m, we get relations

$$\sum_{\mu=(\nu-1)\alpha+1}^{\nu\alpha} a_\mu \gamma_{n1}^\mu \backsim 0 \quad \text{in} \quad P_m; \quad \nu = 1, 2, \cdots, \beta. \tag{7.2e}$$

Let $\bar{\gamma}_{nj}^\nu = \sum_{\mu=(\nu-1)\alpha+1}^{\nu\alpha} a_\mu \gamma_{nj}^\mu$, $j = 1, 2$. The β n-cocycles $\bar{\gamma}_{n2}^\nu$ satisfy a cohomology

$$\sum_{\nu=1}^{\beta} b_\nu \bar{\gamma}_{n2}^\nu \backsim 0 \qquad \text{in } P(x_{m+1}). \tag{7.2f}$$

But by (7.2e),

$$\sum_{\nu=1}^{\beta} b_\nu \sum_{\mu=(\nu-1)\alpha+1}^{\nu\alpha} a_\mu \gamma_{n1}^\mu \backsim 0 \qquad \text{in } P_m,$$

and adding these relations to (7.2f) we get

$$\sum_{\nu=1}^{\beta} b_\nu \sum_{\mu=(\nu-1)\alpha+1}^{\nu\alpha} a_\mu \Gamma_n^\mu \backsim 0 \qquad \text{in } P.$$

Thus the cocycles Γ_n^μ are not independent in P, contradicting the fact that, being linear combinations of the cocycles z_n^i, they must not satisfy any cohomology relation in P.

This completes the proof.

In the material that follows, we have need of the concept of dimension of a space. The one most convenient for our purposes is defined in terms of coverings as follows:

7.3 Definition. The *order* of a covering $\mathfrak{U}$ is the largest integer n such that there are $n + 1$ elements of $\mathfrak{U}$ that have a nonempty intersection. We may appropriately use the term *dimension* of $\mathfrak{U}$ instead of order, if we consider $\mathfrak{U}$ as a complex.

7.4 Definition. A compact space S is said to be *n-dimensional,* or to be *of dimension n,* if n is the smallest integer such that every covering has a refinement of order n. If no such n exists, S is said to be *infinite-dimensional.*

Definition 7.4 may be adapted to the case of locally compact spaces if we make use of the notion of coverings finite relative to a subset: A covering $\mathfrak{U}$ is called finite relative to a set M if $\mathrm{St}(M, \mathfrak{U})$ is finite.

7.5 DEFINITION. A locally compact space S is said to be *n-dimensional* if n is the smallest integer such that for every covering $\mathfrak{U}$ and compact subset F of S, $\mathfrak{U}$ has a refinement of order n finite relative to F.

EXAMPLES. The ordinary circle, S^1, has the property that given any covering $\mathfrak{U}$, there exists a refinement of $\mathfrak{U}$ consisting of a set of open intervals $I_0, I_1, \cdots, I_k$ such that $I_i \cap I_j \neq 0$ only if $|i - j| = 1 \bmod k - 1$.

If S is the cartesian plane, let k be a fixed integer and consider the "lattice" formed by the lines $x = m/k$, $y = m/k$, for all integers m. Each lattice point p is a vertex of 4 squares, but a slight translation in the x direction of the lower two squares will place p on only 3 of the squares. Then if each square is enclosed in an open set which is not too large, a covering of S results which has order 2.

In general, the euclidean n-sphere, S^n, may be shown to be n-dimensional according to the definition 7.4 by using a device similar to that of the preceding paragraph. (See Lebesgue [a].)

For later purposes we note the following two theorems:

7.6 THEOREM. *If the compact space S is n-dimensional, then $H_r(S) = 0$ for all $r > n$.*

(If z_r is a cocycle of S, then there exists a covering $\mathfrak{U}$ of order n on which z_r has a coordinate. Evidently $z_r(\mathfrak{U}) = 0$.)

REMARK. The same type of argument shows that all C-cycles on compact subsets of an n-dimensional, locally compact space S, which are of dimension greater than n, bound on those subsets.

7.7 THEOREM. *If the locally compact space S is n-dimensional, then for every $x \in S$ and $r > n$, $p^r(x) = p_r(x) = g^r(x) = g_r(x) = 0$; hence S is r-lc and r-colc.*

We can now state the following corollary of Theorem 7.2:

7.8 THEOREM. *If the locally compact space S is of dimension n, and $p_n(x)$ is finite or ω for all $x \in S$, then S has property $(P, Q)_n$.*

PROOF. Given P and Q as in Definition 7.1, since every covering has a refinement of order n finite relative to $\bar{Q}$, every $(n + 1)$-cocycle in Q is equivalent to 0 and it is therefore trivial that S has property $(P, Q)_{n+1}$.

7.9 THEOREM. *If the locally compact space S is of dimension n, and $p_r(x)$ is finite or ω for all $x \in S$ and $r \leqq n$, then S is lc^n.*

PROOF. S is n-lc. By hypothesis, if $x \in S$ and P is an open set containing x, then there exists Q such that $x \in Q \subset P$ and such that $p_n(x; P, Q) = p_n(x; P) =$ a finite number k. Let R be an open set such that $x \in R \Subset Q$. Then at most k Čech n-cycles on R are linearly independent with respect to homologies on P. For suppose there exist $k + 1$ C-cycles z_i^n on R that are linearly independent with respect to homologies on P. Then (see the closing remarks on V 18) there

exist cocycles z_n^i mod $S - \bar{P}$ such that $z_n^i \cdot z_i^n = \delta_i^i$. We may assume that the cocycles z_n^i all lie on a covering $\mathfrak{U}$ of order n such that $\mathrm{St}(\bar{R}, \mathfrak{U}) \subset Q$. Then the portion $\bar{z}_n^i(\mathfrak{U})$ of z_n^i on R is a cocycle in Q, since $\delta\bar{z}_n^i(\mathfrak{U}) = 0$. But inasmuch as $p_n(x; P, Q) = k$, there must exist a chain $c_{n-1}(\mathfrak{U})$ in P such that $\delta c_{n-1} = \sum_{i=1}^{k+1} a^i \bar{z}_n^i$. Then since $\sum_{i=1}^{k+1} a^i \bar{z}_n^i$ is a cobounding cocycle, its product with every n-cycle mod $S - P$ must be zero. Now not all the a^i are zero; suppose in particular that $a^1 \neq 0$. But since $\bar{z}_n^i$ is the entire portion of z_n^i on R, and z_1^n is on R, $\bar{z}_n^i \cdot z_1^n = z_n^i \cdot z_1^n = 0$ if $i \neq 1$ and $=1$ if $i = 1$. Hence $(\sum_{i=1}^{k+1} a^i \bar{z}_n^i) \cdot z_1^n = a^1 \bar{z}_n^1 \cdot z_1^n = a^1$. As a result of this contradiction we must conclude that $g^n(x; P, R)$ is finite, and that S is n-lc at x follows from Theorem 6.14.

S is r-lc, $r < n$. Given $x \in S$ and P as before, we select Q so that $p_r(x; P, Q) = p_r(x; P)$, and $\bar{Q}$ is compact. Let U and V be open sets such that $x \in V \Subset U \Subset Q$, and let R denote the open set $Q - \bar{V}$. Let W be an open set such that $R \Supset W \supset (\bar{U} - U)$.

By Theorem 7.8, S has property $(P, Q)_n$. Hence by successive applications of Theorem 7.2, S has property $(P, Q)_{r+1}$. Therefore there exists an integer m such that every m $(r + 1)$-cocycles of W are related by a cohomology relation in R. Let $k = m[p_r(x; P, Q) + 1]$. Then we assert that there are not as many as k r-cycles on V that are linearly independent with respect to homologies on P. For suppose that $z_1^r, \cdots, z_k^r$ are cycles on V that are linearly independent with respect to homologies on P. Then there exist cocycles z_r^j, $j = 1, \cdots, k$, mod $S - \bar{P}$ such that $z_i^r \cdot z_r^j = \delta_i^j$. Denote by $\bar{z}_r^j$ the portion of z_r^j on U. Now we may assume that each z_r^j is on a covering $\mathfrak{U}$ such that $\mathrm{St}(U, \mathfrak{U}) \subset U \cup W$. Then $z_{r+1}^j = \delta\bar{z}_r^j$ is both in $S - \bar{U}$ and $U \cup W$; hence in W. Consequently in groups of m-cocycles each, the cocycles z_{r+1}^j are related by cohomologies in R, and following methods similar to those used in the proof of Theorem 7.2 we show the existence of $k/m = p_r(x; P, Q) + 1$ cocycles Γ_r^ν which lie in Q. Some linear combination $\gamma_r = \sum_\nu a_\nu \Gamma_r^\nu$ of these cocycles must cobound in P, so that $\gamma_r \cdot z_i^r = 0$ for $i = 1, \cdots, k$. But the cycles z_i^r all lie on V and a contradiction is obtained as in the proof of n-lc above.

It is interesting to observe, perhaps, that if we denote by lc_k^n the property of being r-lc for $r = k, k + 1, \cdots, n$, then the above argument also proves:

7.10 THEOREM. *If the locally compact space S is of dimension n and $p_r(x) \leqq \omega$ for all $x \in S$ and all r such that $0 \leqq k \leqq r \leqq n$, then S is lc_k^n.*

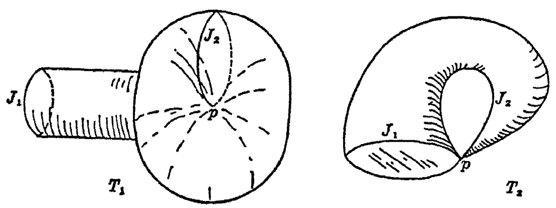
7.11. REMARK. As will be seen later (VII 2.25), Theorem 7.9 has a converse in that every n-dimensional, locally compact, lc^n space has $p_r(x) \leqq \omega$ for all $x \in S$ and $r \leqq n$.

For use in the sequel, we record the following consequence of Theorems 7.2 and 7.8:

7.12 THEOREM. *If S is an n-dimensional, locally compact space such that $p_r(x) \leqq \omega$ for all $x \in S$ and all $r \leqq n$, then S has property $(P, Q)_r$ for all $r \leqq n$.*

8. **Other types of higher dimensional local connectedness.** Although the lc^0 spaces form the special case, $n = 0$, of the lc^n spaces, there exists at least one notable property of the lc^0 spaces that fails to undergo a corresponding strengthening as the degree of local connectedness is raised. From Theorems III 3.3, III 3.10 and V 11.9 it follows that *if a locally compact space is 0-lc, then $x \in S$ and U an open set containing x imply the existence of a strongly connected* (cf. VII 6.1) *open subset V of S such that $x \in V \subset U$*; in brief, the group $h_a^0(U) = 0$. (The group $h^r(S)$ is discussed in full in VIII 2.1. For present purposes it suffices to say that $p^r(U) = k$ implies that there exist k and only k r-cycles on compact subsets of U that are lirh on compact subsets of U.) The question arises, if a locally compact space is 1-lc, or even lc^1, do there exist for each point arbitrarily small neighborhoods that have $h^1(U) = 0$?

Simple examples show the answer to be negative. Such examples may be constructed from either of the configurations of type T_1 or T_2 in the accompanying figure. Each is obtained by an "identification" of two points of a closed



2-cell (II 5); in type T_1, p was formerly (before identification) two distinct interior points of the 2-cell, while in T_2, p was formerly a boundary point and an interior point. In each case, T_i is an irreducible membrane (VII 2.20) relative γ_1^1, where γ_1^1 is the fundamental 1-cycle of the S^1—J_1 in the figure—forming the boundary of the original 2-cell. But each configuration contains a set, J_2, also an S^1, whose fundamental 1-cycle, γ_2^1, fails to bound on T_i.

If two figures of type T_i are joined in such a way that the J_2 of the first becomes the J_1 of the second, then all 1-cycles of the former bound on the new configuration, but the γ_2^1 of the latter is still nonbounding. And if several figures of type T_i are joined serially in this fashion, always the γ_2^1 of the last remains nonbounding, while each γ_2^1 of the preceding bounds only on its successor. Such a configuration as the latter can be used to form a circular ordering of sets of type T_i, by allowing the J_2 of the last to become the J_1 of the first; such a configuration may be called a T_i-cycle of order n, where n is the number of sets of type T_i involved.

In a sufficiently high-dimensional Euclidean space (dimension 4 is certainly high enough in the case of T_2, since a T_2-cycle may be represented in a 3-space), let T be a circle—an S^1—and let T_i-cycles of order n_k, $k = 1, 2, 3, \cdots$, in parallel hyperplanes, converge on T. If this is done in such a way that $n_k \to \infty$, each of the n_k sets of type T_i in the kth T_i-cycle having diameter $< \delta_k$ where $\lim \delta_k = 0$, etc., the resulting configuration will be 1-lc, but small neighborhoods U of points of T will necessarily have $p^1(U) = \infty$.

To obtain an example that is lc^1, one may introduce 2-dimensional sets joining the T_i-cycles of the above configurations to make the set 0-lc, without destroying the 1-lc property. It might also be remarked that these configurations are also locally 1-connected in the sense of homotopy—indeed, LC^∞; cf. Lefchetz [L_1].

Evidently the above suggests a definition of local connectedness of a stronger type than the r-lc property—i.e., one which requires the existence for the point in question of arbitrarily small simply r-connected neighborhoods; or, in the case of locally compact spaces, of arbitrarily small neighborhoods U such that $h^r(U) = 0$. And inasmuch as by Theorem III 3.3, the lc property induces ulc neighborhoods, possibly an even stronger type of local connectedness in case $r > 0$ is obtained by the requirement of arbitrarily small r-ulc neighborhoods (in the sense of compact cycles for the locally compact spaces).

It might be of interest to study the implications of such types of local connectedness—in the homotopy sense as well as in the homology sense.

For further discussion of these matters the reader is referred to IX 7.8; also to Begle [c].

Bibliographical Notes

§1. Definition 1.1 was undoubtedly generally known soon after the introduction of the Vietoris homology theory; see for example the comment in footnote 63 of Alexandroff [c]; also Wilder [n], Čech [d]. Apparently Definition 1.2 appeared first in Čech [d].

§2. The explicit formulation of Definitions 2.1, 2.2, 2.3 given here was stated by Begle [a]; however, it is to be found, in essence, in Lefschetz [d]. An analogous notion, although radically different in one particular, will be found in Wilder [o]. Theorem 2.10 appears in Begle [a]; the proof indicated (without explicit statement) there seems to run into a difficulty, which is avoided in the proof given here (due to Miss K. E. Butcher).

§3. Theorem 3.1; see Lefschetz [d; Th. VIII]; Alexandroff [e]; Begle [a, Th. 5.5]. The conclusion of Theorem 3.4 still holds if instead of assuming "lc^n" only "lc^{n-1} and semi-n-connected" are assumed; cf. Wilder [o]. Corollary 3.8 may be found, for the metric case, in Wilder [o], and for the nonmetric case in Begle [b].

§4. Theorems 4.4 and 4.6 are extensions to nonmetric spaces of theorems given by S. Kaplan [a].

§5. The principal results of this section were abstracted in Wilder [A_3].

§6. Local connectivity numbers were first defined by Alexandroff [f] and Čech [f]. For an essentially different type of local number see Vaughan [a]. Theorems 6.7 and 6.8 are identical with Theorems 6.1 and 6.2 of Begle [a]. The first part of Theorem 6.9 was stated by Čech [f; §I, 3] in a theorem whose latter part appears to be inaccurate.

§7. The principal results of this section were abstracted in Wilder [A_4].

CHAPTER VII

APPLICATION OF HOMOLOGY AND COHOMOLOGY THEORY TO THE THEORY OF CONTINUA

The extension of the notion of cycle from the classical case of the geometric complex to the case of the general space accomplishes, as we have seen in Chapter V, what may have been considered its primary purpose; viz, to attach to the space the invariant $H^r(S)$. It becomes apparent, however, that the theory of homology and cohomology, as extended to the general space, provides a most useful tool for the attack on problems of a set-theoretic character that were not heretofore even considered—that in many cases could not even be formulated because of the lack of the language of cycles and homology in which to express them.

We have seen in Chapter VI in the case of local n-connectedness an instance of the use of homology in extending a familiar point set notion to a more general setting—the local connectedness of the Peano space, for instance, becoming the 0-dimensional case of a general theory of local connectedness. In the present chapter we prepare the ground for further applications. First, we shall set forth a number of lemmas which constitute part of what one might call the "mechanics" of, or "rules of operations" with cycles and cocycles.

1. Fundamental lemmas. Throughout this section, K and M will denote closed subsets of the space S such that $K \subset M$. Čech cycles will be denoted by single symbols, as Z^r, γ^r, $\cdots$, their coordinates on a covering $\mathfrak{U}$ by $Z^r(\mathfrak{U})$, $\gamma^r(\mathfrak{U})$, $\cdots$.

1.1 Lemma. *If $\gamma^r (r > 0)$ is a cycle* mod K *on M, then the collection $\{\partial\gamma^r(\mathfrak{U})\}$ is an $(r - 1)$-cycle on K, which we denote by $\partial\gamma^r$. Evidently $\partial\gamma^r \sim 0$ on M.*

Proof. That γ^r is a cycle mod K implies not only that $\partial\gamma^r(\mathfrak{U})$ is on K, but also that for every pair $\mathfrak{B} > \mathfrak{U}$ there exists a chain $C^{r+1}(\mathfrak{U})$ such that

$$\partial C^{r+1}(\mathfrak{U}) = \gamma^r(\mathfrak{U}) - \pi_{\mathfrak{U}\mathfrak{B}}\gamma^r(\mathfrak{B}) - A^r(\mathfrak{U}), \tag{1.1a}$$

where $A^r(\mathfrak{U})$ is on K. Applying ∂ to (1.1a) and transposing, we get

$$\partial\gamma^r(\mathfrak{U}) - \pi_{\mathfrak{U}\mathfrak{B}}\partial\gamma^r(\mathfrak{B}) = \partial A^r(\mathfrak{U}).$$

Hence $\{\partial\gamma^r(\mathfrak{U})\}$ is a cycle on K which we may denote by $\partial\gamma^r$.

1.2 Lemma. *If γ_1^r and γ_2^r are cycles* mod K *on M, then $\gamma_1^r \sim \gamma_2^r$* mod K *implies that $\partial\gamma_1^r \sim \partial\gamma_2^r$ on K.*

An important special case of Lemma 1.2 is:

1.3 LEMMA. *If γ^r is a cycle* mod K *on* M *such that* $\gamma^r \sim 0$ mod K *then* $\partial\gamma^r \sim 0$ *on* K.

1.4 LEMMA. *If γ^r is a cycle of K such that $\gamma^r \sim 0$ on M, then there exists a cycle γ^{r+1}* mod K *on M such that $\partial\gamma^{r+1} \sim \gamma^r$ on K.*

PROOF. If $\mathfrak{V} > \mathfrak{U}$, then

$$\pi_{\mathfrak{U}\mathfrak{V}}\gamma^r(\mathfrak{V}) \sim \gamma^r(\mathfrak{U}) \quad \text{on } K. \tag{1.4a}$$

And by hypothesis, for each covering $\mathfrak{V}$ there exists a relation $\partial C^{r+1}(\mathfrak{V}) = \gamma^r(\mathfrak{V})$ on M. Consequently we have

$$\partial\pi_{\mathfrak{U}\mathfrak{V}}C^{r+1}(\mathfrak{V}) = \pi_{\mathfrak{U}\mathfrak{V}}\gamma^r(\mathfrak{V}). \tag{1.4b}$$

Relations (1.4a) and (1.4b) imply that $\partial\pi_{\mathfrak{U}\mathfrak{V}}C^{r+1}(\mathfrak{V}) \in [\gamma^r(\mathfrak{U})]^*$, where $[\gamma^r(\mathfrak{U})]^*$ denotes the coset of $H^r(K, 0; \mathfrak{F}, \mathfrak{U})$ (V 7.7) determined by $\gamma^r(\mathfrak{U})$. Note, too, that the "$=$" in (1.4b) may be replaced by "$\sim$ on K" and the statement just made still holds. Let φ be the homomorphism of $Z^r(K, 0; \mathfrak{F}, \mathfrak{U})$ into $H^r(K, 0; \mathfrak{F}, \mathfrak{U})$ mapping $B^r(K, 0; \mathfrak{F}, \mathfrak{U})$ into the coset 0. Consider the homomorphism

$$\varphi\partial : Z^{r+1}(M, K; \mathfrak{F}, \mathfrak{U}) \to H^r(K, 0; \mathfrak{F}, \mathfrak{U}).$$

Evidently $[\gamma^r(\mathfrak{U})]^*$ is the image of some coset of $Z^{r+1}(M, K; \mathfrak{F}, \mathfrak{U})$ under this homomorphism, and we denote this coset by $\{C^{r+1}(\mathfrak{U})\}$—since it contains the given chain $C^{r+1}(\mathfrak{U})$.

We have, then, for each $\mathfrak{U}$, a flat $\{C^{r+1}(\mathfrak{U})\}$ in the space $Z^{r+1}(M, K; \mathfrak{F}, \mathfrak{U})$, such that for each $\mathfrak{V} > \mathfrak{U}$, $\pi_{\mathfrak{U}\mathfrak{V}}\{C^{r+1}(\mathfrak{V})\} \subset \{C^{r+1}(\mathfrak{U})\}$. Consequently, by Theorem V 10.2, there exists a cycle γ^{r+1} mod K on M such that $\gamma^{r+1}(\mathfrak{U}) \in \{C^{r+1}(\mathfrak{U})\}$ and therefore $\partial\gamma^{r+1}(\mathfrak{U}) \sim \gamma^r(\mathfrak{U})$ on K.

1.5 LEMMA. *If γ^r is a cycle* mod K *on M, and there exists a cycle Z^r such that $\gamma^r \sim Z^r$* mod K, *then $\partial\gamma^r \sim 0$ on K.*

PROOF. Let $\gamma^r - Z^r$ be the cycle γ^r of Lemma 1.3.

1.6 LEMMA. *If γ^r is a cycle* mod K *on M such that $\partial\gamma^r \sim 0$ on K, then there exists a cycle Z^r on M such that $\gamma^r \sim Z^r$* mod K *on M.*

PROOF. For each $\mathfrak{U}$ there exists a chain $L^r(\mathfrak{U})$ on K such that $\partial L^r(\mathfrak{U}) = \partial\gamma^r(\mathfrak{U})$. Hence $\gamma^r(\mathfrak{U}) - L^r(\mathfrak{U})$ is a cycle $\Gamma^r(\mathfrak{U})$ of $\mathfrak{U} \wedge M$ such that $\Gamma^r(\mathfrak{U}) \sim \gamma^r(\mathfrak{U})$ mod K.

Let $[\Gamma^r(\mathfrak{U})]^*$ be the set of all absolute cycles of $\mathfrak{U} \wedge M$ such that $\Gamma^r(\mathfrak{U}) \sim \gamma^r(\mathfrak{U})$ mod K. As just shown, this is a nonempty class for every $\mathfrak{U}$. Furthermore, $[\Gamma^r(\mathfrak{U})]^*$ is a flat in the space of r-cycles of $\mathfrak{U} \wedge M$, modulo the subspace of cycles that are homologous to 0 mod K, etc. Hence by Theorem V 10.2, there exists a cycle Γ^r of M such that for each $\mathfrak{U}$, $\Gamma^r(\mathfrak{U}) \in [\Gamma^r(\mathfrak{U})]^*$; and $\Gamma^r \sim \gamma^r$ mod K on M.

By the same method as just used to prove Lemma 1.6, we also have:

1.7 LEMMA. *If A and B are closed subsets of M and γ^r is a cycle* mod $A \cup B$ *on M such that $\partial\gamma^r \sim 0$* mod A *on $A \cup B$, then there exists a cycle Z^r* mod A *on M such that $\gamma^r \sim Z^r$* mod $A \cup B$ *on M.*

[Lemma 1.6 is the case $A = 0$, $B = K$ of Lemma 1.7.]

1.8 LEMMA. *If γ^r is a cycle* mod K *and Z^r is a cycle* mod K *on M such that $\gamma^r \sim Z^r$* mod K, *then $\gamma^r \sim 0$* mod M.

PROOF. For each $\mathfrak{U}$ there exists a relation

$$\partial C^{r+1}(\mathfrak{U}) = \gamma^r(\mathfrak{U}) - Z^r(\mathfrak{U}) - A^r(\mathfrak{U}),$$

where $A^r(\mathfrak{U})$ is on K. Since both $Z^r(\mathfrak{U})$ and $A^r(\mathfrak{U})$ are on M, this relation becomes

$$\partial C^{r+1}(\mathfrak{U}) = \gamma^r(\mathfrak{U}) \qquad \text{mod } M.$$

1.9 LEMMA. *If γ^r is a cycle* mod K *such that $\gamma^r \sim 0$* mod M, *then there exists a cycle Z^r* mod K *on M such that $\gamma^r \sim Z^r$* mod K.

PROOF. By hypothesis, for each covering $\mathfrak{U}$ there exists a chain $C^{r+1}(\mathfrak{U})$ such that

$$\partial C^{r+1}(\mathfrak{U}) = \gamma^r(\mathfrak{U}) - B^r(\mathfrak{U}), \tag{1.9a}$$

where $B^r(\mathfrak{U})$ is on M, and (since $\partial\gamma^r(\mathfrak{U}) = \partial B^r(\mathfrak{U})$) $B^r(\mathfrak{U})$ is a cycle mod K. In particular, then, relation (1.9a) implies that $\gamma^r(\mathfrak{U}) \sim B^r(\mathfrak{U})$ mod K.

Let $[B^r(\mathfrak{U})]^*$ denote the set of all cycles mod K of $\mathfrak{U} \wedge M$ such that if $B^r(\mathfrak{U}) \in [B^r(\mathfrak{U})]^*$, then $\gamma^r(\mathfrak{U}) \sim B^r(\mathfrak{U})$ mod K. Then $[B^r(\mathfrak{U})]^*$ is a coset in the group of cycles of $\mathfrak{U} \wedge M$ mod K modulo the subgroup that are homologous to zero in S mod K, and is a flat satisfying the conditions of the existence Theorem V 10.2. Hence there exists a cycle Z^r on M mod K such that for each covering $\mathfrak{U}$, $Z^r(\mathfrak{U}) \in [B^r(\mathfrak{U})]^*$.

1.10 LEMMA. *If L is a closed subset of a locally compact space S such that $F(L)$ is compact and $\mathfrak{U}$ is an ucos of S, then there exists $\mathfrak{W} > \mathfrak{U}$ such that if the nucleus of a simplex of $\mathfrak{W}$ meets both L and $S - L$, then it meets the boundary of L.*

PROOF. By Lemma V 8.7, there exists $\mathfrak{V} > \mathfrak{U}$ such that $\mathfrak{V} \cap F(L)$ is finite and if the nucleus of a simplex of $\mathfrak{V}$ fails to meet $F(L)$, then its closure fails to meet $F(L)$. It follows that the union, K, of the closures of such nuclei fails to meet $F(L)$, and consequently there exists an open set P such that $F(L) \subset P \subset S - K$.

Denoting the elements of $\mathfrak{V}$ by V_i, for each i let

$$V_i \cap (L \cup P) = V_{i1}, \qquad V_i \cap [(S - L) \cup P] = V_{i2}.$$

Evidently $V_i = V_{i1} \cup V_{i2}$, and $V_{ij} \subset V_i$, $j = 1, 2$. Consequently if we denote the collection of all sets V_{ij} by $\mathfrak{W}$, then $\mathfrak{W}$ is a covering of S and $\mathfrak{W} > \mathfrak{U}$. For later use we note that

$$V_{i1} \cap V_{i2} = V_i \cap P. \tag{1.10a}$$

Let E^r be a simplex of $\mathfrak{W}$ whose nucleus $\mathcal{N}(E^r)$ meets both L and $S - L$. Denote $(S - L) \cap \mathcal{N}(E^r)$ by A and $L \cap \mathcal{N}(E^r)$ by B. Denoting the vertices of E^r by $V_{\nu j}$ ($j = 1$ or 2), let E^m be the simplex of $\mathfrak{V}$ whose vertices are the sets V_ν; the sets V_ν have a nonempty nucleus since $\bigcap V_\nu \supset \bigcap V_{\nu j} = A \cup B$. In particular, then, each vertex V_ν of E^m meets both L and $S - L$. But by the way the sets V_{ij} were constructed, it follows that $\mathcal{N}(E^r) = \bigcap V_{\nu j}$ is either a subset of $L \cup P$ or of $(S - L) \cup P$. In the former case $A \subset (S - L) \cap P$, and in the latter case $B \subset L \cap P$. Hence in either case $\mathcal{N}(E^m)$ has points in P, and therefore $\mathcal{N}(E^m) \cap F(L) \neq 0$.

Let $x \in \mathcal{N}(E^m) \cap F(L)$. Then $x \in V_\nu \cap P$ for every ν. Hence by relation (1.10a), $x \in V_{\nu j}$, where $V_{\nu j}$ is a vertex of E^r.

1.11 DEFINITION. If $C^r(\mathfrak{U})$ is a chain and L is any point set, then by $C^r(\mathfrak{U}) \wedge L$ we denote the portion of $C^r(\mathfrak{U})$ which is on L.

1.12 LEMMA. *If $\mathfrak{V} > \mathfrak{U}$, and $C^{r+1}(\mathfrak{V})$ is a chain of $\mathfrak{V}$, then $\pi_{\mathfrak{U}\mathfrak{V}}[C^{r+1}(\mathfrak{V}) \wedge L] \subset [\pi_{\mathfrak{U}\mathfrak{V}} C^{r+1}(\mathfrak{V})] \wedge L$.*

1.13 LEMMA. *With S locally compact, let L be a closed subset of $S - K$ such that $F(L)$ is compact, and γ^r a cycle on K which bounds on M, or which bounds* mod L *on M. Then there exists a cycle Z^r on $F(L)$ such that $\gamma^r \sim Z^r$ on $\overline{M - L}$.*

PROOF. We shall give the proof only for the case where $\gamma^r \sim 0$ on M. By Lemma 1.4, there exists a cycle γ^{r+1} mod K on M such that $\partial\gamma^{r+1} \sim \gamma^r$ on K. Henceforth in the proof we use only coverings of the type $\mathfrak{W}$ of Lemma 1.10. There exists a relation

$$\partial C^{r+2}(\mathfrak{U}) = \gamma^{r+1}(\mathfrak{U}) - \pi_{\mathfrak{U}\mathfrak{V}}\gamma^{r+1}(\mathfrak{V}) - A^{r+1}(\mathfrak{U}) \tag{1.13a}$$

where $A^{r+1}(\mathfrak{U})$ is on K.

From relation (1.13a) we have:

$$\partial[C^{r+2}(\mathfrak{U}) \wedge L] = \gamma^{r+1}(\mathfrak{U}) \wedge L - [\pi_{\mathfrak{U}\mathfrak{V}}\gamma^{r+1}(\mathfrak{V})] \wedge L - B^{r+1}(\mathfrak{U}), \tag{1.13b}$$

where $B^{r+1}(\mathfrak{U})$ is a chain on L. Moreover, $B^{r+1}(\mathfrak{U})$ is on $F(L)$.

Let $Q^{r+1}(\mathfrak{U}) = [\pi_{\mathfrak{U}\mathfrak{V}}\gamma^{r+1}(\mathfrak{V})] \wedge L - \pi_{\mathfrak{U}\mathfrak{V}}[\gamma^{r+1}(\mathfrak{V}) \wedge L]$. (Cf. Lemma 1.12.) The chain $Q^{r+1}(\mathfrak{U})$ is on L by its definition. We may then restate relation (1.13b) as follows:

$$\partial[C^{r+2}(\mathfrak{U}) \wedge L] = \gamma^{r+1}(\mathfrak{U}) \wedge L - \pi_{\mathfrak{U}\mathfrak{V}}[\gamma^{r+1}(\mathfrak{V}) \wedge L] - Q^{r+1}(\mathfrak{U}) - B^{r+1}(\mathfrak{U}).$$

Applying the operator ∂ again to this relation, and recalling that $\partial^2 = 0$, we obtain the relation:

$$\partial[\gamma^{r+1}(\mathfrak{U}) \wedge L] - \pi_{\mathfrak{U}\mathfrak{V}}\partial[\gamma^{r+1}(\mathfrak{V}) \wedge L] = \partial[Q^{r+1}(\mathfrak{U}) + B^{r+1}(\mathfrak{U})].$$

Hence $\{\partial[\gamma^{r+1}(\mathfrak{U}) \wedge L]\}$ is a cycle of L. We have already noted above that $B^{r+1}(\mathfrak{U})$ is on $F(L)$. The same holds also of Q^{r+1}. Consequently $\{\partial[\gamma^{r+1}(\mathfrak{U}) \wedge L]\} = Z^r$ is a cycle of $F(L)$.

Finally, since for each $\mathfrak{U}$ the chain $\gamma^{r+1}(\mathfrak{U}) - \gamma^{r+1}(\mathfrak{U}) \wedge L$ is on $\overline{M - L}$, and $\partial[\gamma^{r+1}(\mathfrak{U}) - \gamma^{r+1}(\mathfrak{U}) \wedge L] = \partial\gamma^{r+1}(\mathfrak{U}) - Z^r(\mathfrak{U})$, and $\gamma^r(\mathfrak{U}) \sim \partial\gamma^{r+1}(\mathfrak{U})$ on K, we have that $\gamma^r \sim Z^r$ on $\overline{M - L}$.

Remark. In the proof of Lemma 1.13, the portion, $\partial[\gamma^{r+1}(\mathfrak{U}) \wedge L]$, of $\gamma^{r+1}(\mathfrak{U})$ on L, was shown to define a cycle on the special set of coverings of type $\mathfrak{W}$ of Lemma 1.10 and hence, by projection, on all coverings $\mathfrak{U}$. However, for a covering $\mathfrak{U}$ not of type $\mathfrak{W}$, $\partial[\gamma^{r+1}(\mathfrak{U}) \wedge L]$ may fail to be on $F(L)$ In this case, it may be shown that the collection $\{\partial[\gamma^{r+1}(\mathfrak{U}) \wedge L]\}$ forms a cycle approximately on $F(L)$ (V 21).

Consider, for the coverings $\mathfrak{U}$ of type $\mathfrak{W}$ of Lemma 1.10, the chain $\gamma_1^{r+1}(\mathfrak{U}) = \gamma^{r+1}(\mathfrak{U}) - \gamma^{r+1}(\mathfrak{U}) \wedge L$. Then $\gamma_1^{r+1}(\mathfrak{U})$ is in $M - L$ and $\partial\gamma_1^{r+1}(\mathfrak{U}) = \partial\gamma^{r+1}(\mathfrak{U}) - Z^r(\mathfrak{U})$. We may call the collection $\{\gamma_1^{r+1}(\mathfrak{U})\}$ the *portion of* γ^{r+1} *in* $M - L$, *and* $\{\gamma^{r+1}(\mathfrak{U}) \wedge L\}$ *the portion of* γ^{r+1} *on* L.

1.14 Lemma. *In a compact space let A and B be closed point sets, and Z^r a cycle on $A \cup B$. Let Z_A^r denote the portion of Z^r in $A - B$, and suppose that $\partial Z_A^r \sim 0$ on $A \cap B$. Then there exist cycles γ_A^r and γ_B^r on A and B, respectively, such that $\gamma_A^r + \gamma_B^r \sim Z^r$ on $A \cup B$.*

Proof. By Lemma 1.6, there exists a cycle γ_A^r on A such that $\gamma_A^r \sim Z_A^r$ mod $A \cap B$ on A; and the same homology holds mod B on $A \cup B$. Then, since $Z^r \sim Z_A^r$ mod B on $A \cup B$, we have $Z^r \sim \gamma_A^r$ mod B on $A \cup B$. That is, $Z^r - \gamma_A^r$ is a cycle such that $Z^r - \gamma_A^r \sim 0$ mod B on $A \cup B$. Hence by Lemma 1.9 there exists a cycle γ_B^r on B such that $Z^r - \gamma_A^r \sim \gamma_B^r$ on $A \cup B$. Thus, $Z^r \sim \gamma_A^r + \gamma_B^r$ on $A \cup B$.

1.15 Lemma. *With K a compact subset of a locally compact space M, let $M - K = \bigcup_{i=1}^k P_i$ separate, and let γ^r be a cycle* mod K *on M. Then there exist cycles γ_i^r* mod $K \cap \overline{P}_i = F(P_i)$ *on P_i such that $\gamma^r \sim \sum_{i=1}^k \gamma_i^r$* mod K *on M.*

1.16 Corollary. *With K and M as in* 1.15, *if γ^r is a cycle* mod K *on M, then there exists a cycle γ_K^r* mod $F(K)$ *on $M - K$ such that $\gamma_K^r \sim \gamma^r$* mod K *on M, and $\partial\gamma^r \sim \partial\gamma_K^r$ on K. (Here K need not be compact if $F(K)$ is compact.)*

1.17 Lemma. *If γ^r is a cycle* mod K *on M such that $\gamma^r \sim 0$* mod L *on the locally compact S, where $K \subset L$ compact, then there exists a cycle Z^{r+1}* mod $M \cup L$ *on $S - L$ such that $\partial Z^{r+1} \sim \gamma^r$* mod L *on M.*

(Cf. the proof of Lemma 1.4. This lemma is placed here because of its use of the notion of portion of a cycle in the proof.)

2. Existence lemmas regarding carriers of cycles and homologies. In Definition V 7.10, the terms (1) *C-cycle* mod L *on* M and (2) *homologous to zero* mod L *on* M were defined. In case (1) the set M may be called a *carrier* of the C-cycle, and in case (2) the set M may be called a *carrier of the homology*. (And we may say M *carries* the C-cycle in the first case, and *carries the homology* in the second.) In either case, if $H \supset M$, then H is also a carrier; in particular the space S under consideration is always a (unique) *maximal* carrier.

The question of the existence of *minimal* carriers presents a problem, however. In general, we shall inquire only into the existence of minimal *closed* carriers, since (a) as already observed, a set carries a chain if and only if its closure does, and (b) ordinarily minimal carriers fail to exist even when minimal closed carriers do exist. To illustrate (b), one may show that S^1 is the minimal closed carrier of a 1-dimensional C-cycle Z^1, but there exists no minimal subset of S^1 that carries this Z^1.

Even when a minimal closed carrier exists, it is generally not unique. This is easily seen in the case of homology; for example, if $p, q \in S^1$, $p \neq q$, then a nontrivial cycle (V 11.4) on $p \cup q$ is ~ 0 on S^1, and either of the arcs of S^1 with end points p and q is a minimal closed carrier of this homology. For the case of carrier of a cycle, let $\{\mathfrak{U}_k\}$ be a canonical sequence of refinements (V 12.3) of a torus T, and let K_i, $i = 1, 2$, be two disjoint equatorial circles on T. Then T is the union of two continua A and B such that $A \cap B = K_1 \cup K_2$. The set K_i being an S^1, let Z_i^1 be the C-cycle referred to in the preceding paragraph. Let Z^1 be a C-cycle whose coordinate on each $\mathfrak{U}_{2n}$ is $Z_1^1(\mathfrak{U}_{2n})$ and whose coordinate on $\mathfrak{U}_{2n+1}$ is $Z_2^1(\mathfrak{U}_{2n+1})$. Then each of the sets A, B is a minimal closed carrier of Z^1.

(The same type of example may be given on S^1 by using, instead of a 1-dimensional cycle, a nonaugmented 0-cycle on $p \cup q$, where $p, q \in S^1$, $p \neq q$.)

Remarks analogous to the above may be made for the notions of cycle approximately on a set, etc., of V 21. Thus, if a C-cycle γ^r is approximately on a set M, then M may be called an *approximate carrier* of γ^r. A special type of approximate carrier of a γ^r is that of a closed set M such that every open set containing M is a carrier of γ^r; such a set will be called a *locus of concentration* of γ^r. And analogously we may define locus of concentration of a homology.

2.1 LEMMA. *Let γ^r be a C-cycle and $\{F_\rho\}$ a simply-ordered collection of compact sets such that* (1) *if $F_{\rho'} < F_{\rho''}$, then $F_{\rho'} \supset F_{\rho''}$, and* (2) *$F_\rho$ is a locus of concentration of γ^r. Then $\bigcap F_\rho$ is a locus of concentration of γ^r.*

PROOF. Let P be an open set containing $\bigcap F_\rho$, and let F denote a fixed element of $\{F_\rho\}$. Then $F - P$ is a compact subset of $\bigcup (S - F_\rho)$, and there exist finitely many of the sets $S - F_\rho$ covering $F - P$. Hence, in view of the fact that $F_{\rho'} < F_{\rho''}$ implies $S - F_{\rho'} \subset S - F_{\rho''}$, there exists a ρ such that $F - P \subset S - F_\rho$; i.e., $P \supset F_\rho$. Consequently P is a carrier of γ^r.

2.2 LEMMA. *If $\gamma^r \neq 0$ is a C-cycle having a compact locus of concentration F, then some subset of F is a minimal locus of concentration of γ^r.*

PROOF. Let $\{F_\eta\}$ be the collection of all subsets F_η of F that are loci of concentration of γ^r, partially ordered by $\supset$; let $\{F_\rho\}$ be a maximal simply ordered subcollection of $\{F_\eta\}$. Since $\gamma^r \neq 0$, no F_ρ is empty, so that by Theorem I 12.13, the set $K = \bigcap F_\rho \neq 0$. By Lemma 2.1, K is a locus of concentration of γ^r and consequently the last element of $\{F_\rho\}$.

2.3 LEMMA. *If M is a closed subset of a compact space S, and γ^r is a C-cycle*

mod M, *then there exists a minimal closed set* F *containing* M *such that* $\gamma^r \sim 0$ mod F.

PROOF. It is trivial that $\gamma^r \sim 0$ mod S. Let $\{F_\rho\}$ be a simply ordered series of closed sets containing M such that if $F_{\rho'} < F_{\rho''}$ then $F_{\rho'} \supset F_{\rho''}$, and $\gamma^r \sim 0$ mod F_ρ for all ρ. Let $F = \bigcap F_\rho$ (Theorem I 12.13).

By Lemma V 8.7, if $\mathfrak{U}$ is a fcos of S there exist $\mathfrak{B} > \mathfrak{U}$ and open set $Q \supset F$ such that if a simplex of $\mathfrak{B}$ is on Q, then it is on F. Now $S - Q \subset S - F = \bigcup (S - F_\rho)$ and hence there exists ρ such that $S - Q \subset S - F_\rho$; i.e., $Q \supset F_\rho$. By hypothesis, $\gamma^r(\mathfrak{B}) \sim 0$ mod F_ρ, and a fortiori mod $\bar{Q}$, and hence $\gamma^r(\mathfrak{B}) \sim 0$ mod F. Then $\pi_{\mathfrak{U}\mathfrak{B}}\gamma^r(\mathfrak{B}) \sim 0$ mod F. Since $F \supset M$, γ^r is a cycle mod F, so that $\pi_{\mathfrak{U}\mathfrak{B}}\gamma^r(\mathfrak{B}) \sim \gamma^r(\mathfrak{U})$ mod F. Combining homologies we get $\gamma^r(\mathfrak{U}) \sim 0$ mod F. Thus $\gamma^r \sim 0$ mod F. It now follows easily (see proof of Lemma 2.2) that the desired minimal set exists.

2.4 LEMMA. *Let* P *be an open subset of a compact space* S *and let* γ^r *be a C-cycle* mod $S - P$ *such that* $\gamma^r \sim 0$ mod $S - Q$, *where* Q *is some open subset of* P. *Then* P *has a maximal open subset* P_1 *containing* Q *such that* $\gamma^r \sim 0$ mod $S - P_1$.

(This is dual to, and a corollary of, Lemma 2.3.)

2.5 LEMMA. *If* γ^r *is an absolute C-cycle of a compact space* S, *then there exists a minimal closed set* F *such that* $\gamma^r \sim 0$ mod F.

(This is a corollary of Lemma 2.3.)

2.6 LEMMA. *Under the hypothesis of Lemma* 2.3, *if* S *is r-dimensional, then the set* F *is unique and, moreover, a closed carrier of the cycle* γ^r.

PROOF. That $\gamma^r(\mathfrak{U}) \sim 0$ mod F implies $\gamma^r(\mathfrak{U}) = 0$ mod F since $\mathfrak{U}$ may be assumed r-dimensional. This is possible only if all simplexes of $|\gamma^r(\mathfrak{U})|$ meet F. Hence if F_1 and F_2 are two sets of type F, $|\gamma^r(\mathfrak{U})|$ lies on $F_1 \cap F_2$ and $\gamma^r(\mathfrak{U}) \sim 0$ mod $F_1 \cap F_2$; it follows that $F_1 = F_2$ since these sets are minimal.

Also, $\pi_{\mathfrak{U}\mathfrak{B}}\gamma^r(\mathfrak{B}) \sim \gamma^r(\mathfrak{U})$ implies $\pi_{\mathfrak{U}\mathfrak{B}}\gamma^r(\mathfrak{B}) = \gamma^r(\mathfrak{U})$, and hence from the fact that $\gamma^r(\mathfrak{U})$ is on F that $\pi_{\mathfrak{U}\mathfrak{B}}\gamma^r(\mathfrak{B}) \sim \gamma^r(\mathfrak{U})$ on F.

2.7 REMARK. In general, F will not be the carrier of γ^r when dim $S > r$. For example, let $p, q \in S^1$, $p \neq q$, F and K the two arcs of S^1 having in common only their end points p and q. Suppose S^1 imbedded in S^2 and that E is one of the domains of S^2 bounded by S^1. Consider $\bar{E}$ as playing the role of space S. Then there exists a γ^1 mod $p \cup q$ (hence mod F) on K for which F is a minimal closed set such that $\gamma^1 \sim 0$ mod F on $\bar{E}$, but F is neither unique nor is it a carrier of γ^1.

2.8 LEMMA. *If* M *is a closed subset of a compact space* S *and* γ^r *is a C-cycle on* M *such that* $\gamma^r \sim 0$ *on* $K \supset M$, *then there exists a minimal closed set* F *such that* $K \supset F \supset M$ *and such that* $\gamma^r \sim 0$ *on* F.

(Cf. proof of Lemma 2.3.)

2.9 Lemma. *If γ^r is a C-cycle of a space S such that $\gamma^r \sim 0$ on some compact subset M of S, then M contains a minimal locus of concentration of this homology.*

2.10 Definition. If M is a compact set such that $p^r(M) > 0$, but $p^r(M') = 0$ for every proper closed subset M' of M, then we say that M is *irreducible relative to carrying a nonbounding r-cycle.* (Alexandroff [c] calls a set which satisfies this definition and is in addition r-dimensional, an *r-dimensional closed cantorian manifold.*)

2.11 Every n-sphere is irreducible relative to carrying a nonbounding n-cycle. It is not necessary that $p^r(M) = 1$ if M is irreducible relative to carrying a nonbounding r-cycle. As a corollary of duality theorems to be proved in Chapter VIII, it will follow that known examples of continua which form common boundaries of three or more domains in the n-sphere are irreducible relative to carrying a nonbounding $(n - 1)$-cycle, and their $(n - 1)$-dimensional Betti numbers are $\geqq 2$.

2.12 Now a space M may be a minimal closed carrier of one of its nonbounding cycles without being irreducible relative to carrying a nonbounding cycle, as will be pointed out below (2.13). Intermediate between such a space and the type of space defined in 2.10 is the compact space M such that $p^r(M) > 0$ and such that every cycle on a closed proper subset of M bounds *on* M; such a set M will frequently be called *irreducible relative to carrying an r-cycle nonbounding on M.* Such a set need not be r-dimensional (although it must be *at least* r-dimensional; cf. Theorem VI 7.6). This is easily seen by a modification of the following example: Let $M = \{(x, y) \mid (0 < x \leqq 1/\pi) \& (y = \sin 1/x)\} \cup \{(0, y) \mid -1 \leqq y \leqq 1\}$, and let A be an arc in the same cartesian plane with M joining the point $(1/\pi, 0)$ to the point $(0, -1)$ but otherwise not meeting M. Let $M \cup A$ be modified to a configuration in 3-space by replacing the part of M on the y-axis by a square and its interior in a plane perpendicular to the xy-plane and cutting the latter at the points on the y-axis. At the same time we modify the sine curve portion of M so that it has all points on the square plus its interior as limit points. That the so modified set S has the properties desired will follow immediately from the duality theorems cited above; i.e., $p^1(S) > 0$ but every 1-cycle on a proper closed subset of S bounds on S. Notice that the square K carries nontrivial 1-cycles, so that S is not irreducible relative to carrying a nonbounding 1-cycle.

2.13 It should also be emphasized that a set may be a minimal closed carrier of a specific r-cycle, without being irreducible relative to carrying a nonbounding r-cycle. This is exemplified in the case $r = 1$ by the set A and cycle Z^1 of the third paragraph of §2. (See also 3.4). With A imbedded in E^3, suppose we let S_1 and S_2 denote two disjoint 2-dimensional hemispheres such that $S_i \cap A = K_i$. Let $S = A \cup S_1 \cup S_2$. Then $Z^1 \sim 0$ on $S_1 \cup S_2$, although the latter set does not even contain the set A originally designated as carrier of Z^1. Consequently when we designate a set M as carrier of a cycle γ and then speak of γ as bounding on another set S, we do not have a right to assume

that $M \subset S$. However, S will automatically be a carrier of γ, no matter whether M is a subset of S or not:

2.14 LEMMA. *If a cycle γ^r bounds on a set M, then M is a carrier of γ^r.*

PROOF. For each covering $\mathfrak{U}$, there exists a relation $\partial c^{r+1}(\mathfrak{U}) = \gamma^r(\mathfrak{U})$ on M. Hence $\partial[c^{r+1}(\mathfrak{U}) - \pi_{\mathfrak{U}\mathfrak{V}}c^{r+1}(\mathfrak{V})] = \gamma^r(\mathfrak{U}) - \pi_{\mathfrak{U}\mathfrak{V}}\gamma^r(\mathfrak{V})$ on M, as required.

Although, as we have just noted, to assign a carrier to a cycle and then to note that this cycle is ~ 0 on a certain set S does not imply that the original carrier is on S, there is one case in which we can guarantee that a cycle always has the same carrier:

2.15 LEMMA. *If J is irreducible relative to carrying an r-cycle nonbounding on J, and γ^r is a nonbounding cycle of J which bounds on a certain set M, then $J = J \cap \overline{(M - J)}$. In particular, then, $J \subset M$.*

PROOF. By Lemma 1.4 there exists a cycle γ^{r+1} mod J on $J \cup M$ such that $\partial\gamma^{r+1} \sim \gamma^r$ on J. And by Corollary 1.16, there exists a cycle Z^{r+1} mod $J \cap \overline{(M - J)}$ such that $Z^{r+1} \sim \gamma^{r+1}$ mod J on $J \cup M$ and such that $\partial Z^{r+1} \sim \partial\gamma^{r+1}$ on J. Then $\partial Z^{r+1} \sim \gamma^r$ on J; and were J not $= J \cap \overline{(M - J)}$, then would ∂Z^{r+1} be on a proper subset of J and consequently homologous to zero on J. But this would imply $\gamma^r \sim 0$ on J. We must conclude, then, that $J = J \cap \overline{(M - J)}$.

An immediate, but interesting corollary of this lemma is the following:

2.16. COROLLARY. *If in a space S, J is a set which is irreducible relative to carrying an r-cycle nonbounding on J, and some nonbounding r-cycle on J bounds on S, then J has no interior points.*

In particular, we may state then:

2.17 LEMMA. *If S is a space such that $p^r(S) = 0$ and J is a subset of S which is irreducible relative to carrying an r-cycle nonbounding on J, then J has no interior points in S.*

Since the n-sphere has the property that for all $r < n$, $p^r(S^n) = 0$, we can state the following theorem, corollary to Lemma 2.17, regarding the "invariance of dimensionality" of euclidean spaces:

2.18 COROLLARY. *The topological image of an S^n in an S^m, $n < m$, has no interior points.*

2.19 THEOREM. *In a compact space S, let K be a carrier of a cycle Z^r such that $Z^r \nsim 0$ on K. Let K_i, $i = 1, 2$, be closed sets containing K such that $Z^r \sim 0$ on K_i, K_1 being minimal with respect to these properties, and such that $K_1 - K_2 \neq 0$. Then the set $M = K_1 \cup K_2$ carries a cycle Z^{r+1} such that $Z^{r+1} \nsim 0$ on M.*

PROOF. Let $x \in K_1 - K_2$, and let U be an open set such that $x \in U \Subset$

$S - K_2$. Since $Z^r \sim 0$ on K_i, there exists by Lemma 1.4 a cycle C_i^{r+1} mod K on K_i such that

$$\text{(2.19a)} \qquad \partial C_i^{r+1} \sim Z^r \qquad \text{on } K.$$

Evidently $\partial C_1^{r+1} \sim 0$ on K_2; and therefore there exists by Lemma 1.6 a cycle Z^{r+1} such that

$$\text{(2.19b)} \qquad Z^{r+1} \sim C_1^{r+1} \text{ mod } K_2 \text{ on } M; \text{ hence mod } M - U \text{ on } M.$$

Now suppose $Z^{r+1} \sim 0$ on M. Then $Z^{r+1} \sim 0$ mod $M - U$ on M, and therefore, by virtue of (2.19b),

$$\text{(2.19c)} \qquad C_1^{r+1} \sim 0 \text{ mod } M - U \text{ on } M; \text{ hence mod } K_1 - U \text{ on } K_1.$$

Now by Corollary 1.16 there exists a cycle C_{1U}^{r+1} mod $F(U)$ on U such that

$$\text{(2.19d)} \qquad C_{1U}^{r+1} \sim C_1^{r+1} \text{ mod } K_1 - U \qquad \text{on} \qquad K_1,$$

$$\text{(2.19e)} \qquad \partial C_{1U}^{r+1} \sim \partial C_1^{r+1} \qquad \text{on} \qquad K_1 - U.$$

But then by (2.19c) and (2.19d), $C_{1U}^{r+1} \sim 0$ mod $K_1 - U$ on K_1, implying (Lemma 1.3) that

$$\text{(2.19f)} \qquad \partial C_{1U}^{r+1} \sim 0 \qquad \text{on} \qquad K_1 - U.$$

Relations (2.19e) and (2.19f) imply that $\partial C_1^{r+1} \sim 0$ on $K_1 - U$, which with (2.19a) gives $Z^r \sim 0$ on $K_1 - U$. But this contradicts the fact that K_1 is minimal with respect to containing K and carrying the homology $Z^r \sim 0$.

2.20 DEFINITION. If M is a compact point set on which a cycle $Z^r \sim 0$, and $Z^r \nsim 0$ on a proper closed subset of M, then M is called an *irreducible membrane relative to Z^r*.

(The term irreducible membrane is due to Alexandroff [a]. In case $r = 0$ and Z^0 is a nontrivial cycle on a pair of points p and q, then M is called an *irreducible continuum* from p to q.)

2.21 COROLLARY. *If, in a space S, J is a point set which is irreducible with respect to carrying an r-cycle nonbounding on J, and Z^r is a nonbounding cycle of J which bounds on S, and K_1 is an irreducible membrane relative to Z^r; then if K_2 is any set on which $Z^r \sim 0$ and such that $K_1 - K_2 \neq 0$, the set $K_1 \cup K_2$ carries a cycle Z^{r+1} such that $Z^{r+1} \nsim 0$ on $K_1 \cup K_2$.*

PROOF. By Lemma 2.15, $J \subset K_i$, $i = 1, 2$. Then Theorem 2.19 applies.

2.22 THEOREM. *If J and Z^r are as in 2.21, and $Z^r \sim 0$ on some closed set K, then K contains an irreducible membrane M relative to Z^r. The set J is a subset of M.*

[Theorem 2.22 follows from Lemmas 2.8 and 2.15.]

2.23 THEOREM. *If Z^r is a cycle of a compact space S, then there exists a closed subset M of S which is minimal with respect to carrying a cycle γ^r such that $Z^r \sim \gamma^r$ on S. The set M is a minimal closed carrier of γ^r.*

PROOF. By Lemma 2.5, there exists a minimal closed set M in S such that $Z^r \sim 0 \bmod M$. By Lemma 1.9, there exists a cycle γ^r on M such that $Z^r \sim \gamma^r$. No proper closed subset F of M carries a cycle homologous to Z^r, since this would imply $Z^r \sim 0 \bmod F$. Hence the first part of the conclusion is satisfied. And since, if there were a proper closed subset F of M which carries γ^r, we would again have $Z^r \sim 0 \bmod F$, violating the minimal character of M, the second part of the conclusion follows.

We close this section with applications to the theory of local connectedness. First, we obtain an analogue of Theorem IV 2.1:

2.24 THEOREM. *Let S be a locally compact space which is not r-lc at $x \in S$. Then if S is semi-r-connected at x, there exists an open set P containing x such that for every pair of open sets Q, R such that $x \in R \Subset Q \Subset P$, there exist on $F(Q)$ infinitely many r-cycles that are lirh on $P - R$.*

PROOF. By Theorem VI 6.14 there exists an open set P containing x such that for any open set V such that $x \in V \subset P$, infinitely many r-cycles on V are lirh on P. We may suppose P to be taken so that $\overline{P}$ is compact, and that all r-cycles on P bound on S. Let Q, R, V be open sets such that $x \in V \Subset R \Subset Q \Subset P$, and let γ_i^r, $i = 1, 2, 3, \cdots$, be an infinite sequence of cycles on V that are lirh on P.

Let $K = \overline{V}$ and $L = S - Q$. Then by Lemma 1.13 there exists for each i a cycle Z_i^r on $F(L) = F(Q)$ such that $Z_i^r \sim \gamma_i^r$ on $S - L = Q$. Then the cycles Z_i^r are lirh on $P - R$. For suppose there exists a relation

$$\sum c^i Z_i^r \sim 0 \qquad \text{on } P - R. \tag{2.24a}$$

Now $Z_i^r \sim \gamma_i^r$ on Q implies $c^i\gamma_i^r \sim c^i Z_i^r$ on Q. I.e.,

$$\sum c^i\gamma_i^r - \sum c^i Z_i^r \sim 0 \qquad \text{on } Q. \tag{2.24b}$$

Adding relations (2.24a, b) we get $\sum c^i\gamma_i^r \sim 0$ on P, contradicting the fact that the cycles γ_i^r are lirh on P.

Note that in case $r = 0$, the above theorem gives immediately (see Theorem V 11.3a) that there must be infinitely many components of $S \cap (\overline{P} - R)$ that meet $F(Q)$. The connection with Theorem IV 2.1 should be obvious, inasmuch as each such component must have points on $F(P)$ or $F(R)$, and in any case either $\overline{P} - Q$ or $\overline{Q} - R$ will contain continua M_ν such as those whose existence was proved in the theorem cited.

In Remark VI 7.11, it was stated that Theorem VI 7.9 has a converse. The complete form of this theorem may be stated and proved as follows:

2.25 THEOREM. *In order that an n-dimensional, locally compact space S should be lc^n, it is necessary and sufficient that $p_r(x) \leqq \omega$ for all $x \in S$ and $r \leqq n$.*

Proof. The sufficiency having been proved in Theorem VI 7.9, we need show only the necessity of the condition stated.

Suppose $x \in S$ such that $p_r(x) = \infty$. Then there exists an open set P containing x such that $p_r(x; P)$ is infinite. We may assume that $\overline{P}$ is compact and that cycles on $\overline{P}$ bound on S. Let U and V be open sets such that $x \in V \Subset U \Subset P$, and suppose that $r > 0$. By Corollary VI 3.8, there exists an integer m such that every m $(r-1)$-cycles on $F(U)$ satisfy a homology on $\overline{P} - V$.

By Theorem VI 6.7, $p_r(x; P, V) = p^r(x; P, V) = \infty$. Hence there exist cycles γ_i^r, $i = 1, \cdots, m$, mod $S - P$, that are lirh mod $S - V$. By Corollary 1.16, there exist cycles Z_i^r mod $F(U)$ on $\overline{U}$ such that $Z_i^r \sim \gamma_i^r$ mod $S - U$. Evidently the cycles Z_i^r are lirh mod $S - V$. By the choice of m, there exists a homology $\sum_{i=1}^m a^i\partial Z_i^r \sim 0$ on $\overline{P} - V$. Hence by Lemma 1.6, there exists a cycle Γ^r on $\overline{P}$ such that $\Gamma^r \sim \sum_{i=1}^m a^i Z_i^r$ mod $S - V$. But by the choice of P, $\Gamma^r \sim 0$ on S and a fortiori $\Gamma^r \sim 0$ mod $S - V$. Combining homologies, we have that $\sum_{i=1}^m a^i Z_i^r \sim 0$ mod $S - V$, in contradiction of the fact that the cycles Z_i^r are lirh mod $S - V$.

The case $r = 0$ is a direct consequence of Theorem VI 6.9.

In obtaining the above theorem we have also proved the following theorem:

2.26 Theorem. *If S is a locally compact, lc^n space, then $p_r(x) \leqq \omega$ for all $x \in S$ and $r \leqq n$; and if S is semi-$(n+1)$-connected at $x \in S$, then $p_{n+1}(x) \leqq \omega$.*

3. Separations of continua by closed subsets. As we have pointed out elsewhere, the Jordan Curve Theorem has played a central role in the development of topology. And although, as we shall see later, the dualities relating the homology groups of a subset of a manifold with those of its complement are ordinarily considered to be the outstanding extension of the Jordan Curve Theorem, we shall show in this section that the theorem has natural extensions in spaces extremely more general than the manifold.

Aside from certain local "smoothness" properties, the orientable n-manifold, either in the classic sense or in the generalized sense in which we shall use the term in the present work, has as a basic property that it is irreducible relative to carrying a nonbounding n-cycle, and that there is exactly one such n-cycle carried by the space. We have pointed out in 2.11 that a set which is irreducible relative to carrying a nonbounding n-cycle is not necessarily the carrier of only one such cycle, however.

We recall (V 7.10) that if J is a closed subset of a space M, then by $H^n(M; J, 0; \mathfrak{F})$ we denote the vector space of n-cycles of J (using coverings of M) reduced modulo the subspace of cycles that bound on J. Hereafter we use the symbol $H^n(M; J, 0)$, it being understood that a field $\mathfrak{F}$ is the group of coefficients. Now suppose we consider only those elements of $H^n(M; J, 0)$ that correspond to cycles that bound on the space M; these form a subspace $G^n(M; J, 0)$ of $H^n(M; J, 0)$. We denote the dimension of $G^n(M; J, 0)$ by $g^n(M; J, 0)$. Evidently $g^n(M; J, 0) \leqq p^n(M; J, 0)$.

3.1 Example. If M is a torus, and J is an S^1 on M whose basic non-bounding 1-cycle γ^1 bounds on M, then $g^1(M; J, 0) = p^1(M; J, 0) = 1$. However, if J is so situated on M that γ^1 does not bound on M, then $g^1(M; J, 0) = 0$. We note that in the former case J separates M into exactly two components having J as common boundary, whereas in the latter case $M - J$ is connected.

We first prove the following theorem:

3.2 Theorem. *Let M be a compact space carrying cycles*[1] γ_j^n, $j = 1, 2, \cdots, m$, *such that if A and B are closed proper subsets of M and Γ_A^n, Γ_B^n are cycles on A, B respectively, then a homology of the form $\sum a^j\gamma_j^n \sim \Gamma_A^n + \Gamma_B^n$ on M, $a^j \in \mathfrak{F}$, implies that $a^j = 0$ for all j. Then if J is a closed subset of M such that $M - J$ has at least $k + 1$ components, $g^{n-1}(M; J, 0) \geqq km$.*

Proof. Since $M - J$ has at least $k + 1$ components, $M - J = \bigcup_{i=1}^{k+1} P_i$ separate (Theorem I 9.7a). By Lemma 1.15 there exist cycles γ_{ij}^n, mod $F(P_i)$, on P_i, $i = 1, \cdots, k + 1$, $j = 1, \cdots, m$, such that $\gamma_j^n \sim \sum_i \gamma_{ij}^n$ mod J on M. Let

$$\partial\gamma_{ij}^n = Z_{ij}^{n-1}. \tag{3.2a}$$

Then Z_{ij}^{n-1} is on $F(P_i)$ and is evidently in a homology class which is an element of $G^{n-1}(M; J, 0)$.

Suppose $g^{n-1}(M; J, 0) < km$. Then between any km of the cycles Z_{ij}^{n-1} there must exist a homology relation on J; say

$$\sum_{i=1}^{k} \sum_{j=1}^{m} a^{ij} Z_{ij}^{n-1} \sim 0 \qquad \text{on} \qquad J, \qquad\qquad a^{ij} \in \mathfrak{F}. \tag{3.2b}$$

Since not all coefficients a^{ij} are zero, we may suppose $a^{11} \neq 0$. We may rewrite (3.2b) thus:

$$\sum_{j=1}^{m} a^{1j} Z_{1j}^{n-1} \sim \sum_{i=2}^{k} \sum_{j=1}^{m} a^{ij} Z_{ij}^{n-1} \qquad\qquad \text{on } J.$$

And since (3.2a) implies that $\partial a^{ij}\gamma_{ij}^n = a^{ij} Z_{ij}^{n-1}$, we have that

$$\sum_{i=2}^{k} \sum_{j=1}^{m} a^{ij} Z_{ij}^{n-1} \sim 0 \qquad \text{on} \qquad \bigcup_{i=2}^{k} \overline{P}_i\,, \tag{3.2c}$$

so that

$$\sum_{j=1}^{m} a^{1j} Z_{1j}^{n-1} \sim 0 \qquad \text{on} \qquad M - P_1 - P_{k+1}\,. \tag{3.2d}$$

On the other hand, (3.2a) also gives

$$\partial \sum_{j=1}^{m} a^{1j}\gamma_{1j}^n = \sum_{j=1}^{m} a^{1j} Z_{1j}^{n-1} \qquad \text{on} \qquad \overline{P}_1\,. \tag{3.2e}$$

---

[1]Hereafter, "cycle" means "C-cycle" unless indicated otherwise; for example $\gamma^r(\mathfrak{U})$ indicates a cycle of a single covering $\mathfrak{U}$.

By virtue of Lemma 1.6, (3.2d) and (3.2e) imply that there exists a cycle Γ_1^n on $M - P_{k+1}$ such that

$$\Gamma_1^n \sim \sum_{j=1}^{m} a^{1j}\gamma_{1j}^n \qquad \text{mod } M - P_1 .$$

But $\sum_{j=1}^m a^{1j}\gamma_{1j}^n \sim \sum_{j=1}^m a^{1j}\gamma_j^n \text{ mod } M - P_1$, and therefore

$$\sum_{j=1}^{m} a^{1j}\gamma_j^n \sim \Gamma_1^n \qquad \text{mod } M - P_1 .$$

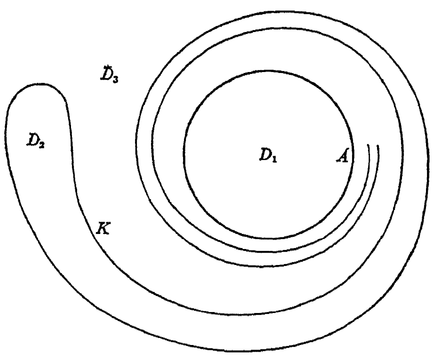
That is, $\sum_{j=1}^m a^{1j}\gamma_j^n - \Gamma_1^n \sim 0 \text{ mod } M - P_1$. Then by Lemma 1.9, there exists a cycle Γ_2^n on $M - P_1$ such that $\sum_{j=1}^m a^{1j}\gamma_j^n - \Gamma_1^n \sim \Gamma_2^n$; that is,

$$\text{(3.2f)} \qquad \sum_{j=1}^{m} a^{1j}\gamma_j^n \sim \Gamma_1^n + \Gamma_2^n ,$$

where Γ_1^n is on $M - P_{k+1}$ and Γ_2^n is on $M - P_1$. However, by hypothesis relation (3.2f) implies that all coefficients a^{1j} are zero; in particular, $a^{11} = 0$. Thus the supposition that $g^{n-1}(M; J, 0) < km$ leads to contradiction.

Evidently a space M which carries at least one nonbounding n-cycle, but is irreducible relative to carrying an n-cycle nonbounding on M, is a space of the type M hypothesized in Theorem 3.2 (but the converse does not hold—see Example 3.4 below). Hence we have:

3.3 Corollary. *If the compact space M is irreducible relative to carrying an n-cycle nonbounding on M, and carries at least m lirh nonbounding n-cycles, and J is a closed subset of M such that $p^{n-1}(M; J, 0) = h$, then $M - J$ has at most $[h/m] + 1$ components.*



3.4 Example. The figure illustrates a continuum M in the euclidean plane composed of (1) a circle A which is the boundary of bounded domain D_1, and (2) a curve K spiraling about A in a double loop in such fashion that $A \cup K = M$ is the common boundary of two domains D_2, D_3 distinct from D_1. As will appear from duality theorems of Chapter VIII, M is minimal closed carrier of a γ_1^1 satisfying the hypothesis of Theorem 3.2 for $n = 1$, $m = 1$;

it is not, however, irreducible relative to carrying a nonbounding 1-cycle, nor is it irreducible relative to carrying a 1-cycle nonbounding on M, of course.

3.5 Remark. Since for the n-sphere, S^n, the number m used above is 1, we have that if J is a closed subset of S^n such that $p^{n-1}(J) = h$, then $M - J$ has at most $h + 1$ components. In particular, if J is an S^{n-1}, then $h = 1$, and $S^n - J$ has at most 2 components—an important portion of the Jordan Curve Theorem when $n = 2$. Although this particular item is superseded later on by the duality theorems cited above, we mention it here to show the interrelations between the dualities on a manifold and the above theorems.

Partly as a means of introducing important new methods utilizing the machinery of cocycles, and partly to give a sort of converse of preceding theorems, we insert at this point some theorems which are of great importance later on in the study of generalized manifolds.

3.6 Lemma. *Let M be a compact space such that if F is a proper closed subset of M, then every n-cycle on F bounds on M. Then if γ^n is a nonbounding cycle of M and P is a nonempty open subset of M, it is impossible that $\gamma^n \sim 0$ mod $M - P$.*

Proof. The relation $\gamma^n \sim 0 \bmod M - P$ would imply, according to Lemma 1.9, that there exists a cycle Z^n on $M - P$ such that $\gamma^n \sim Z^n$ on M. But since, by hypothesis, such a Z^n bounds on M, it would follow that γ^n bounds on M.

3.7 Lemma. *Under the same hypothesis as in Lemma 3.6, if $\gamma_1^n, \cdots, \gamma_m^n$ are lirh on M, then they are lirh mod $S - P$ for every nonempty open subset P of M.*

3.8 Theorem. *Let M be a compact space which carries at least m lirh n-cycles, and such that every n-cycle on a closed proper subset of M bounds on M. Then if J is a closed subset of M, and $M - J$ has at least k components, $p_n(M; M - J, 0) \geqq km$.* [Cf. V 15.4.]

Proof. Since $M - J$ has at least k components, it is the union of k separated sets $D_1, \cdots, D_k$ (Theorem I 9.7a). Let $\gamma_1^n, \cdots, \gamma_m^n$ denote m lirh n-cycles of M. By Lemma 3.7 and Theorems V 18.23, V 18.25, there exist in each open set D_j, $j = 1, \cdots, k$, cocycles $\gamma_n^i(D_j)$, $i = 1, \cdots, m$, such that

$$\gamma_n^i(D_j) \cdot \gamma_h^n = \delta_i^h. \tag{3.8a}$$

We assert that these mk cocycles are lircoh ("lircoh" = "linearly independent relative to cohomology") in $M - J$. For suppose there exists a relation

$$\delta C_{n-1} = \sum a_i^j \gamma_n^i(D_j) \qquad \text{in} \qquad M - J,$$

where we may assume that the chains involved are all on the same covering $\mathfrak{U}$ of M, and at least one a_i^j is not zero. In particular, we may suppose that $a_1^1 \neq 0$.

Now by Lemma V 8.7, there exists $\mathfrak{V}' > \mathfrak{U}$ such that a simplex of $\mathfrak{V}'$ in $M - J$ is also in $M - Q$, where Q is some open set containing J. Let $\mathfrak{V}$ be

the covering of M whose elements are Q and $M - J$, and let $\mathfrak{B} > (\mathfrak{B}', \mathfrak{W})$. Then, since C_{n-1} is in $M - J$ and the chain $\pi^*_{\mathfrak{U}\mathfrak{B}} C_{n-1}$ is likewise, the latter chain must also be in $M - \bar{Q}$. Then we have (V 16)

$$\partial[\pi^*_{\mathfrak{U}\mathfrak{B}} C_{n-1} \frown \gamma^n_1(\mathfrak{B})] = -\pi^*_{\mathfrak{U}\mathfrak{B}} \delta C_{n-1} \frown \gamma^n_1(\mathfrak{B})$$

$$= -[\sum a^i_j \pi^*_{\mathfrak{U}\mathfrak{B}} \gamma^i_n(D_j)] \frown \gamma^n_1(\mathfrak{B}) \qquad \text{in} \qquad M - J.$$

The portion in D_1 of the chain $\pi^*_{\mathfrak{U}\mathfrak{B}} C_{n-1} \frown \gamma^n_1(\mathfrak{B})$ has as its boundary the 0-cycle $-\sum a^1_i \pi^*_{\mathfrak{U}\mathfrak{B}} \gamma^i_n(D_1) \frown \gamma^n_1(\mathfrak{B})$, and the latter must have Kronecker index zero (Lemma V 18.7). But by (3.8a), this Kronecker index is $-a^1_1 \pi^*_{\mathfrak{U}\mathfrak{B}} \gamma^1_n(D_1) \cdot \gamma^n_1(\mathfrak{B}) = -a^1_1 \neq 0$.

3.9 Theorem. *Under the same hypothesis as in Theorem* 3.8, $p^{n-1}(J) \geqq m(k-1)$.

Proof. In the notation of the proof of Theorem 3.8, the cocycles $\gamma^i_n(D_j)$, i fixed, may all be taken in the same cohomology class of M. Consequently each pair $\gamma^i_n(D_1)$, $\gamma^i_n(D_j)$, $j > 1$, satisfies a cohomology on M. There are $m(k-1)$ of these relations:

$$\delta C^{\nu}_{n-1} = \sum_j a^j_{i\nu} \gamma^i_n(D_j), \qquad \nu = 1, 2, \cdots, m(k-1), \tag{3.9a}$$

where only two of the coefficients $a^j_{i\nu}$ in each relation are not zero, etc. Denoting the portion of C^{ν}_{n-1} on J by Z^{ν}_{n-1}, the chains Z^{ν}_{n-1} are cocycles of J—i.e., cocycles mod $M - J$—and we assert they are lircoh on J. For suppose there exists a relation

$$\delta K_{n-2} = \sum c_\nu \pi^*_{\mathfrak{U}\mathfrak{B}} Z^{\nu}_{n-1} + D_{n-1}, \qquad D_{n-1} \quad \text{in} \quad M - J. \tag{3.9b}$$

Applying δ again to (3.9b), and recalling that $\delta^2 = 0$,

$$\sum c_\nu \pi^*_{\mathfrak{U}\mathfrak{B}} \delta Z^{\nu}_{n-1} + \delta D_{n-1} = 0. \tag{3.9c}$$

But then by (3.9a) we have

$$\delta \sum c_\nu \pi^*_{\mathfrak{U}\mathfrak{B}} [C^{\nu}_{n-1} - Z^{\nu}_{n-1}] - \delta D_{n-1} = \sum_{\nu, j} \pi^*_{\mathfrak{U}\mathfrak{B}} c_\nu a^j_{i\nu} \gamma^i_n(D_j) \text{ in } M - J. \tag{3.9d}$$

Suppose c_1, for instance, is not zero. It is the coefficient of δZ^1_{n-1} in relation (3.9c), and Z^1_{n-1} is the portion of C^1_{n-1} on J. Now δC^1_{n-1} is $\gamma^1_n(D_1) - \gamma^1_n(D_2)$, say, so that $a^2_1 = -1$. Hence $-c_1$ is the coefficient of $\gamma^1_n(D_2)$ in relation (3.9d). That is, the cocycles $\gamma^i_n(D_j)$ are not lircoh in $M - J$, contradicting what was proved in Theorem 3.8.

3.10 Theorem. *Under the hypothesis of Theorem* 3.8, *if* $p^{n-1}(M) = 0$ *the following relation holds*:

$$p_n(M; M - J, 0) \geqq p^{n-1}(J) + m.$$

Proof. Let $Z^1_{n-1}, \cdots, Z^h_{n-1}$ be cocycles mod $M - J$ that are lircoh mod

$M - J$. Then $\delta Z_{n-1}^i = Z_n^i$ is a cocycle of $M - J$, $i = 1, \cdots, h$. We assert that the Z_n^i are lircoh in $M - J$.

Assuming that the Z_n^i are all on the same covering $\mathfrak{U}$ of M, suppose that on some covering $\mathfrak{V} > \mathfrak{U}$ there exists a relation

$$\delta C_{n-1}(\mathfrak{V}) = \sum a_i \pi_{\mathfrak{U}\mathfrak{V}}^* Z_n^i \qquad \text{in } M - J. \tag{3.10a}$$

Since $\delta \pi_{\mathfrak{U}\mathfrak{V}}^* Z_{n-1}^i = \pi_{\mathfrak{U}\mathfrak{V}}^* Z_n^i$, we have from (3.10a) that $C_{n-1}(\mathfrak{V}) - \sum \pi_{\mathfrak{U}\mathfrak{V}}^* a_i Z_{n-1}^i$ is a cocycle of M, and as $p_{n-1}(M) = p^{n-1}(M) = 0$, there exists a chain $C^{n-2}(\mathfrak{W})$, $\mathfrak{W} > \mathfrak{V}$, such that $\delta C^{n-2}(\mathfrak{W}) = \pi_{\mathfrak{V}\mathfrak{W}}^* C_{n-1}(\mathfrak{V}) - \pi_{\mathfrak{U}\mathfrak{W}}^* a_i Z_{n-1}^i$. However, this gives at once that $\sum \pi_{\mathfrak{U}\mathfrak{W}}^* a_i Z_{n-1}^i \backsim 0 \bmod M - J$, since C_{n-1} is in $M - J$, contradicting the fact that the cocycles Z_{n-1}^i are lircoh mod $M - J$. We must conclude, then, that the cocycles Z_n^i are lircoh in $M - J$.

Next, suppose that there exists a relation

$$\delta C_{n-1}(\mathfrak{V}) = \sum c_i \pi_{\mathfrak{U}\mathfrak{V}}^* Z_n^i + \sum a_i \pi_{\mathfrak{U}\mathfrak{V}}^* \gamma_n^i(D_1) \qquad \text{in} \qquad M - J. \tag{3.10b}$$

Not all the a_i are zero, since we already know the Z_n^i to be lircoh in $M - J$. But then, since $\delta \pi_{\mathfrak{U}\mathfrak{V}}^* Z_{n-1}^i = \pi_{\mathfrak{U}\mathfrak{V}}^* Z_n^i$, we get $\delta[C_{n-1}(\mathfrak{V}) - \sum c_i \pi_{\mathfrak{U}\mathfrak{V}}^* Z_{n-1}^i] = \sum a_i \pi_{\mathfrak{U}\mathfrak{V}}^* \gamma_n^i(D_1)$, contradicting the fact that the cocycles $\gamma_n^i(D_1)$ are lircoh on M.

3.11 Definition. If a cycle γ^r of a space S has a closed carrier K such that $\gamma^r \not\sim 0$ on K, then γ^r will be called a *nontrivial* r-cycle of S. (Evidently the case $r = 0$ conforms with the usage of the term in V 11.4.)

3.12 Definition. If M is a point set, and F is a closed subset of M carrying a cycle γ^r which bounds on M, then a point x of $M - F$ will be called a *barrier* to γ^r in M if no closed subset of $M - x$ carries a homology $\gamma^r \sim 0$.

3.13 Theorem. *Let M be a compact space such that $p^n(F) = 0$ for every closed proper subset F of M, and let J be a closed subset of M such that $p^{n-1}(J) = 1$ and J is irreducible relative to carrying an $(n - 1)$-cycle nonbounding on J. Denoting by γ^{n-1} a nonbounding cycle of J, let γ^{n-1} bound on M and no point of $M - J$ be a barrier to γ^{n-1}. Then $M - J$ is the union of two disjoint domains having J as common boundary.*

Proof. By Theorem 2.22, there exists $M_1 \subset M$ such that M_1 is an irreducible membrane relative to γ^{n-1}. Since J is irreducible relative to carrying an $(n - 1)$-cycle nonbounding on J, and γ^{n-1} is nonbounding, J must be the carrier of γ^{n-1}. Hence there exists $x \in M_1 - J$, and since x is not a barrier to γ^{n-1}, there exists a closed subset M_2 of $M - x$ such that $\gamma^{n-1} \sim 0$ on M_2. Then by Theorem 2.21, there exists on M a nontrivial n-cycle γ^n, and since $p^n(F) = 0$ for every closed proper subset F of M, the space M must be the irreducible carrier of γ^n, and $M = M_1 \cup M_2$. By Corollary 3.3, $M - J$ has at most two components.

In order to prove that J separates M, let us first replace M_2 by a subset of M_2 which is an irreducible membrane relative to γ^{n-1}; such a set exists by Theorem 2.22. We shall then prove that $M - M_1$ is nonempty and connected.

It is certainly nonempty, since otherwise M_2 would be a subset of $M_1 - x$ and consequently M_1 would not be an irreducible membrane relative to γ^{n-1}. Suppose $M - M_1 = A \cup B$ separate. Then $F(A) \subset J$. For suppose $y \in (M_1 - J) \cap F(A)$. Since y is not a barrier to γ^{n-1}, there exists a closed subset M_3 of $M - y$ such that $\gamma^{n-1} \sim 0$ on M_3. But arguing as in the preceding paragraph we see that $M = M_1 \cup M_3$, which is impossible since there exist points of A in a neighborhood of y that belong to neither M_1 nor M_3. Therefore $(M_1 - J) \cap F(A) = 0$.

By Corollary 1.16, there exists a cycle γ_A^n mod J on A such that $\gamma_A^n \sim \gamma^n$ mod $M - A$ on M. Let $Z^{n-1} = \partial\gamma_A^n$. Suppose $Z^{n-1} \sim 0$ on J. Then by Lemma 1.6, there exists a cycle Z^n on $J \cup A$ such that $\gamma_A^n \sim Z^n$ mod $M - A$. But then $\gamma^n \sim Z^n$ mod $M - A$, and by Lemma 1.9 there exists a cycle Γ^n on $M - A$ such that $\gamma^n - Z^n \sim \Gamma^n$. That is, $\gamma^n \sim Z^n + \Gamma^n \sim 0$, since Z^n and Γ^n are on closed proper subsets of M. We must conclude then that Z^{n-1} does not bound on J, and therefore that $Z^{n-1} \sim a\gamma^{n-1}$ on J, $a \in \mathfrak{F}$, since $p^{n-1}(J) = 1$. But then $\gamma^{n-1} \sim 0$ on $\bar{A}$, a closed proper subset of M_2, contradicting the fact that M_2 is an irreducible membrane relative to γ^{n-1}.

Thus $M - M_1 = H$ is connected. That $F(H) \subset J$ follows from the same argument used to prove $F(A) \subset J$ above. Then it easily follows that H is a component of $M - J$, since H is closed in $M - M_1$ and open in M. And since $M - J - H \neq 0$, $M - J$ is not connected.

We have proved, then, that $M - J$ is the union of exactly two disjoint domains D and E. It remains to prove that J is their common boundary. Suppose that $J \cap F(D) = J'$ is a proper subset of J. Now by Corollary 1.16, there exists a cycle γ_D^n mod J' on $\bar{D}$ such that $\gamma^n \sim \gamma_D^n$ mod $M - D$. Since J' is a proper subset of J, $\partial\gamma_D^n \sim 0$ on J, and there exists by Corollary 1.6 a cycle Z^n on $D \cup J$ such that $Z^n \sim \gamma_D^n$ mod J. But $Z^n \sim 0$ on $D \cup J$ and therefore $\gamma_D^n \sim 0$ mod $M - D$, implying $\gamma^n \sim 0$ mod $M - D$. This contradicts Lemma 3.6.

3.14 Remark. Theorem 3.13 might be called a "generalized Jordan-Brouwer separation theorem", inasmuch as it contains Brouwer's extension of Jordan's Theorem to the separation of euclidean n-space by an $(n - 1)$-manifold as a special case.

3.15 It is interesting to note the relation of the theorem just proved to the particular case of the 2-sphere. For example, in Theorem II 4.3 we proved that if a Peano continuum contains at least one S^1 and satisfies the Jordan Curve Theorem, then it is a 2-sphere. Now it is easy to see that if M is a Peano continuum which satisfies the conditions (1) M contains at least one S^1, and every nontrivial 1-cycle on an S^1 bounds on M, (2) $p^2(F) = 0$ for every closed proper subset of M, and (3) no point is a barrier to any 1-cycle, then M is a 2-sphere. For if J is any S^1 in such a space M, then the conditions of Theorem 3.13 are satisfied for $n = 2$ and $M - J$ is the union of two disjoint domains having J as common boundary. Before proceeding further in this direction, however, we shall introduce some concepts important for the sequel.

4. **Non-r-cut and r-avoidable points.**

4.1 Definition. A point p of a space S will be called a *non-r-cut point* of S if every r-cycle of any compact subset of $S - p$ is homologous to zero on a compact subset of $S - p$.

4.2 Definition. A space S will be called *r-avoidable at* $p \in S$ (and p will be called an *r-avoidable point* of S) if for every open set U containing p there exists an open set V such that $p \in V \Subset U$ and such that every r-cycle on $F(U)$ bounds on $S - V$.

4.3 Definition. A space S will be called *locally r-avoidable at* $p \in S$ (and p will be called a *locally r-avoidable point* of S) if for every open set U containing p there exist open sets V and W such that $p \in W \Subset V \Subset U$ and such that if γ^r is a cycle on $F(V)$ then $\gamma^r \sim 0$ on $S - W$.

In order to investigate some of the relations between the above definitions, we shall prove a theorem on what might be called "set-avoidability."

4.4 Definitions. A countable set, $\mathfrak{B}^r$, of cycles of a compact metric space M such as that whose existence was proved in Theorem V 12.4 will be called a *metric fundamental system of r-cycles* of M. The normal sequence of refinements used in that proof will be called *the normal sequence of refinements associated with* $\mathfrak{B}^r$. A set of cycles $Z^r_{n(k)+1}, \cdots, Z^r_{n(k+1)}$ will be called an *interval* of $\mathfrak{B}^r$; more specifically, the $(k + 1)$*st interval of* $\mathfrak{B}^r$.

4.5 Theorem. *If A and B are disjoint closed subsets of a compact metric space M such that every r-cycle on A bounds on a closed subset of $M - B$, then there exists $\epsilon > 0$ such that every r-cycle on A bounds on $M - S(B, \epsilon)$.*

Proof. Let $Z^r_1, \cdots, Z^r_{n(k)}, \cdots$ be a metric fundamental system, $\mathfrak{B}^r$, of r-cycles of A. We may assume the normal sequence of refinements associated with $\mathfrak{B}^r$ to consist of coverings of M. We shall first prove the theorem for cycles of $\mathfrak{B}^r$. The result will then follow. For suppose γ^r is any cycle of A. Then for any covering $\mathfrak{U}_k$ of the normal sequence of refinements associated with $\mathfrak{B}^r$, $\gamma^r(\mathfrak{U}_k) \sim \sum_{i=1}^{n(k)} a^i Z^r_i(\mathfrak{U}_k)$ on $\mathfrak{U}_k \wedge A$, $a^i \in \mathfrak{F}$, and as each $Z^r_i(\mathfrak{U}_k) \sim 0$ on $M - S(B, \epsilon)$, $\gamma^r(\mathfrak{U}_k) \sim 0$ on $M - S(B, \epsilon)$.

Suppose the theorem not true for the cycles of $\mathfrak{B}^r$. Let $\delta_1 < 1$, and let $Z^r_{t(1)}$ be the first cycle of $\mathfrak{B}^r$ that fails to bound on $M - S(B, \delta_1)$. Let $\mathfrak{U}_{m(1)}$ be the first covering of the normal sequence of refinements associated with $\mathfrak{B}^r$ such that $Z^r_{t(1)}(\mathfrak{U}_{m(1)}) \nsim 0$ on $M - S(B, \delta_1)$. Now let $\delta_2 < \min(\delta_1, 1/2)$ be such that if Z^r_i is a cycle of $\mathfrak{B}^r$ of subscript $i \leqq t(1)$, or in the same interval of $\mathfrak{B}^r$ as $Z^r_{t(1)}$, then $Z^r_i \sim 0$ on $M - S(B, \delta_2)$. Let $Z^r_{t(2)}$ be the first cycle of $\mathfrak{B}^r$ that fails to bound on $M - S(B, \delta_2)$ and $\mathfrak{U}_{m(2)}$ a covering of the normal sequence of refinements associated with $\mathfrak{B}^r$ such that $Z^r_{t(2)}(\mathfrak{U}_{m(2)}) \nsim 0$ on $M - S(B, \delta_2)$. Evidently $Z^r_{t(2)}(\mathfrak{U}_{m(1)}) \sim 0$ on A.

Proceeding in the above manner we obtain a sequence of cycles of $\mathfrak{B}^r$, $Z^r_{t(1)}, \cdots, Z^r_{t(k)}, \cdots$, such that (1) $t(k) < t(k + 1)$, (2) $Z^r_{t(k)}(\mathfrak{U}_{m(k)}) \nsim 0$ on $M -$

$S(B, \delta_k)$, where $\delta_1, \cdots, \delta_k, \cdots$, is a sequence of positive numbers monotonically decreasing to zero, (3) $Z^r_{t(k+1)}(\mathfrak{U}_{m(k)}) \sim 0$ on A, and (4) $Z^r_{t(i)} \sim 0$ on $M - S(B, \delta_k)$ if $i < k$.

Now let $\gamma^r(\mathfrak{U}_{m(1)}) = Z^r_{t(1)}(\mathfrak{U}_{m(1)})$, and generally $\gamma^r(\mathfrak{U}_{m(k)}) = Z^r_{t(1)}(\mathfrak{U}_{m(k)}) + Z^r_{t(2)}(\mathfrak{U}_{m(k)}) + \cdots + Z^r_{t(k)}(\mathfrak{U}_{m(k)})$. Then the collection $\gamma^r = \{\gamma^r(\mathfrak{U}_{m(k)})\}$ is a C-cycle on A on the complete family $\{\mathfrak{U}_{m(k)}\}$. For $\gamma^r(\mathfrak{U}_{m(k)}) - \pi^k_{k+1}\gamma^r(\mathfrak{U}_{m(k+1)}) = \sum_{i=1}^k[Z^r_{t(i)}(\mathfrak{U}_{m(k)}) - \pi^k_{k+1}Z^r_{t(i)}(\mathfrak{U}_{m(k+1)})] - \pi^k_{k+1}Z^r_{t(k+1)}(\mathfrak{U}_{m(k+1)})$. (We abbreviate $\pi_{\mathfrak{U}_j\mathfrak{U}_i}$ to π^j_i for typographical reasons.) For each i, the cycle designated by the expression in the brackets is homologous to zero on A, since $Z^r_{t(i)}$ is a C-cycle of A. By (3) above, $Z^r_{t(k+1)}(\mathfrak{U}_{m(k)}) \sim 0$ on A, and hence we have that $\pi^k_{k+1}Z^r_{t(k+1)}(\mathfrak{U}_{m(k+1)}) \sim Z^r_{t(k+1)}(\mathfrak{U}_{m(k)}) \sim 0$ on A. Consequently γ^r is a C-cycle on A.

We assert $\gamma^r \not\sim 0$ on $M - B$. For suppose the contrary. Then there exists k such that $\gamma^r \sim 0$ on $M - S(B, \delta_k)$. But $\gamma^r(\mathfrak{U}_{m(k)}) - Z^r_{t(k)}(\mathfrak{U}_{m(k)}) = Z^r_{t(1)}(\mathfrak{U}_{m(k)}) + \cdots + Z^r_{t(k-1)}(\mathfrak{U}_{m(k)}) \sim 0$ on $M - S(B, \delta_k)$ by (4) above, so that $Z^r_{t(k)}(\mathfrak{U}_{m(k)}) \sim 0$ on $M - S(B, \delta_k)$, contradicting (2) above. Thus $\gamma^r \not\sim 0$ on $M - B$ and the assumption that the theorem is not true for the cycles of $\mathfrak{B}^r$ leads to contradiction.

THEOREM 4.5 may be generalized as follows:

4.6 THEOREM. *If A and B are disjoint closed subsets of a compact metric space M, and there exist on A cycles $\gamma^r_1, \cdots, \gamma^r_k$, finite in number, such that every cycle γ^r on A is homologous on a closed subset of $M - B$ to a linear combination of the cycles $\gamma^r_1, \cdots, \gamma^r_k$, then there exists $\epsilon > 0$ such that every cycle γ^r on A is homologous on $M - S(B, \epsilon)$ to a linear combination of the cycles $\gamma^r_1, \cdots, \gamma^r_k$.*

REMARK. Evidently the above theorems remain valid if instead of assuming M compact, only A is assumed compact; the essential modification in the argument would consist of obtaining $\mathfrak{B}^r$ by means of coverings $\mathfrak{U}_k$ of M, finite in some neighborhood of A and such that each $\mathfrak{U}_{k+1}$ is a normal refinement of $\mathfrak{U}_k$ rel $(A, 0)$.

As a corollary of Theorem 4.5 we have:

4.7 COROLLARY. *If M is a compact metric space and p is a non-r-cut point of M, then p is an r-avoidable point of M.*

4.8 LEMMA. *In any space, an r-avoidable point is locally r-avoidable.*

4.9 REMARKS. It is evident that the requirement that every point of a space S be a non-r-cut point places severe restriction on the character of the r-dimensional connectivity of S. In particular, if a compact S is not the minimal closed carrier of any r-cycle, then the above requirement implies that $p^r(S) = 0$ and no point be a barrier to any r-cycle. Conversely, if no point of a compact S is a barrier to any r-cycle and $p^r(S) = 0$, then every point of S is a non-r-cut point of S.

It is easy to see that the properties defined in 4.1–4.3 are actually successively

weaker. For example, in the cartesian plane, let $M = \{(x, y) \mid (x^2 + y^2 = 1) \vee (1 < x \leqq 2, y = 0)\}$. The point $(2, 0)$ is a 1-cut point and a 1-avoidable point, since for every open set containing $(2, 0)$ the boundary contains no nontrivial 1-cycles. And that a point may be a locally 1-avoidable point while not 1-avoidable is shown by the example, in cartesian 3-space, of the set $M = \{(x, y, z) \mid (x^2 + y^2 + z^2 \leqq 1) \vee (x^2 + y^2 \leqq 4, z = 0)\} - \{(x, y, z) \mid (x - 3/2)^2 + y^2 \leqq 1/4, z = 0\}$, with $p = (0, 0, 0)$.

However, if a compact metric space is simply r-connected, the above distinctions disappear.

4.10 Lemma. *In a simply r-connected compact space S, a locally r-avoidable point is both r-avoidable and a non-r-cut point.*

Proof. Let p be a locally r-avoidable point of S, and let γ^r be a cycle on a compact subset K of $S - p$. Letting $S - K = U$, there exist open sets V and W such that $p \in W \Subset V \Subset U$ and such that if Z^r is a cycle on $F(V)$, then $Z^r \sim 0$ on $S - W$. Now by Lemma 1.13 there exists a cycle Z^r on $F(V)$ such that $\gamma^r \sim Z^r$ on $S - V$, and it follows that $\gamma^r \sim 0$ on $S - W$. Hence p is a non-r-cut point of S.

The proof that p is r-avoidable is similar.

4.11 Corollary. *In a simply r-connected, compact metric space the non-r-cut points, r-avoidable points and locally r-avoidable points are all identical.*

4.12 Corollary. *In a metric continuum, the non-*0*-cut points,* 0*-avoidable points and locally* 0*-avoidable points are identical.*

We can now obtain a characterization of the 2-sphere which does not include the assumption that the space is peanian—specifically, in the obvious characterization given in 3.15 in the remark following Theorem 3.13, we eliminate the assumption "M is a Peano continuum."

4.13 Theorem. *Let M be a compact metric space such that $p^2(F) = 0$ for every closed proper subset F of M, which contains at least one* 1*-sphere, and all of whose points are non-*1*-cut points. Then if M is not a* 1*-sphere, it is a* 2*-sphere.*

For the proof, we notice that in proving Theorem 3.13 we also proved the following lemma.

4.14 Lemma. *Let M be a compact metric space such that $p^n(F) = 0$ for every closed proper subset F of M, and let J be a closed subset of M which carries a nontrivial cycle γ^{n-1} but is irreducible relative to carrying an $(n - 1)$-cycle nonbounding on J. Then if $\gamma^{n-1} \sim 0$ on M, and no point of $M - J$ is a barrier to γ^{n-1}, the number $p^n(M) > 0$.*

Now in order to prove Theorem 4.13 it is only necessary to show that M is lc, assuming M is not a 1-sphere. Let $x \in M$, and U an open set containing x. Since by Corollary 4.7 every point of M is 1-avoidable, there exists an open

set V such that $x \in V \Subset U$ and such that every 1-cycle of $F(U)$ bounds on $M - V$. Suppose that not all points of V are in the same quasi-component of U. Then $U = A \cup B$ separate, where $x \in A$ and $V \cap B \neq 0$.

By the above lemma, $p^2(M) > 0$. Let γ^2 be a nonbounding cycle of M. Then by Corollary 1.16, there exists a cycle γ_A^2 mod $F(A)$ such that $\gamma_A^2 \sim \gamma^2$ mod $M - A$. Let $Z^1 = \partial\gamma_A^2$. Then Z^1 is a cycle of $F(A) \subset F(U)$, and is ~ 0 on $M - V$. By Lemma 1.6 (with $K = M - V$) there exists a cycle Z^2 on $M - B \cap V$ such that $Z^2 \sim \gamma_A^2$ mod $M - V \cap A$. Then $Z^2 \sim \gamma^2$ mod $M - V \cap A$, and hence by Lemma 1.9, $\gamma^2 - Z^2 \sim \Gamma^2$, where Γ^2 is on $M - V \cap A$. But Z^2 and Γ^2 are on closed proper subsets of M, implying $\gamma^2 \sim 0$ on M. But γ^2 is a nonbounding cycle of M. We must conclude, then, that V lies in one quasi-component of U; hence that M is lcq at every point and therefore lc (Theorem II 1.8).

Now the assumption in the hypothesis of Theorem 4.13 that M contains an S^1 is a strong assumption, especially in view of the fact that M is not assumed to be a Peano space. It is well to notice, then, that if we assume that M has a closed proper subset J such that J is irreducible relative to carrying a 1-cycle nonbounding on J, then the proof that M is lc goes through as before. And as M has no cut point by Corollary 3.3 and is therefore cyclicly connected (Corollary III 3.32a), there exists a 1-sphere in M. We can therefore state:

4.15 Theorem. *Let M be a compact metric space such that $p^2(M') = 0$ for every closed proper subset M' of M, which contains a closed subset J such that J is irreducible relative to carrying a 1-cycle nonbounding on J, and which has no 1-cut points. Then if $M \neq J$, M is a 2-sphere.*

A characterization of the S^1 can be given analogous to the characterization of the S^2 given in Theorem 4.15:

4.16 Theorem. *Let M be a compact metric space such that $p^1(M') = 0$ for every closed proper subset M' of M, and which has no 0-cut points. Then if M has at least 3 points, it is a 1-sphere.*

Proof. Since the only point set J which has the property that J is irreducible relative to carrying a 0-cycle nonbounding on J is that which consists of exactly two points; and since M has at least three points, Lemma 4.14 applies. The proof that M is lc is as in Theorem 4.13. Then since M is a Peano continuum with no cut point, it contains an S^1, and since $p^1(M) > 0$ irreducibly, this S^1 must constitute all of M.

It is natural now to ask whether the closed 2-manifold can be characterized in the same order of ideas. First let us note the following theorem which is now easily proved:

4.17 Theorem. *Let M be a compact, metric lc space carrying m cycles Z_i^2, $i = 1, \cdots, m$, $m \geqq 1$, such that if A and B are closed proper subsets of M and γ_1^2, γ_2^2 cycles on A, B respectively, then a homology of the form $\sum a^i Z_i^2 \sim \gamma_1^2 + \gamma_2^2$*

on M implies $a^i = 0$ for all i. Then if M is separated by each simple closed curve of M of diameter less than some fixed positive number ϵ, M is a closed 2-manifold.

PROOF. That M is not separated by any closed subset J such that $p^1(J) = 0$ follows from Theorem 3.2. Then M is connected; M is cyclicly connected (III 3.32a); and if J is a simple closed curve of M, every component of $M - J$ has J as complete boundary. It follows that M contains a simple closed curve of diameter $< \epsilon$ (cf. III 6.5), and the theorem follows from Theorem III 6.1.

4.18 COROLLARY. *If M is a compact, metric lc space such that* (1) *M is irreducible relative to carrying a 2-cycle nonbounding on M, and* (2) *for some positive number ϵ, M is separated by each of its simple closed curves of diameter $<\epsilon$, then M is a closed 2-manifold.*

We also note that in the same way in which we proved the lc property in Theorem 4.13, we may prove:

4.19 LEMMA. *Let M be a compact metric space which is irreducible relative to carrying an n-cycle nonbounding on M, and suppose that all points of M are locally $(n - 1)$-avoidable. Then M is a Peano continuum.*

(This lemma is much weaker than Theorem 5.4 below. It is stated here because it is needed immediately below and its proof is obvious.)

4.20 DEFINITION. A C-cycle Z^r will be said to be of diameter $< \epsilon$ if it has a carrier of diameter $< \epsilon$.

Now in setting up a theorem characterizing the closed 2-manifold analogous to Theorem 4.15, it would be natural to assume that there exists a number ϵ such that 1-cycles of diameter $< \epsilon$ bound and no point is a barrier to any 1-cycle of diameter $< \epsilon$. Incidentally, in view of Theorem 4.5 this condition will imply the local 1-avoidability of all points, for the case under consideration here.

4.21 THEOREM. *Let M be a compact metric space such that $p^2(M') = 0$ for every closed proper subset M' of M, and let ϵ be a positive number such that 1-cycles of diameter $< \epsilon$ bound on M and no point of M is a barrier to any 1-cycle of M of diameter $< \epsilon$. If M contains some proper subset J of diameter $< \epsilon$ which is irreducible relative to carrying a 1-cycle nonbounding on J, and M is not a 1-sphere, then M is a closed 2-manifold.*

PROOF. By Lemma 4.14, $p^2(M) > 0$, and since, as remarked above, M is locally 1-avoidable, it follows from Lemma 4.19 that M is lc. That M is separated by every 1-sphere of diameter $< \epsilon$ now follows from Theorem 3.13. Hence M is a Peano continuum which contains a 2-cycle irreducibly, and is separated by every 1-sphere of diameter $< \epsilon$; M is therefore a closed 2-manifold (Corollary 4.18). We may also state the following three theorems, enumerated so as to indicate their forerunners above:

4.15a THEOREM. *Let M be a compact metric space such that $p^2(M) > 0$ irreducibly, and which has no 1-cut points. Then M is a 2-sphere.*

PROOF. That M is lc follows from Corollary 4.7 and Lemma 4.19. By Corollary 3.3, M has no cut points, and is therefore cyclicly connected and contains an S^1. Now M cannot be an S^1, since $p^2(M) > 0$, and the theorem follows from Theorem 4.13.

4.16a THEOREM. *Let M be a compact metric space such that $p^1(M) > 0$ irreducibly, and which has no 0-cut points. Then M is a 1-sphere.*

4.21a THEOREM. *Let M be a compact metric space such that $p^2(M) > 0$ irreducibly, and let $\epsilon > 0$ be such that 1-cycles of diameter $< \epsilon$ bound on M and no point of M is a barrier to any 1-cycle of diameter $< \epsilon$. Then M is a closed 2-manifold.*

The ϵ-condition here corresponds to the condition given in Theorem 4.17 that every 1-sphere of diameter $< \epsilon$ separate space. However, the remark made above to the effect that the former condition implies local 1-avoidability throughout the above argument raises the question as to the actual relationship between these two properties. That the latter property is not generally as strong a condition as the former is shown by the well-known example of the "sphere with infinitely many handles;" this configuration, M, satisfies the condition that $p^2(M; \mathfrak{F}) > 0$ irreducibly as well as the condition that every point be locally 1-avoidable. Consequently Theorem 4.21a, for instance, would not hold if the avoidability condition were substituted for the ϵ-condition used therein. However, noting in the case of the example just cited that the 1-dimensional Betti number is infinite, let us consider adding to the avoidability condition the condition that M be semi-1-connected.

4.22 LEMMA. *In a compact metric space M, the conditions* (a) *that there exist $\epsilon > 0$ such that r-cycles of diameter $< \epsilon$ bound and no point of M is a barrier to any r-cycle of diameter $< \epsilon$ and* (b) *that M be semi-r-connected and locally r-avoidable are equivalent.*

PROOF. We have already remarked above that (a) implies the avoidability condition. That the semi-r-connectedness is implied is obvious.

To see that (b) implies (a), since M is compact there exists (cf. V 19.4) $\epsilon > 0$ such that every r-cycle of M of diameter $< \epsilon$ bounds on M. Let γ^r be a cycle on some closed set J of diameter $< \epsilon$, and let $p \in M - J$. Let U be an open set containing p such that $J \cap \overline{U} = 0$, and let V and W be open sets as in the definition (4.3) of local r-avoidability. By Lemma 1.13, there exists a cycle Z^r on $F(V)$ such that $\gamma^r \sim Z^r$ on $M - V$. And since $Z^r \sim 0$ on $M - W$, it follows that $\gamma^r \sim 0$ on $M - W$, and p is therefore not a barrier to γ^r.

In view of Lemma 4.22 we may state:

4.23 THEOREM. *Let M be a compact metric space such that $p^2(M) > 0$ irreducibly, which is semi-1-connected and is locally 1-avoidable at every point. Then M is a closed 2-manifold.*

5. **r-extendibility.** In close relation to the local properties employed above stands a local "relative" property employed by Čech [g] in his studies on manifolds:

5.1 Definition. A space M will be said to be *r-extendible*[2] at $p \in M$ if for arbitrary open set U containing p there exists an open set V such that $p \in V \Subset U$ and such that if γ^r is any cycle mod $M - U$, then there exists a cycle Z^r on M such that $\gamma^r \sim Z^r$ mod $M - V$.

5.2 Lemma. *If the locally compact M is locally $(r - 1)$-avoidable at $p \in M$, then M is r-extendible at p.*

Proof. Given an open set U containing p, let V and W be as in Definition 4.3. Let γ^r be a cycle mod $M - U$. Then a fortiori γ^r is a cycle mod $M - V$. By Corollary 1.16, there exists a cycle γ_V^r mod $F(V)$ such that $\gamma^r \sim \gamma_V^r$ mod $M - V$, and such that $\partial\gamma^r \sim \partial\gamma_V^r$ on $M - V$. Since $\partial\gamma_V^r$ is on $F(V)$, $\partial\gamma_V^r \sim 0$ on $M - W$. Then $\partial\gamma^r \sim 0$ on $M - W$, and by Lemma 1.6 there exists a cycle Z^r on M such that $\gamma^r \sim Z^r$ mod $M - W$.

5.3 Lemma. *If the locally compact space M is r-extendible and semi-$(r - 1)$-connected at $p \in M$, $r > 0$, then M is locally $(r - 1)$-avoidable at p.*

Proof. Given any open set P containing p, let U be an open set such that $p \in U \Subset P$. We may assume that U is such that every $(r - 1)$-cycle on $F(U)$ bounds on M. Let V be an open set as in Definition 5.1. Let γ^{r-1} be a cycle on $F(U)$. Since $\gamma^{r-1} \sim 0$ on M, there exists by Lemma 1.4 a cycle γ^r mod $F(U)$ such that $\partial\gamma^r \sim \gamma^{r-1}$ on $F(U)$. And by the way V was selected, there exists a cycle Z^r on M such that $Z^r \sim \gamma^r$ mod $M - V$. Then by Lemma 1.2, $\partial\gamma^r \sim \partial Z^r = 0$ on $M - V$. Since $\partial\gamma^r \sim \gamma^{r-1}$ on $M - U$, and a fortiori on $M - V$, we have then that $\gamma^{r-1} \sim 0$ on $M - V$.

Čech defined (in the Princeton notes cited above) an *n-pseudomanifold* as a compact S of dimension n, such that $p^n(S) > 0$ but every n-cycle on a proper closed subset of S bounds on S, and which is n-extendible at every point, and investigated thoroughly the separation properties of such a space. In particular he proved:

5.4 Theorem. *If the compact space M is n-extendible at every point and such that $p^n(M) > 0$ but every n-cycle on a closed proper subset of M bounds on M, then M is lc.*

Proof. (Compare proof of Lemma 4.14.) Let $p \in M$, U an open set containing p, and let V be as in Definition 5.1. Suppose that V is not in one quasi-component of U. Then $U = A \cup B$ separate, where $p \in A$ and $B \cap V \neq 0$. Let γ^n be a nonbounding cycle of M. Then by Corollary 1.16 there exists a cycle γ_A^n mod $F(A)$ such that

[2]The term is our own; Čech used no name for the property.

(5.4a) $$\gamma_A^n \sim \gamma^n \quad \text{mod } M - A.$$

By the way V was chosen, there exists a cycle Z^n such that

(5.4b) $$\gamma_A^n \sim Z^n \quad \text{mod } M - V.$$

Then evidently $Z^n \sim 0 \text{ mod } M - B \cap V$, since γ_A^n is on A. But then by Lemma 1.9 there exists a cycle Γ^n on $M - B \cap V$ such that $Z^n \sim \Gamma^n$. As a cycle on a proper closed subset of M, $\Gamma^n \sim 0$; hence $Z^n \sim 0$. Then (5.4b) implies that $\gamma_A^n \sim 0 \text{ mod } M - V$, and a fortiori mod $M - A$. From (5.4a) we get $\gamma^n \sim 0$ mod $M - A$, which is impossible by Lemma 3.6.

It now follows that:

5.5 Corollary. *If the compact metric space M is* 1*-extendible at every point, and $p^1(M) > 0$ irreducibly, then M is a* 1*-sphere.*

5.6 Corollary. *Let M be a compact metric space such that $p^2(M, \mathfrak{F}) > 0$ irreducibly, which is* 2*-extendible at every point and semi-*1*-connected. Then M is a closed* 2*-manifold.*

[Cf. Theorem 4.23.]

5.7 Theorem. *Let M be a compact space which is minimal closed carrier of exactly m nonbounding n-cycles γ_i^n, all n-cycles on proper closed subsets of M being bounding cycles of M. Let J be a closed subset of M such that* (1) *$M - J$ has a finite number, k, of components, and* (2) *M is n-dimensional and n-extendible at every point of $M - J$. Then $p_n(M; M - J, 0) = mk$.*

Proof. Let D be a component of $M - J$; as k is finite, D is open. Let γ_n^i, $i = 1, \cdots, m$, be cocycles in D such that $\gamma_n^i \cdot \gamma_j^n = \delta_j^i$. Suppose γ^n is any cycle mod $M - D$, and let $p \in D$. As M is n-extendible at p, there exists an open set V such that $p \in V \subset D$ and a cycle Z^n such that $\gamma^n \sim Z^n \text{ mod } M - V$. Since the γ_i^n form a base for M, there exists a relation $Z^n \sim \sum a^i\gamma_i^n$. Hence $\gamma^n \sim \sum a^i\gamma_i^n \text{ mod } M - V$. By Lemma 2.4, D has a maximal open subset P such that $\gamma^n \sim \sum a^i\gamma_i^n \text{ mod } M - P$.

We assert that $P = D$. For suppose not. As D is connected, there exists $x \in \bar{P} \cap (D - P)$ and, by the argument employed above, an open set V' such that $x \in V' \subset D$ and such that $\gamma^n \sim \sum b^i\gamma_i^n \text{ mod } M - V'$. Then $\sum a^i\gamma_i^n \sim \sum b^i\gamma_i^n \text{ mod } M - P \cap V'$ and, by Lemma 3.6, $a^i = b^i$ for all i. However, if we restrict ourselves to coverings that are n-dimensional in D, the homologies $\gamma^n \sim \sum a^i\gamma_i^n$, etc., become identities, so that the identity $\gamma^n = \sum a^i\gamma_i^n$ extends over $P \cup V'$; i.e., $\gamma^n \sim \sum a^i\gamma_i^n \text{ mod } M - (P \cup V')$, contradicting the fact that P was maximal for this homology.

Then $\gamma^n \sim \sum a^i\gamma_i^n \text{ mod } M - D$, and $p^n(M; M, M - D) = p_n(M; D, 0) = m$. Hence cocycles γ_n^i in D such that $\gamma_n^i \cdot \gamma_j^n = \delta_j^i$ form a base for cocycles in D relative to cohomology in D. Now any cocycle in $M - J$ can be decomposed into cocycles in the respective k components of $M - J$, so that the cocycles $\gamma_n^i(D_j)$ of Theorem 3.8 form a base for cohomology in $M - J$; i.e., $p_n(M; M - J, 0) = mk$.

5.8 Corollary. *Under the same hypothesis as in Theorem 5.7, except that M is lc and k may be infinite, the cardinality of $p_n(M; M - J, 0)$ is mk.*

Proof In this case the components of $M - J$ are open and the above proof holds when k is infinite.

Remark. An easy consequence is that if M is n-dimensional and n-extendible at all points, hence lc by Theorem 5.4, then $p_n(M; M - J, 0) = mk$, no matter what the cardinality of k.

5.9 Theorem. *Let M be an n-dimensional compact space such that $p^{n-1}(M) = 0$, which is n-extendible at all points, which carries exactly m ($\geqq 1$) nonbounding n-cycles, and such that all n-cycles on proper closed subsets of M bound on M. Let J be any closed subset of M. Then if the number of components of $M - J$ is a finite number k,*

$$p^{n-1}(J) = m(k - 1). \tag{5.9a}$$

Furthermore, if either $p^{n-1}(J)$ or k is infinite, then both are infinite.

Proof. By Theorem 3.10, $p_n(M; M - J, 0) \geqq p^{n-1}(J) + m$, so that if $p^{n-1}(J)$ is infinite, so too is $p_n(M; M - J, 0)$; and hence in this case k is infinite by Corollary 5.8. Conversely, if k is infinite, $p^{n-1}(J)$ is infinite by Theorem 3.9.

On the other hand, if k is finite, $p^{n-1}(J) \geqq m(k - 1)$ by Theorem 3.9. But also, since by Theorem 5.7, $p_n(M; M - J, 0) = mk$, and by Theorem 3.10, $p_n(M; M - J, 0) \geqq p^{n-1}(J) + m$, we have that $p^{n-1}(J) \leqq mk - m = m(k - 1)$.

Remark. In such theorems as 5.9 above which hypothesize that (1) the space under consideration carries exactly m nonbounding cycles, all n-cycles on proper closed subsets of M being bounding cycles of M—i.e., that M is irreducible relative to carrying a cycle nonbounding on M and $p^n(M)$ is finite—and (2) M is n-extendible at all points, the condition (2) may be replaced by the condition that $p^n(x) \leqq m$ for all $x \in M$:

5.10 Theorem. *If M is a compact space such that $p^n(M) = m$ finite, all n-cycles on proper closed sets of M bound on M, and x is a point of M such that $p^n(x) \leqq m$,*[3] *then M is n-extendible at x.*

Proof. If U is any open set containing x, then U contains open sets V and W such that $x \in W \Subset V$ and $p^n(x; V, W) = p^n(x)$. Let Γ_i^n, $i = 1, \cdots, m$, constitute a base for n-cycles of M relative to homologies on M, and suppose γ^n is a cycle mod $M - U$. By Lemma 3.7, the cycles Γ_i^n are lirh mod $M - W$, and may, since $p^n(x) \leqq m$, be taken as a base for n-cycles relative to homologies mod $M - W$. Consequently $\gamma^n \sim \sum a^i\Gamma_i^n$, $a^i \in \mathfrak{F}$, mod $S - W$.

As a consequence of Theorems 5.4 and 5.10 we have:

---

[3]Evidently, by virtue of Lemma 3.7, $p^n(x)$ will have to be exactly equal to m under these conditions.

5.11 THEOREM. *If M is a compact space such that $p^n(M) = m$ finite, all n-cycles on closed proper subsets of M being bounding cycles of M, and $p_n(x) \leqq m$ for all $x \in M$, then M is lc.*

6. Non-cut points and avoidable points. In the last section we introduced the notions of non-r-cut point and of r-avoidable and locally r-avoidable points. In the present section we propose to inquire further into these notions as well as to introduce some important related notions.

First, let us consider the 0-dimensional cases of the above-mentioned concepts. For example, what is the relation between *non-cut point* as defined in I 5.11, and *non-0-cut point*? In defining the former, we stated that it was to be assumed that the space, or set, M, under consideration is connected, and that a non-cut point p was then to be characterized by the fact that the set $M - p$ is connected. On the other hand, a non-0-cut point is a point p such that every 0-cycle on a compact subset of $M - p$ bounds on a compact subset of $M - p$.

6.1 Let us define a set M to be *strongly connected* if it has the following property: If $x, y \in M$, then there exists a continuum K such that $x, y \in K \subset M$. For example, as a result of Theorem IV 3.1, the domains of a locally compact, lc space are strongly connected. We now prove:

6.2 THEOREM. *If p is a non-0-cut point of a point set M, then $M - p$ is strongly connected; but the converse does not in general hold (even when M is a metric continuum).*

PROOF. If p is a non-0-cut point of M and $x, y \in M - p$, then by Theorem V 11.5 there exists a nontrivial cycle γ^0 on $x \cup y$; and if $\gamma^0 \sim 0$ on a compact subset of $S - p$, then by Corollary V 11.11, x and y lie in a continuum in $M - p$.

To see that the converse does not generally hold, we note first the following lemma:

6.3 LEMMA. *If A is a compact subset of an open subset P of a compact metric space M such that every 0-cycle on A bounds on a compact subset of P, then A lies in a subcontinuum of P.*

PROOF. By Theorem 4.5, there exists an open set U containing $M - P$ such that every 0-cycle on A bounds on $M - U$. Let $x, y \in A$. Then there exists a nontrivial cycle γ^0 on $x \cup y$, and γ^0 bounds on a compact subset of $M - U$. Consequently by Corollary V 11.11, x and y lie in a subcontinuum of $M - U$. Hence, if C is the component of $M - U$ determined by a fixed point p of A, C is a continuum since $M - U$ is compact, and $C \supset A$.

6.4 To complete the proof of Theorem 6.2, consider the following example in the polar coordinate plane: Let $M_n = \{(\rho, \theta) \mid (0 \leqq \rho \leqq 1) \& (\theta = \pi/4^n)\}$, $M_0 = \{(\rho, \theta) \mid (0 \leqq \rho \leqq 1) \& (\theta = 0)\}$, $K = \{(\rho, \theta) \mid (\rho = \tan \theta) \& (0 \leqq \theta \leqq \pi/4)\}$, and let T be the set of points on some arc which joins the point $(1, 0)$ to the point $(1, \pi/4)$, but which otherwise fails to meet any of the sets M_n or K. Let

$M = T \cup K \cup \bigcup_{n=0}^{\infty} M_n$, and $p = (0, 0)$. Then $M - p$ is strongly connected. However, p is a 0-cut point of M. For consider the set A consisting of $(1, 0)$ and all points of the form $(1, \pi/4^n)$. If every 0-cycle on A bounds on a compact subset of $M - p$, then by Lemma 6.3, there exists a continuum in $M - p$ which contains A. Clearly no such continuum exists, and consequently there must be a 0-cycle on A which fails to bound on a compact subset of $M - p$. (It is an instructive exercise to find this cycle.)

The space of the above example, it will be noticed, is not lc. As a matter of fact, we can prove:

6.5 THEOREM. *If M is a locally compact, lc space, and p is a non-cut point of M, then p is a non-0-cut point of M.*

PROOF. Let γ^0 be a cycle on a compact subset K of $M - p$. By Theorem IV 3.3, K lies in a subcontinuum C of $M - p$. Hence by Theorem V 11.2 $\gamma^0 \sim 0$ on $C \subset M - p$.

As a corollary of Theorems 6.2 and 6.5 we have:

6.6 COROLLARY. *Among the locally compact, lc spaces, the non-cut points and non-0-cut points are identical.*

6.7 REMARK. In view of Corollary 6.6 and Theorem I 12.15 it follows that *every Peano continuum has at least two non-0-cut points.*

Turning to the notions of r-avoidability and local r-avoidability, we give the following theorems, provable by methods used above:

6.8 THEOREM. *In order that a point p of a compact space M should be a 0-avoidable point of M, it is necessary and sufficient that if U is an open set containing p, there exist an open set V such that $p \in V \Subset U$ and such that $F(U)$ lies in a subcontinuum of $M - V$.*

6.9 THEOREM. *In order that a point p of a compact space M should be a locally 0-avoidable point of M it is necessary and sufficient that if U is an open set containing p, there exist open sets V and W such that $p \in W \Subset V \Subset U$ such that $F(V)$ lies in a subcontinuum of $M - W$.*

As a corollary of Corollary 4.7 and Theorem 6.8 we have:

6.10 COROLLARY. *If M is a compact metric space and p is a non-0-cut point of M, then for any open set U containing p, there exists an open set V such that $p \in V \Subset U$ and such that $F(U)$ lies in a subcontinuum of $M - V$.*

Next, let us consider the following definitions:

6.11 DEFINITION. A space S will be said to have $p \in S$ as a *local non-r-cut point* (and p will be called a local non-r-cut point of S) if for arbitrary open set U containing p there exists an open set V such that $p \in V \Subset U$ such that every r-cycle on a compact subset of $V - p$ bounds on a compact subset of $U - p$.

6.12 DEFINITION. A space S will be called *completely r-avoidable* at $p \in S$ (and p will be called a completely r-avoidable point of S) if for arbitrary open set U containing p there exist open sets V and W such that $p \in W \Subset V \Subset U$ and such that every r-cycle on $F(V)$ bounds on $\overline{U} - W$.

6.13 The similarity between Definition 6.11 and the definition of "r-lc at p" suggests that the relation between the two be investigated. Consider the example, in the cartesian plane, of the set $M = \bigcup_{n=1}^{\infty} M_n$, where $M_n = \{(x, y) \mid (x - 1/n)^2 + y^2 = 1/n^2\}$. Let $p = (0, 0)$. Then p is a local non-1-cut point of M, but M is not 1-lc at p. And if S is the euclidean plane and $p \in S$, then S is 1-lc at p, but p is not a local non-1-cut point of S.

Let us consider the case $r = 0$ of Definition 6.11, however. In the first place, it is easy to prove that:

6.14 THEOREM. *If p is a local non-0-cut point of a space S, and U is an open set containing p, then there exists an open set V such that $p \in V \Subset U$ and such that if $x, y \in V - p$, then there exists a continuum K such that $x, y \in K \subset U - p$.*

The property stated in the conclusion of Theorem 6.14 is precisely that which, when stated in metric terms, Urysohn used [d; §3, p. 301] to define "avoidable" (*vermeidbar*) point of a metric continuum.

As a matter of fact, we may prove:

6.15 THEOREM. *A necessary and sufficient condition that a point p of a locally compact, lc space S should be an "avoidable" point in Urysohn's sense is that p be a local non-0-cut point of S.*

PROOF. The sufficiency follows from Theorem 6.14. To prove the necessity, let U be an open set containing p, and let V be an open set such that $p \in V \Subset U$, and such that if $x, y \in V - p$, then there exists a continuum C such that $x, y \in C \subset U - p$. Let γ^0 be a cycle on a compact subset K of $V - p$. Then by Corollary IV 3.4, K lies in a finite number of continua $C_1, \cdots, C_k$ of $V - p$, which (see proof of Theorem 6.5) lie in a subcontinuum C of $U - p$. Hence γ^0 bounds on the compact subset C of $U - p$ and p is therefore a local non-0-cut point of S.

Urysohn showed that if p is an "avoidable" point of a metric continuum, then M is lc at p. Inasmuch as the natural extension of the "continuum" condition in higher dimensions is semi-r-connectedness, we might consider the implications of the local non-r-cut point condition in semi-r-connected spaces.

6.16 THEOREM. *If a locally compact metric space S is semi-r-connected and p is a local non-r-cut point of S, then S is r-lc at p.*

Theorem 6.16 is a consequence of the following two theorems:

6.17 THEOREM. *If M is a locally compact metric space and p is a local non-r-cut point of M, then M is completely r-avoidable at p.*

(This is a consequence of Theorem 4.5.)

6.18. THEOREM. *If a locally compact space S is completely r-avoidable at $p \in S$ and semi-r-connected at p, then S is r-lc at p.*

(This is a consequence of Theorem 2.24.)

From Theorem 6.17 and Lemma 6.3 we have:

6.19 THEOREM. *If p is a local non-0-cut point of a locally compact metric space M, then for every open set U containing p there exist open sets V and W such that $p \in W \Subset V \Subset U$ and such that all points of $F(V)$ lie in a subcontinuum of $\overline{U} - W$.*

6.20 COROLLARY. *If p is a local non-0-cut point of a locally compact metric space M and $\epsilon > 0$ is arbitrary, then there exists a continuum K in $S(p, \epsilon)$ such that $M - K = A \cup B$ separate, where $p \in A \subset S(p, \epsilon)$.* (Compare Urysohn, loc. cit.)

Now Theorem 6.17 has a converse of the following form:

6.21 THEOREM. *If the locally compact space M is semi-r-connected and completely r-avoidable at p, then p is a local non-r-cut point of M.*

PROOF. Let U be any open set containing p. Then there exists open sets V and W such that $p \in W \Subset V \Subset U$ and such that every r-cycle on $F(V)$ bounds on $\overline{U} - W$. And since M is semi-r-connected at p we may assume that W is such that all r-cycles on W bound on M.

Consider any cycle γ^r on a compact subset K of $W - p$. Let U' be an open set such that $p \in U' \Subset W - K$, and let V', W' be open sets such that $p \in W' \Subset V' \Subset U'$ and such that all r-cycles on $F(V')$ bound on $\overline{U'} - W'$.

Now $\gamma^r \sim 0$ on M, and if we let $L = \overline{V'} \cup (M - V)$, there exists by Lemma 1.13 a cycle Z^r on $F(L) = F(V') \cup F(V)$ such that

$$\text{(6.21a)} \qquad \gamma^r \sim Z^r \qquad \text{on } \overline{V} - V'; \text{ a fortiori on } \overline{U} - W'.$$

Now $Z^r = Z_1^r + Z_2^r$, where Z_1^r and Z_2^r are cycles on $F(V')$ and $F(V)$ respectively; and $Z_1^r \sim 0$ on $\overline{U'} - W'$, $Z_2^r \sim 0$ on $\overline{U} - W$. Consequently

$$\text{(6.21b)} \qquad Z^r = Z_1^r + Z_2^r \sim 0 \text{ on } \overline{U} - W'$$

and relations (6.21a) and (6.21b) imply that $\gamma^r \sim 0$ on $\overline{U} - W'$. Hence we have proved that every cycle on a compact subset of $W - p$ bounds on a compact subset of $\overline{U} - p$, and p is a local non-r-cut point of M.

6.22 COROLLARY. *If M is a locally compact, connected space, and p is a completely 0-avoidable point of M, then p is a local non-0-cut point of M.*

And as a corollary of Theorems 6.16 and 6.21 we have:

6.23 THEOREM. *In the locally compact, semi-r-connected metric spaces, the local non-r-cut points and the completely r-avoidable points are identical.*

6.24 THEOREM. *In order that a point p of a locally compact, connected metric space should be a local non-0-cut point, it is necessary and sufficient that for every $\epsilon > 0$ the point p be ϵ-separatable*[4] *by means of a continuum.*[5]

PROOF. The necessity follows from Corollary 6.20 and the sufficiency proceeds as follows: Given ϵ, let K be a continuum in $S(p, \epsilon)$ such that $M - K = A \cup B$ separate, where $p \in A \subset S(p, \epsilon)$. Then if W is an open set such that $p \in W \Subset A$, we have that all 0-cycles on $F(A)$ bound on $K \subset S(p, \epsilon) - W$; i.e., p is completely 0-avoidable. Then by virtue of Corollary 6.22, p is a local non-0-cut point of M.

6.25 Now the above avoidability and non-cut notions may be modified by allowing in each case a finite number of "exceptions to the rule;" for example, in the case of r-avoidability, we may define a space S to be *almost r-avoidable* at $p \in S$ if for arbitrary open set U containing p there exists an open set V such that $p \in V \Subset U$ and such that only a finite number of r-cycles on $F(U)$ are lirh on $S - V$. *Almost locally r-avoidable* and *almost completely r-avoidable* are defined by analogous alteration of the corresponding definitions above. And in the case of non-cut points, we define a point p of S to be *almost a non-r-cut point* of S if there exist at most a finite number of r-cycles on compact subsets of $S - p$ that are lirh on compact subsets of $S - p$; *almost a local non-r-cut point* is defined by analogous change of the corresponding definition above.

We may then state theorems for the "almost" categories similar to the theorems above. Not all of the expected analogies hold, however, so that we shall state precisely those which do. In doing so, we shall use the preceding enumeration followed by an upper index "a"; thus, the analogue of Theorem 2.6 is designated below by 2.6^a.

4.7^a, 4.8^a, 4.10^a THEOREM. *If M is a compact metric space and p is almost a non-r-cut point of M, then p is an almost r-avoidable point of M. In any space, if p is an almost r-avoidable point, then it is an almost locally r-avoidable point. And in a simply r-connected, locally compact space, an almost locally r-avoidable point is almost r-avoidable.*

PROOF. The first sentence is a corollary of Theorem 4.6. The second is obvious. To prove the third, let p be an almost locally r-avoidable point of a simply r-connected space S, and let U be an open set containing p. There exist open sets V and W such that $p \in W \Subset V \Subset U$ and such that for some number k, every k r-cycles on $F(V)$ satisfy a homology in $S - W$. Let $\gamma_1^r, \cdots, \gamma_k^r$ be cycles on $F(U)$. As S is simply r-connected, each $\gamma_i^r \sim 0$ on S, $i = 1, \cdots, k$. Hence by Lemma 1.13 there exists a cycle Z_i^r on $F(V)$ such that $\gamma_i^r \sim$

[4]A point p is called ϵ-separatable by a set K in a metric space S if $S - K = A \cup B$ separate, where $p \in A \subset S(p, \epsilon)$.

[5]Theorem 6.24 was proved by Urysohn (loc. cit. Satz V) for an lc continuum using "avoidable point" instead of "local non-0-cut point." In view of Theorem 6.15, Urysohn's theorem follows from Theorem 6.24.

Z_i^r on $M - V$, and since the cycles Z_i^r are related by a homology in $M - W$, the same holds for the cycles γ_i^r.

4.11[a] Corollary. *In a simply r-connected, locally compact space, the almost r-avoidable points and the almost locally r-avoidable points are identical.*

4.12[a] Corollary. *In any continuum, the almost* 0*-avoidable points and the almost locally* 0*-avoidable points are identical.*

6.26 Remark. It will be noted that whereas in Lemma 4.10 it was proved that in a simply r-connected compact space, a locally r-avoidable point is a non-r-cut point, we have not stated in 4.10[a] that in a like space an almost locally r-avoidable point is almost a non-r-cut point. Such a statement would not hold, as the following example shows: In the polar coordinate plane, let $M_n = \{(\rho, \theta) \mid (0 \leqq \rho \leqq 1/n)\&(\theta = \pi/n)\}$, and let $M = \bigcup_{n=1}^{\infty} M_n$. Then if $p = (0, 0)$, the compact metric space M is a continuum and therefore simply 0-connected, and is almost locally 0-avoidable at p. However, $M - p$ has infinitely many lirh nontrivial 0-cycles.

6.2[a] Theorem. *If p is almost a non-*0*-cut point of a point set M, then $M - p$ has only a finite number of constituants.* [See Index.]

Proof. By hypothesis there exists a finite set of 0-cycles, $\gamma_1^0, \cdots, \gamma_k^0$ on a compact subset K of $M - p$ such that every cycle γ^0 on a compact subset of $M - p$ is homologous on a compact subset of $M - p$ to a linear combination of the cycles γ_i^0. Now suppose there exist points $x_1, x_2, \cdots, x_n, \cdots$ of $M - p$ such that no pair x_i, x_j, $i < j$, are joined by a subcontinuum of $M - p$. For each pair x_1, x_n, $n > 1$, there exists a nontrivial cycle γ_n^0 on $x_1 \cup x_n$, and by hypothesis there exists a relation

$$\gamma_n^0 \sim \sum_{i=1}^{k} a_n^i \gamma_i^0 = \Gamma_n^0, \qquad a^i \in \mathfrak{F}, \qquad \text{on} \qquad K_n, \tag{6.2a}$$

where K_n is a compact subset of $M - p$.

Now at most k of the right-hand members of relations (6.2a) are linearly independent in the algebraic sense, and therefore there exists a relation

$$\sum_{n=1}^{k+1} b^n \Gamma_n^0 = 0, \qquad \text{not all } b^n = 0. \tag{6.2b}$$

Assuming $b^1 \neq 0$, we get that

$$\Gamma_1^0 = \sum_{n=2}^{k+1} c^n \Gamma_n^0. \tag{6.2c}$$

From relations (6.2a) and (6.2c) it follows that $\gamma_1^0 \sim \sum_{n=2}^{k+1} c^n \gamma_n^0$ on $\bigcup_{n=1}^{k+1} K_n$, that is, there exists a homology of the form $\sum_{n=1}^{k+1} d^n \gamma_n^0 \sim 0$ on $\bigcup_{n=1}^{k+1} K_n$. Now $\bigcup_{n=1}^{k+1} K_n$ is compact, hence its components are continua and a fortiori constituants. That no linear form such as $\sum_{n=1}^{k+1} d^n \gamma_n^0$ can bound on $\bigcup_{n=1}^{k+1} K_n$ may be shown by methods similar to those employed in V 11.

6.5[a] Theorem. *If M is a locally compact, lc space, and p is a point of M such that $M - p$ has only a finite number of components, then p is almost a non-0-cut point of M.*

Proof. Suppose $M - p$ has k components. Let $\gamma_1^0, \cdots, \gamma_k^0$ be k 0-cycles on a compact subset K of $M - p$. By Corollary IV 3.4, we may assume that K has at most k components, and by Theorem V 11.3a, $p^0(K, \mathfrak{F}) \leqq k - 1$. Hence there exists a relation $\sum_{i=1}^{k} a^i\gamma_i^0 \sim 0$ on K.

6.6[a] Corollary. *If M is a locally compact, lc space, then in order that a point p of M should be almost a non-0-cut point of M, it is necessary and sufficient that $M - p$ have only a finite number of components.*

6.8[a] Theorem. *In order that a point p of a compact space M should be an almost 0-avoidable point of M, it is necessary and sufficient that if U is an open set containing p, there exists an open set V such that $p \in V \Subset U$ and $F(U)$ lies in a finite number of subcontinua of $M - V$.*

6.27 Remark. It follows from Theorem 6.8a that an almost 0-avoidable point of a continuum M is a point at which M is semi-locally-connected in the sense of Whyburn; and conversely.

(Cf. G. T. Whyburn [Wh, p. 19]. Whyburn defines a metric, connected set M to be semi-locally-connected at $x \in M$ if for arbitrary $\epsilon > 0$ there exists a neighborhood V of x of diameter $< \epsilon$ such that $M - V$ has only a finite number of components. It follows easily from the conclusion of Theorem 6.8[a] that the open set V, augmented by components of $M - V$ that do not meet $F(U)$, is an open set $V' \subset U$ such that $M - V'$ is a finite number of connected sets.)

6.9[a] Theorem. *In order that a point p of a compact space M should be an almost locally 0-avoidable point of M it is necessary and sufficient that if U is any open set containing p then there exist open sets V and W such that $p \in W \Subset V \Subset U$ and such that $F(V)$ lies in a finite number of subcontinua of $M - W$.*

6.17[a] Theorem. *If M is a locally compact metric space and p is almost a local non-r-cut point of M, then M is almost completely r-avoidable at p.*

Remark. The analogue of Theorem 6.21 does not hold; see the example of 6.26.

6.18[a] Lemma. *If a locally compact space S is semi-r-connected at $p \in S$ as well as almost completely r-avoidable at p, then S is r-lc at p.*

[This is a direct consequence of Theorem 2.24.]

6.18[b] Corollary. *If a locally compact connected space S is almost completely 0-avoidable at $p \in S$, then S is lc at p.*

6.28 Theorem. *In order that a locally compact space S should be lc^n it is necessary and sufficient that S be semi-r-connected and almost completely r-avoidable at all points for $r = 0, 1, \cdots, n$.*

[The necessity follows from Corollary VI 3.8.]

6.16[a] Theorem. *If a locally compact metric space M is semi-r-connected at $p \in S$ and p is almost a local non-r-cut point of M, then S is r-lc at p.*

[This is a consequence of Theorem 6.17[a] and Lemma 6.18[a].]

6.16[b] Corollary. *If p is almost a local non-0-cut point of a locally compact, metric, connected space S, then S is lc at p.*

6.24[a] Theorem. *In order that a point p of a locally compact, metric space M should be almost a local non-0-cut point of M, it is necessary and sufficient that for every open set U containing p there exist open sets V and W such that $p \in W \Subset \overline{V} \Subset U$ and such that all points of $F(V)$ lie in a finite number of subcontinua of $\overline{U} - W$; or, in other words, that for every $\epsilon > 0$, p be ϵ-separatable by finitely many continua.*

[The necessity follows from Theorem 6.17[a] and Theorem V 11.10; the sufficiency follows as in Theorem 6.24.]

6.24[b] Corollary. *If the point p of the locally compact, metric, connected space M is ϵ-separatable by a finite number of continua for every ϵ, then M is lc at p.*

[The proof uses Corollary 6.16[b] and Theorem 6.24[a].]

6.29 Theorem. *In order that a locally compact, connected space S should be lc it is necessary and sufficient that for every $p \in S$ and open set U containing p, there exist a finite set K of continua in $U - p$ such that $S - K = A \cup B$ separate, where $p \in A \Subset U$.* (Compare Urysohn, loc. cit., Satz I.)

The necessity follows from Corollary IV 3.4: Select V and W such that $p \in W \Subset \overline{V} \Subset U$, and let $F(V) = K$; then K lies in a finite number of continua of $U - \overline{W}$. The sufficiency follows from Theorem 2.24: If S is not lc at p, select $U = P$ as defined in that theorem, and K as in the sufficiency condition above. Then let W be an open set such that $p \in W \Subset A$. Then only finitely many 0-cycles on $F(A)$ ($\subset K$) are lirh on K ($\subset U - \overline{W}$).

6.30 Corollary. *In order that a locally compact, metric, connected space S should be lc at $p \in S$, it is necessary and sufficient that p be almost a local non-0-cut point of S.*

[This is a corollary of Theorems 6.24[a] and 6.29.]

7. **Property** S_n. In IV 3.6 we introduced the notion of property S, and showed that for compact spaces it is equivalent to the lc property. We now propose to define a property which we call S_n, which for $n = 0$ reduces to property S, and which plays an analogous role for the higher-dimensional local connectedness. (The position of the index n will serve to distinguish property S_n from the n-sphere S^n. Furthermore, S_n is used only with the word "property.")

7.1 Definition. By a *pair* (U, V) we mean an ordered pair of point sets

U and V such that $U \supset V$. By $h^n(U, V)$ we denote the maximum number of n-cycles on compact subsets of V that are lirh on compact subsets of U. A set of pairs $\{(U, V)\}$ will be said to cover a point set M if for each $x \in M$ there exists at least one pair in the set such that $x \in V$.

7.2 Definition. If $\mathcal{P}$ is a set of pairs (U, V), then by $\bigcup_U \mathcal{P}$ we denote the set $\bigcup U$, $U \in (U, V) \in \mathcal{P}$; and by $\bigcup_V \mathcal{P}$ we denote the set $\bigcup V$, $V \in (U, V) \in \mathcal{P}$. The number $h^n(\bigcup_U \mathcal{P}, \bigcup_V \mathcal{P})$ we abbreviate to $h^n(\mathcal{P})$. A point set M will be said to be the *union* $\bigcup \mathcal{P}$ of a set $\mathcal{P}$ of pairs (U, V) if $M = \bigcup_U \mathcal{P} = \bigcup_V \mathcal{P}$.

7.3 Definition. If $\mathfrak{C}$ is any covering of the space under consideration, then a pair (U, V) will be said to be of diameter $< \mathfrak{C}$ if U is of diameter $< \mathfrak{C}$.

7.4 Definition. A point set M in a space S will be said to have *property* S_n if given an arbitrary covering $\mathfrak{C}$ of S, the set M is the union of a finite set $\mathcal{P}$ of pairs of diameter $< \mathfrak{C}$, and such that for every subset $\mathcal{P}'$ of $\mathcal{P}$ the number $h^n(\mathcal{P}')$ is finite. In particular, then, *a set cannot have property* S_n *unless its n-dimensional Betti number is finite.*

7.5 Remark. In order to prevent confusion with closely related properties to be introduced later on, it seems advisable to emphasize here that in the determination of the numbers h^n (Definition 7.1) only cycles and chains on compact carriers of the sets U, V, etc., are considered. It may be objected that in the case where the set in question is not only noncompact, but has few compact subsets of any significance, then property S_n can be of little use. The answer to this objection is that (1) the most significant applications of the property are either to compact spaces or to open subsets thereof; and that (2) later we shall find it convenient to drop the requirement of compact carriers altogether, in which case the objection loses force entirely.

7.6 It will also be desirable at times to restrict the group of cycles under consideration; we have already done this sort of thing in §3, when defining the group $G^n(M; J, 0)$. For example, *property* S_n *rel.* G, where G is a special group of n-cycles, indicates that in the determination of the numbers h^n only cycles of G are employed. In the next section we shall make similar conventions in regard to avoidability properties.

We first give some "justification" theorems.

7.7 Theorem. *If a point set M has property* S_0, *then it has property* S.

Proof. Let $\mathfrak{C}$ be an arbitrary covering of space. Then M is the union of a finite set $\mathcal{P}$ of pairs $(U, V)_i$ each of which is of diameter $< \mathfrak{C}$ and such that for every subset $\mathcal{P}'$ of $\mathcal{P}$ the number $h^0(\mathcal{P}')$ is finite. In particular, each $h^0(U, V)_i$ is finite, so that by Theorem V 11.10, V lies in a finite number m_i of components of U. Since $M = \bigcup_V \mathcal{P}$, it follows that M is the union of at most $\sum_{i=1}^k m_i$ connected sets of diameter $< \mathfrak{C}$, where k is the number of elements in $\mathcal{P}$.

7.8 Theorem. *If an open subset M of a locally compact space has property* S, *then M has property* S_0.

PROOF. Let $\mathfrak{C}$ be any covering of space. By Theorem IV 4.5 M is the union of a finite collection of domains, M_i, of M of diameter $< \mathfrak{C}$. For each i, let $(U, V)_i$ denote a pair such that $U = V = M_i$, and let $\mathcal{P}$ be the set of all such pairs. Let $\mathcal{P}' \subset \mathcal{P}$; then $\bigcup_U \mathcal{P}' = \bigcup_V \mathcal{P}'$ is the union of disjoint domains $D_1, \cdots, D_k$, finite in number. By Theorem IV 4.2, M is lc. Hence by Corollary IV 3.4, if K is any compact subset of $D = \bigcup_{i=1}^k D_i$, then K is contained in the union of at most k continua. It follows that $h^0(\mathcal{P}') \leqq k$.

We now state a most important theorem, embodying the definition of a property which is equivalent to property S_r:

7.9 THEOREM. *Let S be a compact space and M a subset of S. Then in order that M have property S_r rel. G^r, where G^r is any group of cycles of M, it is necessary and sufficient that if P and Q are open subsets of S such that $P \Supset Q$, then at most a finite number of cycles of G^r on compact subsets of $M \cap Q$ are lirh on compact subsets of $M \cap P$.*

PROOF. To prove the necessity, let P' be an open set such that $S - P \subset P' \Subset S - \bar{Q}$, and let $\mathfrak{C}$ denote the covering of S consisting of the open sets P, P'. Since M has property S_r rel. G^r, there exists a finite set of pairs $\{P_i, Q_i\}$ of diameter $< \mathfrak{C}$ such that $M = \bigcup P_i = \bigcup Q_i$, etc. Denote the union of those sets Q_i which meet $\bar{Q}$ by V, and the union of the corresponding sets P_i by U. Then $M \cap Q \subset V \subset U \subset M \cap P$, and at most a finite number of cycles of G^r on compact subsets of V are lirh on compact subsets of U. A fortiori, at most a finite number of cycles of G^r on compact subsets of $M \cap Q$ are lirh on compact subsets of $M \cap P$.

To prove the sufficiency, let $\mathfrak{C}$ be an arbitrary covering of S formed by a finite collection of open sets U_i, $i = 1, 2, \cdots, k$. By Lemma V 8.2, $\mathfrak{C}$ has a closure refinement, $\mathfrak{D}$, consisting of sets V_i such that $U_i \Supset V_i$. Denote those elements of $\mathfrak{D}$ that meet M by $V_{i(j)}$, $j = 1, \cdots, h$. Let $P_j = M \cap U_{i(j)}$, $Q_j = M \cap V_{i(j)}$ and consider the set of pairs $\{P_j, Q_j\}$. Obviously their union is M. And if V' is the union of some of the sets Q_j, and U' the union of the corresponding sets P_j, then the respective unions V, U of the corresponding sets $V_{i(j)}$, $U_{i(j)}$ form open subsets of S such that $V \Subset U$, and consequently at most a finite number of cycles of G^r on compact subsets of $M \cap V$ are lirh on compact subsets of $M \cap U$—implying that at most a finite number of cycles of G^r on compact subsets of V' are lirh on compact subsets of U'. Thus M has property S_r rel. G^r.

An almost identical argument also shows:

7.10 THEOREM. *Let S be a compact space and M a subset of S. Then in order that M have property S_r rel. G^r, where G^r is any group of cycles, it is necessary and sufficient that if P and F are an open and a closed subset, respectively, of S such that $P \supset F$, then at most a finite number of cycles of G^r on compact subsets of $M \cap F$ are lirh on compact subsets of $M \cap P$.*

An interesting corollary of Theorem 7.10 (cf. also Theorems V 11.10, and 7.7, 7.8 above) is the following:

7.11 Corollary. *Let M be a subcontinuum of a compact space S. Then in order for M to have property* S, *it is necessary and sufficient that if P and F are respectively open and closed subsets of S such that $P \supset F$, then $M \cap F$ is contained in a finite number of components of $M \cap P$.*

The property which is proved equivalent to property S_r rel. G^r in Theorem 7.9 is strictly analogous to property $(P, Q)_n$ of VI 7.1, the only difference being that in the present case, cycles instead of cocycles are employed. Because of the importance of the property, we introduce a special term for it:

7.12 Definition. A subset M of a space S will be said to have *property* $(P, Q)^n$ rel. G^n, where G^n is a set of n-cycles, if for every pair P, Q of open subsets of S such that $P \supset Q$, $\bar{Q}$ compact, at most a finite number of cycles of G^n on compact subsets of $M \cap Q$ are lirh on compact subsets of $M \cap P$. [If $G^n = Z^n(M; \mathfrak{F})$ the "rel. G^n" may be omitted.]

By Theorem 7.9, then, a subset of a compact space has property S_r rel. G^r if and only if it has property $(P, Q)^r$ rel. G^r.

Now in regard to the relationship between property S_n and the n-lc property, we have:

7.13 Theorem. *If a closed subset M of a compact space S has property S_n, then M is n-lc; but the converse does not necessarily hold if $n > 0$.* (Cf. Corollary IV 3.8.)

Proof. The first part of the theorem follows from Theorem 7.10 and VI 6.14. To see that a compact space may have property 1-lc, for instance, and not property S_1, consider the following example:

7.14 Example. In the coordinate plane let p_n denote the point $(1/n, 0)$, $n = 1, 2, 3, \cdots$. Let K_n be the set of all points on an ellipse with center p_n, major axis parallel to the y-axis and length 2, and minor axis of length less than $\rho(p_n, p_{n+1})/2$. Let K_0 denote the set of all points $(0, y)$ such that $-1 \leqq y \leqq 1$. Then the point set $M = \bigcup_{n=0}^{\infty} K_n$ is compact and locally 1-connected, but does not have property S_1; in particular, $p^1(M)$ is infinite.

Consider also the following example:

7.15 Example. In cartesian 3-space let $K_n = \{(x, y, z) \mid (0 \leqq x \leqq 1) \& (y = 1/n) \& (0 \leqq z \leqq 1)\}$, for $n = 1, 2, 3, \cdots$. Let K_0 be the surface, excepting the base in the xy-plane, of the unit cube of whose faces three lie in the coordinate planes and of which $(1, 1, 1)$ is a vertex. Let $M = \bigcup_{n=0}^{\infty} K_n$. Here again M is compact and 1-lc, and does not have property S_1. In this case, however, $p^1(M) = 0$. If $\mathcal{P}$ were a set of pairs of diameters less than 1/2, whose union is M, and $\mathcal{P}'$ were that subset of $\mathcal{P}$ consisting only of those pairs containing points for which $z = 0$, then $h^1(\mathcal{P}')$ would be infinite.

It will be noted in each of the above examples that the set M is not 0-lc. We found in Theorem IV 3.7 that the properties S (hence S_0) and 0-lc are equivalent for compact spaces. What is the situation as regards the lc^n property and the possession of property S_i for all values of i from 0 to n?

7.16 Definition. A set M will be said to have property S_j^k, $j \leqq k$, if it has property S_r for all r such that $j \leqq r \leqq k$.

7.17 Theorem. *In order that a compact space M should be lc^n, it is necessary and sufficient that it have property S_0^n.*

The necessity follows from Theorem VI 6.16 and Theorem 7.10, and the sufficiency from Theorem 7.13.

Theorem 7.17, which is the natural generalization of Corollary IV 3.8, is also the generalization of the Sierpinski theorem characterizing Peano continua (Corollary IV 3.9).

In Remark IV 4.3 it was pointed out that property S is stronger than the lc property, for nonclosed subsets of a compact space. Likewise, property S_0^n is stronger than lc^n for such sets. We shall see later, for example (Corollary XI 2.22), that the boundary of a domain D in S^n complementary to a continuum is peanian if D has property S_0^{n-2}; generally this is not the case, although an open subset of S^n is always lc^{n-2}.

8. Set-avoidability. This section deals with the obvious generalization of the notions of avoidable point and non-cut point, wherein we replace "point" by "closed set". The generalization yields important connections with the local connectedness and property S_n concepts. We are interested particularly in the almost local avoidability and almost complete avoidability generalizations: For example, if K is a closed subset of a space S, then we say that K is almost locally r-avoidable if, given an open set $U \supset K$, there exist open sets V, W such that $K \subset W \Subset V \Subset U$ and such that at most a finite number of r-cycles of $F(V)$ are lirh on $S - W$.

8.1 Note first that the *points* of a compact space may be almost locally r-avoidable without the *closed sets* being almost locally r-avoidable. For example, in the cartesian plane let $M_n = \{(x, y) \mid (x = 1/n)\&(0 \leqq y \leqq 1)\}$, $M_0 = \{(x, y) \mid (x = 0)\&(0 \leqq y \leqq 1)\}$, $A = \{(x, y) \mid (0 \leqq x \leqq 1)\&(y = 1)\}$, $B = \{(x, y) \mid (0 \leqq x \leqq 1)\&(y = 0)\}$. Let $S = \bigcup_{n=0}^{\infty} M_n \cup A \cup B$, and let $a = (0, 1)$, $b = (0, 0)$. With the cartesian metric, S is a compact metric space, and is almost locally 0-avoidable at all points (this can be seen by trial, but will follow as a result of theorems to be proved later). However, if $K = a \cup b$ and $U = S(a, 1/4) \cup S(b, 1/4)$, then V and W are not obtainable so as to satisfy the definition of almost local 0-avoidability of K.

8.2 Theorem. *If M is any space and G^r is any group of r-cycles of M, then the property of almost local avoidability of the closed subsets of M relative to G^r is equivalent to the property of almost complete avoidability of the closed subsets of M relative to G^r.*

Proof. That almost complete avoidability implies almost local avoidability is obvious.

Suppose the closed subsets of M are almost locally avoidable rel. G^r, and let K be any closed subset of M. Then there exists, for any open set U containing K, a pair of open sets V and W such that $K \subset W \Subset V \Subset U$. The set $K' = (M - U) \cup \overline{W}$ is a closed subset of M, and the set $U' = M - F(V)$ is an open subset of M containing K'. Since the closed subsets of M are almost locally avoidable rel. G^r, there exist open sets V' and W' such that $K' \subset W' \Subset V' \Subset U'$ and such that only a finite number of cycles of G^r on $F(V')$ are lirh on $M - W'$.

Now $M - W' \subset U - W$. For $W' \supset K' \supset W$, implying that $M - W' \subset M - W$; and $W' \supset K' \supset M - U$, implying $M - W' \subset U$. Therefore we have

$$M - W' \subset U \cap (M - W) = U - W. \tag{8.2a}$$

Consider the open set $V_1 = V \cap V'$. Evidently $U \Supset V_1 \Supset W \supset K$; for $U \supset \overline{V}$, implying $U \supset \overline{V}_1$, and relations $V \supset \overline{W}$, $V' \supset K' \supset \overline{W}$ imply $V \cap V' \supset \overline{W}$. Also, since $M - F(V) = U' \Supset V'$, it follows that $M - F(V) \supset F(V')$ and hence, since $F(V \cap V') \subset F(V) \cup F(V')$, that

$$F(V_1) = F(V \cap V') \subset F(V'). \tag{8.2b}$$

The cycles of G^r that lie on $F(V_1)$ form, then, a subgroup G_1^r of the group of cycles of G^r that lie on $F(V')$. And since at most a finite number of the latter are lirh on $M - W'$, it follows from (8.2a) and (8.2b) that at most a finite number of cycles of G_1^r are lirh on $U - W$. Thus, given $U \supset K$, we have found open sets V_1 and W such that $K \subset W \Subset V_1 \Subset U$ and such that at most a finite number of the cycles of G^r on $F(V_1)$ are lirh on $U - W$, and therefore K is almost completely avoidable rel. G^r.

If $\mathfrak{E}$ is a covering of the space, G^r any group of cycles, and $\mathcal{P}$ is a covering of a set M by a finite set of pairs (U, V) of diameter $< \mathfrak{E}$ such that for every $\mathcal{P}' \subset \mathcal{P}$, the number of cycles of G^r on $\bigcup_V \mathcal{P}'$ that are lirh on $\bigcup_U \mathcal{P}'$ is finite, then we call $\mathcal{P}$ an $S(\mathfrak{E}, G^r)$ covering of M.

8.3 Theorem. *If a compact space M has property* S_r *relative to some group G^r of cycles, then the closed subsets of M are almost completely avoidable rel. G^r.*

Proof. Given a closed subset K of M and an open set U containing K, let V and W be arbitrary open sets such that $K \subset W \Subset V \Subset U$. By Theorem 7.10 at most a finite number of cycles of G^r on $F(V)$ are lirh on $\overline{U} - W$.

8.4 The question arises, does the converse of Theorem 8.3 hold? That the answer is negative is easily seen from the following example: In the polar coordinate plane let $K_0 = (0, 0)$, $K_n = \{(\rho, \theta) \mid \rho = 1/n\}$, and $M = \bigcup_{n=0}^{\infty} K_n$. Then the closed subsets of M are almost completely avoidable relative to the group of all 1-cycles, but M does not have property S_1 since $p^1(M) = \infty$.

8.5 It appears, then, from Theorem 8.3, that the S-properties are in general a strengthening of the almost complete avoidability properties. And since possession of property S_r implies finite p^r and hence semi-r-connectedness

(Corollary V 19.5), the sequence "*Property* $S_r \to$ *semi-r-connectedness and almost complete r-avoidability at all points* $\to$ *r-lc*" (Lemma 6.18[a]) reveals the intermediate character of the avoidability properties between the S and lc properties.

Now there is one case, important in the sequel, in which the converse of Theorem 8.3 does hold; namely, where $G^r = B^r(M, \mathfrak{F})$, the group of bounding r-cycles of M.

8.6 THEOREM. *In every compact space M, Property S_r rel. $B^r(M, \mathfrak{F})$ is equivalent to the almost complete avoidability of the closed subsets of M rel. $B^r(M, \mathfrak{F})$.*

PROOF. Let P and Q be open sets such that $P \supset \bar{Q}$. If M has the almost complete avoidability property, there exist open sets U and V such that $Q \Subset V \Subset U \Subset P$ and such that at most a finite number of cycles of $B^r(M, \mathfrak{F})$ on $F(U)$ are lirh on $\bar{P} - V$. If $Z^r \in B^r(M, \mathfrak{F})$ on $\underline{Q}$, then by Lemma 1.13 there exists a cycle γ^r on $F(U)$ such that $Z^r \sim \gamma^r$ on $\bar{U}$. The theorem now follows from Theorem 7.9.

8.7 Further light may be thrown on the relationships between the S, avoidability, and lc properties by the extension of the local connectivity numbers to the connectivity numbers about a set, already referred to in VI 7. We refer particularly to the numbers "g". If $P \supset Q$ are open sets containing the fixed set K, then a number $g^r(K; P, Q)$, in analogy with $g^r(x; P, Q)$, is defined as the number of C-cycles on Q lirh on P. The number $g^r(K; P)$ is the greatest cardinal number which is $\leqq g^r(K; P, Q)$ for all Q, and $g^r(K)$ is the least cardinal number $\geqq g^r(K; P)$ for all P. If $g^r(K; P)$ is infinite for some P, we write $g^r(K) = \infty$, but if $g^r(K; P)$ is always finite while $g^r(K)$ is infinite, we write $g^r(K) = \omega$. That, unlike the case where K is a point, the various cases $g^r(K) = 1, 2, \cdots, \omega$ can occur when K is a general closed point set is easily seen from examples. For instance, if K is an S^1 in S^2, then $g^1(K) = 1$. If for each natural number n, K_n is an S^1 of diameter $< 1/n$ in S^2 such that lim sup $\{K_n\}$ is a single point x, then $g^1(\bigcup K_n \cup x) = \omega$.

8.8 REMARK. The numbers $g^r(K)$ are clearly dependent not upon the set K alone, but also upon the space in which K is imbedded. For example, if K is a point in an S^2, then $g^1(K) = 0$, but if the space is the set M of the example in 8.4, then for the point K_0, $g^1(K_0) = \infty$. It would be appropriate, then, to include the imbedding space in the symbol for the number—as for instance $g^r(K; S^2)$ for a set K in S^2—but unless more than one space is involved during an argument, we use the abbreviated symbol $g^r(K)$.

In case only cycles of some special group, G^r, are taken into consideration, we denote the numbers g^r by $g(K; G^r)$, $g(K; P; G^r)$, etc.

8.9 THEOREM. *In every compact space M, Property S rel. G^r is equivalent to the relation $g(K; G^r) \leqq \omega$ for all closed subsets K of M.*

Theorem 8.9 follows easily from Theorem 7.9.

From Theorems 8.3, 8.6 and 8.9 we also have:

8.10 THEOREM. *In a compact space M, if $g^r(K; G^r) \leqq \omega$ for every closed subset K of M, then the closed subsets of M are almost completely avoidable rel. G^r. And for $G^r = B^r(M, \mathfrak{F})$, the two properties are equivalent.*

9. An addition theorem.

The following theorem is an important generalization, from S^n to general topological spaces, of Theorem II 5.18.

9.1 THEOREM. *In a normal space S let A_1 and A_2 be compact point sets and K a compact subset of $S - A_1 - A_2$ which carries a cycle γ^r that is homologous to zero on compact subsets of $S - A_1$ and $S - A_2$. Suppose Z_i^{r+1} is a cycle* mod *K on compact K_i, $i = 1, 2$, where $S - A_i \supset K_i \supset K$ and $\partial Z_i^{r+1} \sim \gamma^r$ on K, and that γ^{r+1} is a cycle on $K_1 \cup K_2$ such that $\gamma^{r+1} \sim Z_1^{r+1} - Z_2^{r+1}$* mod *$K$ on $K_1 \cup K_2$.*[6] *If γ^{r+1} can be so chosen that $\gamma^{r+1} \sim 0$ on a compact subset M of $S - (A_1 \cap A_2)$, then $\gamma^r \sim 0$ on a compact subset of $S - A_1 - A_2$.*

PROOF. We may assume that $M \supset K_1 \cup K_2$. Let $M \cap A_i = B_i$, and let P be an open set such that $B_1 \subset P \Subset S - A_2 - K_1$. Let $H = (K_2 - P) \cup (M \cap F(P))$. Then $H \subset S - A_1 - A_2$, and we shall prove that $\gamma^r \sim 0$ on H.

By Lemma 1.4 there exists on M a cycle Z^{r+2} mod $K_1 \cup K_2$ such that $\partial Z^{r+2} \sim \gamma^{r+1}$ on $K_1 \cup K_2$. By Lemma 1.16 there exists a cycle Z_K^{r+2} mod the boundary (rel. M) of $M - (K_1 \cup K_2 \cup \overline{P})$ such that $Z_K^{r+2} \sim Z^{r+2}$ mod $K_1 \cup K_2 \cup \overline{P}$ on M and such that $\partial Z_K^{r+2} \sim \partial Z^{r+2}$ on $K_1 \cup K_2 \cup \overline{P}$. Now the relations $\partial Z^{r+2} \sim \gamma^{r+1}$ on $K_1 \cup K_2$, and $\gamma^{r+1} \sim Z_1^{r+1} + Z_2^{r+1}$ mod K on $K_1 \cup K_2$ (as given by hypothesis), together with the second relation of the preceding sentence, imply that $\partial Z_K^{r+2} \sim Z_1^{r+1} + Z_2^{r+1}$ mod K on $K_1 \cup K_2 \cup \overline{P}$.

Let $\mathfrak{U}$ be any covering none of whose elements meets both P and K_1. There exists a chain $C^{r+2}(\mathfrak{U})$ on $K_1 \cup K_2 \cup \overline{P}$ such that $\partial C^{r+2}(\mathfrak{U}) = \partial Z_K^{r+2}(\mathfrak{U}) - Z_1^{r+1}(\mathfrak{U}) - Z_2^{r+1}(\mathfrak{U}) + K^{r+1}(\mathfrak{U})$, where $K^{r+1}(\mathfrak{U})$ is on K. As defined above, $\partial Z_K^{r+2}(\mathfrak{U})$ is on $K_1 \cup (K_2 - P) \cup (M \cap F(P))$. Now $\partial[C^{r+2}(\mathfrak{U}) \wedge (K_2 \cup \overline{P})] = [\partial Z_K^{r+2}(\mathfrak{U})] \wedge (K_2 \cup \overline{P}) - Z_1^{r+1}(\mathfrak{U}) \wedge (K_2 \cup \overline{P}) - Z_2^{r+1}(\mathfrak{U}) + K^{r+1}(\mathfrak{U}) + N^{r+1}(\mathfrak{U})$, where $N^{r+1}(\mathfrak{U})$ is a chain on $K_2 - P$. Applying the operator ∂ to this relation and transposing, we get

$$\begin{aligned}(9.1a)\qquad \partial Z_2^{r+1}(\mathfrak{U}) = \partial\{[\partial Z_K^{r+2}(\mathfrak{U})] \wedge (K_2 \cup \overline{P})\} - \partial[Z_1^{r+1}(\mathfrak{U}) \wedge (K_2 \cup \overline{P})] \\ + \partial K^{r+1}(\mathfrak{U}) + \partial N^{r+1}(\mathfrak{U}).\end{aligned}$$

Now $K^{r+1}(\mathfrak{U})$ is on K and $Z_1^{r+1}(\mathfrak{U}) \wedge (K_2 \cup \overline{P})$ is on $K_1 \cap (K_2 \cup \overline{P}) = K_1 \cap K_2 \subset K_2 - \overline{P}$. Also, $[\partial Z_K^{r+2}(\mathfrak{U})] \wedge (K_2 \cup \overline{P})$ is on the set $[K_1 \cup (K_2 - P) \cup (M \cap F(P))] \cap (K_2 \cup \overline{P}) = [K_1 \cap (K_2 \cup \overline{P})] \cup (K_2 - P) \cup [(M \cap F(P))] \cap (K_2 \cup \overline{P}) \subset (K_2 - \overline{P}) \cup (M \cap F(P)) = H$. Hence relation (9.1a) implies the existence on H of a chain $L^{r+1}(\mathfrak{U})$ such that $\partial Z_2^{r+1}(\mathfrak{U}) = \partial L^{r+1}(\mathfrak{U})$. But since $\partial Z_2^{r+1} \sim \gamma^r$ on K, there exists a chain $C^{r+1}(\mathfrak{U})$ on K such that $\partial C^{r+1}(\mathfrak{U}) =$

[6]Under the hypothesis stated in the preceding sentence, such chains as Z_i^{r+1}, γ^{r+1} exist by Lemmas 1.4, 1.6.

$\gamma^r(\mathfrak{U}) - \partial Z_2^{r+1}(\mathfrak{U})$ and consequently $L^{r+1}(\mathfrak{U}) + C^{r+1}(\mathfrak{U})$ is a chain on H with $\gamma^r(\mathfrak{U})$ as boundary.

An important consequence of Theorem 9.1 is a generalization of the Phragmen-Brouwer property of S^n (Theorem II 5.19).

9.2 THEOREM. *If A and B are disjoint, closed subsets of a normal space S, x, $y \in S - A - B$ lie in single constituants of $S - A$ and $S - B$, respectively, and all 1-cycles on compact subsets of S bound on compact subsets of S, then x and y lie in the same constituant of $S - A - B$.*

By virtue of the theorems of II 4, we then have:

9.3 COROLLARY. *If S is an lc continuum such that $p^1(S) = 0$, then S has all of the properties* I, I′, II-V of II 4.1; *in particular, then, S is unicoherent. And if S is completely normal, then it has property* V′.

REMARK. Evidently Corollary 9.3 still holds if instead of assuming S to be a continuum, etc., it is assumed that S is locally compact, normal, lc, and such that every 1-cycle on a compact subset of S bounds on some compact subset of S.

9.4 THEOREM. *In a space S let A and K be compact, disjoint point sets, and let γ^r be a cycle on K that bounds on the compact sets K_1, K_2 (each of which contains K) respectively, but bounds on no compact subset of $S - A$. If γ^{r+1} is a cycle as defined in Theorem* 9.1, *and M is a compact set (containing $K_1 \cup K_2$) on which $\gamma^{r+1} \sim 0$, then some component of $M \cap A$ contains points of both K_1 and K_2.*

PROOF. If $K_1 \cap K_2 \cap A \neq 0$, the theorem is trivial. We suppose, then, that the sets $K_1 \cap A$, $K_2 \cap A$ are disjoint, and that no component of $M \cap A$ contains points of both K_1 and K_2. Let A_i' be the set $K_i \cap A$ together with all points of $M \cap A$ that are c-equivalent in $M \cap A$ to points of K_i, $i = 1, 2$. The sets A_i' are disjoint and closed (Corollary IV 1.7), and by Theorem IV 1.5 $M \cap A = A_1 \cup A_2$ separate, where $A_i \supset A_i'$, $i = 1, 2$. Now with M as the space, a direct application of Theorem 9.1 shows that $\gamma^r \sim 0$ on a compact subset of $M - A_1 - A_2 = M - (M \cap A) \subset S - A$, contradicting the hypothesis.

In concluding this chapter, we give a characterization of the 2-cell which will be useful in the sequel.

9.5 THEOREM. *Let M be a compact, metric, lc space containing a simple closed curve J such that* (1) *each arc of M spanning J separates M, and* (2) *if γ^1 is a nontrivial 1-cycle of J, then M is an irreducible membrane rel. γ^1. Then M is a closed 2-cell.*

PROOF. That M is connected follows easily from condition (2). (See Lemma IX 5.2.) If we can prove that a union of two disjoint arcs A_1 and A_2, each having one end point on J, cannot separate M, then the theorem will follow from Theorem III 5.1 (condition 5.1a of the latter will follow from the fact that A_1 and A_2 may be degenerate). Hence suppose, on the contrary, that

$M - (A_1 \cup A_2) = X \cup Y$ separate. If both X and Y contain points of J, let $A_i \cap J = a_i$, $i = 1, 2$, and let $x, y \in J - (a_1 \cup a_2)$ that are separated by a_1 and a_2 on J. Denote the two arcs of J containing a_1 and a_2 by K_1 and K_2, respectively, and let γ^0 be a nontrivial cycle on $x \cup y$. With $K = x \cup y$ and the "A" of Theorem 9.4 replaced by $A_1 \cup A_2$, it follows from that theorem that some component of A contains points of both A_1 and A_2, which is impossible. The case $X \cap J = 0$ is left to the reader.

Bibliographical comment

§1. Many of these lemmas will be found in the works of Čech; several are stated, for example, in Čech [g].

§2. An analogue of Theorem 2.19 for the metric case was stated as Lemma 1, p. 179, of Wilder [s]. Theorem 2.26 was announced for the metric case in Wilder [A_{12}].

§3. Theorem 3.2 has an interesting history. Urysohn showed in [e_1; p. 123] that a common boundary of two domains in E^3 cannot be separated by a closed subset of dimension zero; in [e_2; pp. 311-313] he discussed the problem of showing that an analogous set in E^n, $n > 3$, cannot be separated by a closed subset of dimension $\leqq n - 3$. The latter was proved by Alexandroff in [c; p. 154], who showed further that if M is such a set in E^n, $K \subset M$ and $p^{n-2}(K, 2) = 0$, then $M - K$ is connected. In Wilder [k; p. 282] it was shown that if $p^{n-2}(K, 2) = k$, then $M - K$ has at most $k + 1$ components; also (p. 281) that if M is any n-dimensional metric continuum that is irreducible relative to carrying a nonbounding n-cycle, and K is a closed subset of M such that $p^{n-1}(K) = k$, then $M - K$ has at most $k + 1$ components; and (p. 285) if $p^n(M) > 1$, $k > 0$, then $M - K$ has at most k components. In [h] Čech announced generalizations of the latter results; details may be found in [g], in which Theorem 3.2 appears as Theorem I.

Example 3.4 was given by Kline [b; p. 167].

Theorem 3.13 was given in Wilder [s; p. 178, Theorem J], but only for Peano spaces.

Theorems 3.8-3.10 evidently bear a close relation to theorems of Čech on n-pseudomanifolds in [g].

§4. Definitions 4.1—4.3 (as well as 6.11, 6.12 of §6) were given for the metric case, and relations discussed, in Wilder [p]. Theorem 4.5 was first proved (using Vietoris cycles) by S. Kaplan in his dissertation [a].

§5. Theorem 5.4; compare with Theorem II of Čech [g].

§7. See Wilder [q].

§9. Theorem 9.5 was first given by Whitney [b]. See also van Kampen [a; 8.1].

CHAPTER VIII

GENERALIZED MANIFOLDS; DUALITIES OF THE POINCARÉ AND ALEXANDER TYPE

1. **General properties.** A locally compact space S will be called a *generalized manifold of dimension n* if the following axioms are satisfied:

A. *The dimension of S is n.*

B. *S is colc^{n-1}.*

C. *For each $x \in S$, $p_n(x) = 1$.*

For sake of brevity, we use the symbol *n-gm* for such a space. If the n-gm is compact, then we call it an *n-dimensional generalized closed manifold,* and symbolize it by *n-gcm*. When compactness is not assumed we may emphasize the fact by calling the n-gm an "open generalized manifold."

By virtue of Theorems VI 7.9 and VI 7.12, we have immediately the following theorem:

1.1 THEOREM. *Every generalized manifold is locally connected in all dimensions, and has property $(P, Q)_r$ in all dimensions.*

In particular, an n-gm is lc in the sense of Chapter I, and the n-gcm can therefore have only a finite number of components.

The euclidean n-sphere, S^n, is a special case of an n-gcm. It is, moreover, a special case of what we call the "sphere-like" generalized manifold.

1.2 DEFINITION. An n-gcm is called *sphere-like* if its homology groups are isomorphic with the corresponding groups of S^n.

The classical manifolds, such as those of polyhedral character with elements so grouped as to form an n-cell at each vertex, as well as the generalized "manifolds" such as those of van Kampen [b] (see also Lefschetz [L_2]), which again are polyhedral in character but with the star of each vertex satisfying special conditions, are all special cases of gcm's. And the same holds for the Brouwer manifolds, which were defined among the topological spaces with each point having a neighborhood homeomorphic with the interior of the closed sphere in euclidean space.

As a consequence, the theory which we develop concerning the generalized manifolds holds equally well for the classical types. In particular, the positional properties of point sets, of which the Alexander duality theorem and its special corollary, the Jordan-Brouwer separation theorem, are classical examples, will be proved for the gcm. And we shall develop a natural extension of the methods for recognizing special continua, such as were developed for the case of the

plane by Schoenflies and the later set-theoretic topologists of the first third of the century.

One thing should be made clear at the start, namely that the gcm is actually a generalization in the sense that even among the separable metric spaces it includes spaces not found among any of the classical types of manifolds. An example of such is the following, due to van Kampen:

We first construct a "Poincaré space." (See H. Seifert and W. Threlfall, [S-T, p. 218, and references in appendix note 33 therein].) In euclidean 3-space, let K be an S^1 in which has been put a knot—say a trefoil knot. Expand K slightly so as to become a knotted anchor ring with torus surface T. The set M obtained by deleting the interior of T from euclidean 3-space, compactified by the addition of an ideal point at infinity, will be called a space of type M, and the set T will be called its boundary.

The Poincaré space which we have in mind is formed by combining two sets of type M along their boundaries, in such a way that (1) their only common points are the points of their boundaries, and (2) in combining (identifying) their boundaries, the meridional axis of one goes into the equatorial axis of the other. The effect of this is to render it impossible to deform continuously to a point a circle C which lies in one of the sets of type M and "links" its boundary. And this is true despite the fact that the resulting 3-dimensional space is "spherelike". For as soon as C is deformed onto the boundary of the type M set containing it, it becomes essentially an equatorial axis of the torus boundary of the other.

To construct the gcm which we mentioned above, we now proceed as follows: In cartesian 3-space E^3, for each natural number n let $S_n = \{(x, y, z) \mid (x - 1/n)^2 + y^2 + z^2 < 1/[3n(n + 1)]^2\}$. Denote the sphere which is the boundary of S_n by B_n. Let P_n be a Poincaré space of the type just described, but with a 3-cell deleted—the boundary sphere B_n' of this 3-cell being allowed to remain in P_n, however. Let $S' = E^3 - \bigcup S_n$. Then let S be the space obtained by adding the P_n's to S' in such a way that each P_n meets S' only in its boundary B_n', which is identified with B_n, and in such a manner that if $x_n \in P_n$, then p, the origin of coordinates in the original E^3, is the sequential limit point of the sequence $\{x_n\}$. The space S has the necessary local homology characteristics, but it contains arbitrarily small simple closed curves (in the P_n's) which are not continuously deformable to a point in the neighborhood of the origin p—an irregularity property not present in a finite complex whether a manifold or not.

The reason for giving a 3-*dimensional* example will be clear as soon as we show that for the 1- and 2-dimensional separable metric cases, the gcm and the classical closed manifolds are the same. Thus, for $n = 1$, a gcm is an S^1. and for $n = 2$ a gcm is an ordinary 2-dimensional manifold—every point having a neighborhood homeomorphic with the euclidean plane. These facts furnish additional justification for the Definition 1, of course. Naturally one can give examples of nonseparable n-gm's for $n = 1$ and 2.

2. **Orientability.** The classical manifolds are divided broadly into two types: the *orientable* and the *nonorientable*. In the 2-dimensional case, the closed orientable manifolds are the ordinary closed surfaces without "singularities" such as the S^2, the torus, and surfaces of higher genus. (See Hilbert and Cohn-Vossen, *Anschauliche Geometrie,* Berlin, 1932, Chapter VI; Kerékjártó [K]; also A. W. Tucker, *Elementary topology,* Princeton lecture notes, 1935–1936, which contains a remarkable set of illustrations; Veblen [V].) The nonorientable surfaces such as the projective plane, Klein bottle, etc., are nonimbeddable in 3-space.

The distinguishing characteristic of the classical orientable n-manifold is the existence of an n-cycle over the group of integers which is nonbounding and which is destroyed by the deletion of an n-cell. The existence of this cycle means that the n-cells σ_i^n of the complex constituting the manifold can be so oriented that the chain $\sum \sigma_i^n$ is a cycle. Inasmuch as the gcm is only a topological space with special local properties, and, as the above example shows, is generally not decomposable into cells constituting a "locally finite" complex, we shall require for orientability simply the existence of an n-cycle which has the whole space as minimal closed carrier.

Before going further, however, it will be necessary to extend the concepts of cycle and cocycle to fit the "in the large" situation of the open gm. In dealing with local properties, we were able to avoid so doing, and where the open gm is an open subset of a gcm, as it will frequently be in the sequel, we can still avoid this necessity. But where the open gm is not necessarily a subset of a compact space, we need the following extension (in which, incidentally, we describe only the "absolute" cases):[1]

2.1 The space S being locally compact, let $\{F_\nu\}$ be the collection of all compact subsets of S. It becomes a directed system (13.1) if we let $F_\mu < F_\nu$ mean that $F_\mu \subset F_\nu$. Let the vector spaces[2] $Z^r(S; F_\nu, 0)$, $B^r(S; F_\nu, 0)$ be denoted by z_ν^r, b_ν^r, respectively, and if $F_\mu < F_\nu$, so that $F_\mu \subset F_\nu$, and z^r is a cycle of F_μ, let $\omega_{\mu\nu}^* z^r = z^r$, where the latter is considered as an element of F_ν —i.e., $\omega_{\mu\nu}^*$ is the identity mapping. Then $\{z_\nu^r ; \omega_{\mu\nu}^*\}$, $\{b_\nu^r ; \omega_{\mu\nu}^*\}$ are direct systems of groups, and their respective $\lim_{\rightarrow}$'s we denote by $z^r(S)$ and $b^r(S)$, respectively. The factor group $h^r(S) = z^r(S)/b^r(S)$ we call the *r-dimensional compact homology group* of S. The elements of $z^r(S)$ are called *compact cycles*; the elements in z_ν^r may be called *coordinates* of compact cycles. It will be noted that since $\{z_\nu^r, \omega_{\mu\nu}^*\}$ is a direct system, a compact cycle is determined, up to homology, by any coordinate in a z_ν^r. Consequently we shall in practice usually identify the compact cycle with one of its coordinates, treating it as a C-cycle on a compact subset of S. And when we speak of a compact cycle bounding *in* an open set P, we shall mean that, considered as a C-cycle on a compact set, it

[1] An alternative procedure might be to introduce compactification of a manifold M by addition of an ideal set C, and use of cycles mod C instead of the "infinite" cycles and cocycles described below.

[2] See V 7.10; the symbol $\mathfrak{F}(= G)$ is deleted since we consider only the case of field coefficients.

bounds on a compact subset of P. Evidently the group $h^r(S)$ is isomorphic with the group of all C-cycles of S that have compact carriers, reduced modulo the subgroup of those that bound on compact subsets of S.

The group $h^r(S)$ could also be defined as $\lim_{\rightarrow} \{h_\nu^r ; \varphi_{\mu\nu}^*\}$, where $h_\nu^r = z_\nu^r/b_\nu^r$ and $\varphi_{\mu\nu}^*$ maps $\{z^r\} \in h_\mu^r$ into $\{\gamma^r\} \in h_\nu^r$ if $\{z^r\} \subset \{\gamma^r\}$ in h_ν^r.

2.2 Next, let H_r^ν denote the r-dimensional cohomology group of S mod $S - F_\nu$—i.e., $H_r(S; S, S - F_\nu)$ in the symbolism of V 15.4—and let $\varphi_{\mu\nu}$ be a mapping of H_r^ν into H_r^μ which carries an element $\{Z_r^\nu\}$ of H_r^ν into $\{\gamma_r^\mu\} \in H_r^\mu$ if $\{Z_r^\nu\} \subset \{\gamma_r^\mu\}$ in H_r^μ. Then $\mathfrak{H}_r(S) = \lim_{\leftarrow} \{H_r^\nu ; \varphi_{\mu\nu}\}$ will be called the *r-dimensional infinite cohomology group* of S; and a collection $\{Z_r^\nu\}$ will be called an *infinite r-cocycle* of S if $\varphi_{\mu\nu} Z_r^\nu \backsim Z_r^\mu \bmod S - F_\mu$, the Z_r^ν being the *coordinates*.

2.3 In order to define a cap product between compact cycles and infinite cocycles, consider a compact cycle γ^r and an infinite cocycle Γ_p. Then γ^r is by definition a collection $\{\gamma_\nu^r\}$ where γ_ν^r is a C-cycle of F_ν, and Γ_p is a collection $\{\Gamma_p^\nu\}$ where Γ_p^ν is a cocycle mod $S - F_\nu$. Now for a fixed ν, we already have a definition (V 17.3) of the C-cycle $\gamma_\nu^{r-p} = \Gamma_p^\nu \frown \gamma_\nu^r$. The compact cycle determined by γ_ν^{r-p} we let be the cap product $\Gamma_p \frown \gamma^r$. To see that it is independent of the choice of F_ν, consider any F_α such that $F_\nu < F_\alpha$. Then $\gamma_\alpha^{r-p} = \Gamma_p^\alpha \frown \gamma_\alpha^r = \Gamma_p^\alpha \frown \omega_{\nu\alpha}^*\gamma_\alpha^r$, since the ω^*'s are identical mappings, etc., and hence on F_ν becomes $\Gamma_p^\nu \frown \gamma_\nu^r = \gamma_\nu^{r-p}$. Thus $\omega_{\nu\alpha}^*\gamma_\nu^{r-p} = \gamma_\alpha^{r-p}$.

2.4 Between compact cycles and infinite cocycles of the same dimension r we may also define a dot product, or multiplication relative to $\mathfrak{F}$, as in the case of the C-cycles and cocycles, by first determining a cap product $\Gamma_r^\nu \frown \gamma_\nu^r$ as above, and then letting $\Gamma_r \cdot \gamma^r = \mathrm{Ki}(\Gamma_r^\nu \frown \gamma_\nu^r)$. That both $\frown$ and $\cdot$ can be extended to products between cohomology and homology classes of infinite cocycles and compact cycles follows from the compact case. And we may prove:

2.5 Theorem. *The vector spaces $\mathfrak{H}_r(S)$ and $h^r(S)$ for a locally compact lc^r space S form an orthogonal dual pair relative to the $\cdot$ multiplication and the field $\mathfrak{F}$.*

Proof. If Γ_r is an infinite noncobounding cocycle, and Γ_r^ν is one of its coordinates such that $\Gamma_r^\nu \not\backsim 0 \bmod S - F_\nu$, then by Theorem V 18.19 there exists a C-cycle γ_ν^r of F_ν such that $\Gamma_r^\nu \cdot \gamma_\nu^r \neq 0$. If γ^r is the compact cycle determined by γ_ν^r, then $\Gamma_r \cdot \gamma^r \neq 0$.

In order to show that if a compact cycle $\gamma^r \not\sim 0$, there exists an infinite cocycle Γ_r such that $\Gamma_r \cdot \gamma^r \neq 0$ we again use the machinery we utilized in the compact case in Chapter V. Given an F_ν, let P be an open set containing F_ν such that $\bar{P}$ is compact. Since S is lc^r, it follows from Corollary VI 3.9 that $H_r(S: S, S - \bar{P}; S, S - F_\nu)$ is finite. If P' is any compact set containing P, let $G(P')$ denote the set of all cocycles mod $S - F_\nu$ that are also cocycles mod $S - P'$. A cocycle mod $S - F_\nu$ that is also a cocycle mod $S - F'$ for every compact set $F' \supset F_\nu$ may be called essential, and evidently these are the cocycles of $\bigcap_{P'} G(P')$. Now just as in the proof of Theorem V 10.7 we can show the existence of a $P' \supset P$ such that every element of $G(P')$ is essential.

The rest of the proof is merely an adaptation of Theorems V 10.1, 10.2 to the present case.

2.6 In case the space S is already compact, $\mathfrak{H}_r(S)$ and $h^r(S)$ are still definable as above, but in this case they become $H_r(S)$ and $H^r(S)$ as defined in V, since the directed system $\{F_\nu\}$ has a maximal element, S.

2.7 Paralleling the above definitions, let us consider the set $\{U_\nu\}$ of all open sets U_ν such that each $\overline{U}_\nu$ is compact. It becomes a directed system if we let $U_\mu < U_\nu$ mean that $U_\mu \subset U_\nu$.

Let $H_r^\nu = H_r(S : U_\nu, 0)$. For $U_\mu < U_\nu$, let $\omega_{\mu\nu}^* : H_r^\mu \to H_r^\nu$ be defined by the stipulation that $Z_r^\mu \in H_r^\mu$ maps into $Z_r^\nu \in H_r^\nu$ if the elements of Z_r^μ all belong to Z_r^ν. Then $h_r(S) = \lim_{\rightarrow} \{H_r^\mu ; \omega_{\mu\nu}^*\}$ is called the *r-dimensional compact cohomology group* of S. If $\{Z_r^\nu\} \in h_r(S)$, then a collection $\{\gamma_r^\nu\}$, where $\gamma_r^\nu \in Z_r^\nu$, is called a *compact cocycle* of S if γ_r^ν and γ_r^μ are identical for $U_\mu < U_\nu$; evidently such a "representative" cocycle exists for each element of $h_r(S)$ since $\omega_{\mu\nu}^*$ is defined so that $Z_r^\mu \subset \omega_{\mu\nu}^*(Z_r^\mu)$. And evidently every cocycle γ_r in an open set with compact closure determines a compact cocycle, so that as in the case of compact cycles we shall in practice find it convenient to identify the compact cocycle with one of its elements γ_r^ν, ν fixed. Any one of the open sets U_ν (containing γ_r^ν) will be called a *carrier* of γ_r.

2.8 If $Z^r(U_\nu)$ is the group of cycles mod $S - U_\nu$, and $B^r(U_\nu)$ the subgroup of those that bound mod $S - U_\nu$, then with $\omega_{\mu\nu}$ as the identity mapping we denote $\lim_{\leftarrow} \{Z^r(U_\nu); \omega_{\mu\nu}\}$ and $\lim_{\leftarrow} \{B^r(U_\nu); \omega_{\mu\nu}\}$ by $Z^r(S)$ and $B^r(S)$, respectively. The elements of $Z^r(S)$ we call *infinite cycles* of S, and the group $\mathfrak{H}^r(S) = Z^r(S)/B^r(S)$ we call the *r-dimensional infinite homology group of S*. The meaning of the statement that a given infinite cycle "bounds mod $S - U_\nu$," or that certain infinite cycles are "lirh mod $S - U_\nu$," etc., should be clear.

2.9 The definition of a cap product between compact cocycles and infinite cycles is made in the usual manner: Given a compact cocycle $\gamma_p = \{\gamma_p^\nu\}$ and $Z^r \in Z^r(S)$, we choose any ν and obtain the cap product $Z_\nu^{r-p} = \gamma_p^\nu \frown Z_\nu^r$, where Z_ν^r is the coordinate of Z^r on U_ν. Then Z_ν^{r-p} is a cycle on U_ν, and determines a compact cycle Z^{r-p}. The definition of a dot product between compact cocycles and infinite cycles of the same dimension parallels that for the infinite cocycle-compact cycle case above, and we have the theorem:

2.10 THEOREM. *The vector spaces $\mathfrak{H}^r(S)$ and $h_r(S)$ of an open generalized manifold S form an orthogonal dual pair relative to the · multiplication and the field $\mathfrak{F}$.*

(Actually all that is needed is a locally compact, lc^r space since only the $(P, Q)_r$ property is employed. That such a space has property $(P, Q)_r$ is proved in Chapter XI.)

PROOF. If Z^r is an infinite cycle of S not in $B^r(S)$, then for some U_ν, $Z_\nu^r \not\sim 0$ mod $S - U_\nu$, Z_ν^r being the coordinate of Z^r on U_ν. Then there exists a cocycle

γ_r^ν in U_ν such that $\gamma_r^\nu \cdot Z_\nu^r \not\sim 0$, and consequently the compact cocycle γ_r determined by γ_r^ν satisfies the condition that $\gamma_r \cdot Z^r \not\sim 0$.

If a compact cocycle $\gamma_r \not\sim 0$, then each corresponding $\gamma_r^\mu \not\sim 0$ in U_μ. Now if we fix U_μ, then for any $U_\nu \supset U_\mu$, we have that $p_r(S: U_\mu, 0; U_\nu, 0)$ is finite since S has property $(P, Q)_r$ for all r (Theorem 1.1). Hence by Theorem V 18.31 the dimensions of $H^r(S: S, S - U_\nu; S, S - U_\mu)$ and $H_r(S: U_\mu, 0; U_\nu, 0)$ are equal and finite.

From here on the proof of the existence of an infinite cycle Z^r such that $\gamma_r \cdot Z^r \not\sim 0$ is strictly parallel to the analogous case in Theorem 2.5.

2.11 In case S is compact, $\mathfrak{H}^r(S)$ and $h_r(S)$ are the same as the $H^r(S)$ and $H_r(S)$ respectively, of Chapter V.

For the sequel it is important to notice what the groups just defined become when the open n-gm S is an open subset of another n-gm T. For inasmuch as the conditions $A - C$ are purely local, any open subset of an n-gm is itself an n-gm.

2.12 Theorem. *If T is a locally compact space and S is an open subset of T such that $\overline{S}$ is compact, then $H_r(T; S, 0)$ is isomorphic with $h_r(S)$.*

Proof. If $z_r(\mathfrak{U})$ is a cocycle of T in S, let $\mathfrak{B} > \mathfrak{U}$ be such that a simplex of $\mathfrak{B}$ in S is also in $T - \overline{Q}$, where Q is an open set containing $T - S$ (Lemma V 8.7). Then, since $\pi_{\mathfrak{U}\mathfrak{B}}^* z_r(\mathfrak{U})$ is in S, it is also in $T - \overline{Q}$ and accordingly determines a compact cocycle of S. If $z_r(\mathfrak{U}) \sim 0$ in S, then (by definition) for some $\mathfrak{W} > \mathfrak{B}$, $\mathfrak{W}$ again being chosen according to Lemma V 8.7, $\pi_{\mathfrak{U}\mathfrak{W}}^* z_r(\mathfrak{U}) \sim 0$ on $\mathfrak{W}$ in S, hence $\pi_{\mathfrak{B}\mathfrak{W}}^* \pi_{\mathfrak{U}\mathfrak{B}}^* z_r(\mathfrak{U}) \sim \pi_{\mathfrak{U}\mathfrak{W}}^* z_r(\mathfrak{U}) \sim 0$ in S, and as this cohomology is on $\mathfrak{W}$, it is also in an open subset of S whose closure in S is compact. Consequently the correspondences $z_r(\mathfrak{U}) \to \pi_{\mathfrak{U}\mathfrak{B}}^* z_r(\mathfrak{U})$ induce a homomorphism of $H_r(T; S, 0)$ into $h_r(S)$, and this homomorphism is clearly both onto and one-to-one, inasmuch as every compact cocycle of S is a cocycle of T in S, etc.

We defer the proof of the isomorphism of $H^r(T; T, T - S)$ and $\mathfrak{H}^r(S)$ (cf. Lemma 6.1).

2.13 By *orientability* of a manifold we shall mean the existence of a nonbounding n-cycle of the suitable type; in the case of the n-gcm, we shall ask that there exist a C-cycle γ^n which is nonbounding, and for the open n-gm we shall ask that there exist an infinite n-cycle which is not homologous to any infinite n-cycle on a closed proper subset.

It should be noticed that whether the given manifold is orientable or not is dependent upon the field $\mathfrak{F}$ employed. For example, the projective plane is an orientable 2-gcm when $\mathfrak{F}$ is the field of integers mod 2, but is not orientable when $\mathfrak{F}$ is the field of integers mod 3.

In the sequel, when dealing with compact sets, C-cycles will be employed as heretofore except when some other type of cycle is specified.

3. **The orientable n-gcm.** The n-gcm will in the sequel constitute the most important case. The spherelike n-gcm is the natural generalization of the euclidean n-sphere, so far as homology properties are concerned, and our use of the open manifold will frequently be for the case where it is an open subset of the orientable n-gcm.

It will be noticed that there is nothing in the purely local conditions A-C defining the n-gm, even after the addition of compactness, to prevent the space consisting of, say, a pair of nonintersecting n-spheres. The condition which we add to the former conditions in the case of an n-gcm, which leads most directly to the type of space to which we wish to restrict considerations, is:

D. *If F is a proper closed subset of S, then every n-cycle on F bounds on S.*

(Since S is n-dimensional, having therefore a complete family of n-dimensional coverings, and since $Z^n(\mathfrak{U}) \sim 0$ on an n-dimensional $\mathfrak{U}$ implies $Z^n(\mathfrak{U}) = 0$, condition D is evidently equivalent to the assumption that if F is a proper closed subset of S then $p^n(F) = 0$.)

3.1 Theorem. *If S is an n-gcm satisfying* D, *then* $p^n(S) \leqq 1$.

Proof. Suppose γ_1^n, γ_2^n are cycles lirh on S. Let $x \in S$, and P, Q open sets such that $x \in Q \subset P$ and $p_n(x; P, Q) = 1$—such sets exist by condition C. Since, by VI 6.7, $p^n(x; P, Q) = p_n(x; P, Q)$, there must exist a relation

$$a_1\gamma_1^n + a_2\gamma_2^n \sim 0 \qquad \text{mod } S - Q. \tag{3.1a}$$

But as γ_1^n, γ_2^n are lirh on S, the cycle $a_1\gamma_1^n + a_2\gamma_2^n$ is a nonbounding cycle of S and by Lemma VII 3.6 relation 3.1a is impossible.

3.2 Corollary. *A necessary and sufficient condition that an n-gcm, S, be orientable is that $p^n(S) = 1$; and a necessary and sufficient condition that S be nonorientable is that $p^n(S) = 0$.*

Since we must have $p^n(S) = 1$ if S is an orientable n-gcm, we may henceforth assume that every orientable n-gcm has a unique nonbounding n-cycle-γ^n, which we shall call its *fundamental cycle.* And if P is an open subset of such a space, there exists in P a cocycle γ_n such that $\gamma^n \cdot \gamma_n = 1$, which we shall call a *fundamental cocycle* in P. It should be noticed that fundamental cocycles are necessarily chosen from a unique cohomology class. For consider fundamental cocycles γ_n^1, γ_n^2 (in any open subsets); then by definition $\gamma^n \cdot \gamma_n^1 = \gamma^n \cdot \gamma_n^2 = 1$. Now since $p_n(S) = p^n(S) = 1$, there must exist a relation $\gamma_n^1 \backsim a\gamma_n^2$, $a \in \mathfrak{F}$, and since the dot product is invariant within the cohomology class, it follows that $\gamma^n \cdot (a\gamma_n^2) = \gamma^n \cdot \gamma_n^1 = 1$. But $\gamma^n \cdot (a\gamma_n^2) = a(\gamma^n \cdot \gamma_n^2) = a$ and consequently $a = 1$.

(It should be recalled that in general the relation $\gamma^n \cdot \gamma_n^1 = \gamma^n \cdot \gamma_n^2$ does not imply that $\gamma_n^1 \backsim \gamma_n^2$. For instance, if $\{Z_i^n\}$, $\{Z_n^i\}$ are finite collections of cycles and cocycles such that $Z_i^n \cdot Z_n^j = \delta_i^j$ (as in the case of orthogonal pairings for a space of finite Betti number), then $Z_1^n \cdot (Z_n^1 + Z_n^2) = Z_1^n \cdot Z_n^1 + Z_1^n \cdot Z_n^2 = Z_1^n \cdot Z_n^1 = 1$, but obviously $Z_n^1 + Z_n^2$ is not in the same cohomology class as Z_n^1.)

3.3 THEOREM. *All fundamental cocycles of an orientable n-gcm S lie in the same cohomology class of S.*

It is natural to ask, at this point, if there is any direct relationship between the fundamental cocycles and the orientability. By condition C, even in the case of the nonorientable manifold, there exist for every $x \in S$ canonical pairs (VI 6.11) P, Q such that $p^n(x; P, Q) = 1$, so that in Q there is a cocycle Z_n that fails to cobound in P (although $Z_n \backsim 0$ on S, to be sure). That is, the local situation as regards the existence of nonbounding cocycles is the same for the nonorientable case as for the orientable case. The question raised here has been neatly settled by Begle [a] in the following manner:

3.4 Let S be an n-gcm, and $\mathfrak{E}$ a covering of S. Since $p_n(x) = 1$ for all $x \in M$, there exist refinements $\mathfrak{E}_1$ and $\mathfrak{E}_2$ of $\mathfrak{E}$ such that each element of $\mathfrak{E}_2$ is a Q of a canonical pair, $P \supset Q$, relative to n and the local Betti number, and each element of $\mathfrak{E}_1$ is a P corresponding to one of the Q's of $\mathfrak{E}_2$. In each $Q_i \in \mathfrak{E}_2$ there is a cocycle $\gamma_n^i \not\backsim 0$ in the corresponding $P_i \in \mathfrak{E}_1$. Suppose $Q_i, Q_j \in \mathfrak{E}_2, Q_i \cap Q_j \neq 0$. Then there exists a canonical pair P, Q in $Q_i \cap Q_j$, and in Q a cocycle $\gamma_n \not\backsim 0$ in P. Since γ_n is in Q_i, there exists a relation $a_i\gamma_n^i \backsim \gamma_n$ in P_i; and since γ_n is in Q_j, a relation $a_j\gamma_n^j \backsim \gamma_n$ in P_j, where $a_i, a_j \in \mathfrak{F}$.

3.5 THEOREM. *The n-gcm S is orientable if and only if there exist, for each covering $\mathfrak{E}$ of S, coverings $\mathfrak{E}_1$, $\mathfrak{E}_2$, and cocycles γ_n^i, as defined in* 3.4, *such that for any choice of canonical pairs P, Q in the intersection of elements of $\mathfrak{E}_2$, the ratios a_i/a_j are all* 1.

PROOF. Suppose S is an orientable n-gcm and γ^n is its fundamental n-cycle. Then, being given $\mathfrak{E}_1$ and $\mathfrak{E}_2$, we choose in each Q_i a fundamental cocycle γ_n^i. The cohomology $a_i\gamma_n^i \backsim a_j\gamma_n^j$ implies that $\gamma^n \cdot (a_i\gamma_n^i) = \gamma^n \cdot (a_j\gamma_n^j)$ and hence $a_i = a_j$; $a_i \neq 0$ since $\gamma_n \backsim aZ_n$ in P, $a \neq 0$, where Z_n is a fundamental cocycle, and evidently $a_i = a$.

To prove the converse, let S be an n-gcm and $\mathfrak{E}$ any covering of S. Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be given as above, and let $\mathfrak{U}_2 \gg \mathfrak{E}_2$ as in Lemma V 8.3. Let $\mathfrak{U}$ be a covering such that $\mathrm{St}(\mathfrak{U}_2, \mathfrak{U}) > \mathfrak{E}_2$, and let Σ' be a complete family of coverings all of which are refinements of $\mathfrak{U}$ and n-dimensional. Hereafter cycles and cocycles will be considered only on Σ'.

Suppose $E_i^1, E_j^1 \in \mathfrak{E}_2$ and that the corresponding elements E_i^2, E_j^2 of $\mathfrak{U}_2$ are such that $\overline{E}_i^2 \cap \overline{E}_j^2 \neq 0$. By hypothesis there exist cocycles γ_n^i in the sets E_i^1 that are $\not\backsim 0$ in the corresponding elements E_i of $\mathfrak{E}_1$ and such that if P, Q are a canonical pair for a point of $\overline{E}_i^2 \cap \overline{E}_j^2$ such that $P \subset E_i^1 \cap E_j^1$, then in Q there is a cocycle γ_n such that $\gamma_n \backsim \gamma_n^i$ in E_i, $\gamma_n \backsim \gamma_n^j$ in E_j.

There exists a cycle γ_i^n mod $S - E_i$ such that $\gamma_i^n \cdot \gamma_n^i = 1$, and since $\gamma_n^i \backsim \gamma_n$ in E_i, also $\gamma_i^n \cdot \gamma_n = 1$. Similarly there is a cycle γ_j^n mod $S - E_j$ such that $\gamma_j^n \cdot \gamma_n = 1$. Since P, Q form a canonical pair, $\gamma_i^n \sim a\gamma_j^n$ mod $S - Q$, $a \in \mathfrak{F}$. However, this homology implies that $\gamma_i^n \cdot \gamma_n = (a\gamma_j^n) \cdot \gamma_n$ and hence that $a = 1$, and accordingly $\gamma_i^n \sim \gamma_j^n$ mod $S - Q$. As each element of Σ' is n-dimensional,

homology between n-cycles of elements of Σ' implies identity and therefore $\gamma_i^n = \gamma_j^n \bmod S - Q$. And this holds for every point of $\overline{E}_i^2 \cap \overline{E}_j^2$.

In order to construct a nonbounding cycle γ^n on S, we now introduce coverings $\mathfrak{U}_4 \gg \mathfrak{U}_3 \gg \mathfrak{U}_2$ as in Lemma V 8.3, and assume that $\mathfrak{U} > \mathfrak{U}_4$ as well as such that $\mathrm{St}(\mathfrak{U}_4, \mathfrak{U}) > \mathfrak{U}_3$, $\mathrm{St}(\mathfrak{U}_3, \mathfrak{U}) > \mathfrak{U}_2$. More specifically, we stipulate that $\mathrm{St}(U_i^4, \mathfrak{U})$ lies in $U_i^3 \in \mathfrak{U}_3$ and $\mathrm{St}(U_i^3, \mathfrak{U})$ lies in U_i^2. For each such $\mathfrak{U}$ we define a $\gamma^n(\mathfrak{U})$ as follows: Let σ_k^n be any n-cell of $\mathfrak{U}$. It is on some $\overline{U}_i^3$, where $U_i^3 \in \mathfrak{U}_3$, and in $\gamma^n(\mathfrak{U})$ we assign to σ_k^n the coefficient which it has in $\gamma_i^n(\mathfrak{U})$. If σ_k^n is also on $\overline{U}_j^3$, where $U_j^3 \in \mathfrak{U}_3$ and $i \neq j$, then it is also on $\overline{E}_i^2 \cap \overline{E}_j^2$, and since $\gamma_i^n = \gamma_j^n$ on this set, the coefficient of σ_k^n is not dependent on the particular U_i^3 chosen.

To see that $\gamma^n(\mathfrak{U})$ is a cycle of $\mathfrak{U}$, let σ_j^{n-1} be any $(n-1)$-cell of $\mathfrak{U}$. It is on some $\overline{U}_i^4$, $U_i^4 \in \mathfrak{U}_4$, and every n-cell with which it is incident is on $\overline{U}_i^3$ by the choice of $\mathfrak{U}_4$ and $\mathfrak{U}$. These n-cells have the same coefficients in $\gamma^n(\mathfrak{U})$ as they have in $\gamma_i^n(\mathfrak{U})$, and $\partial\gamma_i^n(\mathfrak{U})$ is on $S - E_i$, so that σ_j^{n-1} must have a zero coefficient in $\partial\gamma^n(\mathfrak{U})$. Thus $\gamma^n(\mathfrak{U})$ is a cycle.

Let $\gamma^n = \{\gamma^n(\mathfrak{U})\}$. To see that γ^n is a C-cycle, we need merely notice that if $\mathfrak{V} > \mathfrak{U}$, then $\pi_{\mathfrak{U}\mathfrak{V}}\gamma_i^n(\mathfrak{V}) = \gamma_i^n(\mathfrak{U})$, so that $\pi_{\mathfrak{U}\mathfrak{V}}\gamma^n(\mathfrak{V}) = \gamma^n(\mathfrak{U})$.

Finally, $\gamma^n \not\sim 0$ on S since, for any i, $\gamma^n \cdot \gamma_n^i = \gamma_i^n \cdot \gamma_n^i = 1$, and S is therefore orientable.

4. **The Poincaré duality for an orientable n-gcm.** The equality $p^r(S) = p^{n-r}(S)$, $0 \leqq r \leqq n$ (using nonaugmented homology theory), S an orientable n-gcm, will first be established, and the case of the orientable open manifold considered later. In each case the notion of "cochain realization" is useful, and we first give a lemma concerning such realizations, in a form which is adaptable to both cases. *Throughout we employ nonaugmented complexes.*

Consider a finite complex K of dimension $\leqq n$, and let $\mathfrak{U}$ be a covering of an orientable n-gm S, whose fundamental cycle is denoted by γ^n. A function τ^* which assigns to each chain C^r of K a chain $\tau^*(C^r) = C^{n-r}$ of $\mathfrak{U}$ is called a *cochain realization* of K on $\mathfrak{U}$ if

(a) τ^* is linear,

(b) $\tau^*\partial C^r = \delta\tau^*C^r$,

(c) $\mathrm{Ki}(C^0) = \mathrm{Ki}(\tau^*C^0 \frown \gamma^n)$ for every 0-chain C^0 of K.

Definitions of *partial cochain realization* and *norm* parallel the previous cases in connection with chain realizations (in Definition VI 2.9, "on E" is replaced by "in E").

4.1 Lemma. *Let S be an orientable n-gm,*[3] *L a compact subset of S and P a neighborhood of L such that $\overline{P}$ is compact. For each covering $\mathfrak{E}$ of S there is a refinement $\mathfrak{E}_n = \mathfrak{E}_n(\mathfrak{E}; L, P)$ and for each covering $\mathfrak{D}$ of S there is a refinement $\mathfrak{D}_n = \mathfrak{D}_n(\mathfrak{D}, \mathfrak{E}; L, P)$ such that if $\tau^{*\prime}$ is a partial cochain realization on $\mathfrak{D} \wedge L$*

[3] This lemma is stated for the general n-gm to avoid repeating it when needed in §5; the material in the early part of §5 is needed for the proof in this case, however.

of norm less than $\mathfrak{E}_n$ *of an n-dimensional complex K, then there is a cochain realization* τ^* *of K in P on* $\mathfrak{D}_n$ *of norm less than* $\mathfrak{E}$ *such that* $\tau^* C^r = \pi^*_{\mathfrak{D}\mathfrak{D}_n}\tau^{*\prime} C^r$ *wherever the latter is defined.*

(The proof parallels that of Theorem VI 3.6, and for the n-gm employs conditions B and C of the definition of n-gm, and Theorem 3.3, in place of the lc^n condition. However, stated as it is for the general n-gm (in order to avoid repeating in §5), the material in the early part of §5 is needed for the proof.)

4.2 THEOREM. *If S is an orientable n-gcm, then for* $0 \leqq r \leqq n$, *the vector spaces* $H^r(S)$ *and* $H^{n-r}(S)$ *are linearly isomorphic.*

PROOF. By Theorem 1.1 and Corollary VI 3.2, the vector spaces $H^r(S)$ are all of finite dimension. We shall show that for each r, there exists a linear homomorphism[4] of $H^r(S)$ onto $H^{n-r}(S)$, and conversely; it will then follow that these vector spaces are of the same finite dimension and linearly isomorphic. Inasmuch as $H^r(S)$ and $H_r(S)$ are linearly isomorphic (Theorem V 18.18; see also V 9.1), it will be sufficient if we define a linear homomorphism of $H_r(S)$ onto $H^{n-r}(S)$ for all r.

For any cocycle γ_r of S, let $\varphi(\gamma_r) = \gamma_r \frown \gamma^n$, where γ^n is the fundamental cycle of S. Then $\varphi(\gamma_r)$ is an $(n - r)$-cycle of S (cf. V 17.2). And if $\gamma_r \backsim 0$ on S, then $\varphi(\gamma_r) \sim 0$ on S (by use of V 16.9). Then φ induces a linear homomorphism Φ of $H_r(S)$ into $H^{n-r}(S)$. We shall show that Φ is "onto."

Let $\mathfrak{U}$ be any covering of S, and $\mathfrak{U}_0 >^* \mathfrak{U}^*_n(\mathfrak{U})$, where $\mathfrak{U}^*_n(\mathfrak{U})$ is the covering defined in Theorem VI 2.10; incidentally, then, $\mathfrak{U}_0$ may be considered as the same covering which was designated by the same symbol in the proof of Theorem VI 3.1. Also, let $\mathfrak{U}_1$ be an n-dimensional refinement of $\mathfrak{U}_0$ which is $>\mathfrak{B}^*_n(\mathfrak{U}, \mathfrak{U}_0)$ as well as $>^*\mathfrak{E}_n(\mathfrak{U}_0\,; S, S)$, the latter being the covering defined in Lemma 4.1. In each element of $\mathfrak{U}_1$ there is a fundamental cocycle of S, and we may assume that each of these lies on a fixed covering $\mathfrak{U}_2 > \mathfrak{U}_1$. Let $\tau^{*\prime}$ assign to each element of $\mathfrak{U}_1$ the corresponding cocycle on $\mathfrak{U}_2$. Then $\tau^{*\prime}$ is a partial cochain realization of the complex $\mathfrak{U}_1$ on $\mathfrak{U}_2$ of norm less than $\mathfrak{E}_n(\mathfrak{U}_0\,; S, S)$. Consequently, by Lemma 4.1, there exists a cochain realization τ^* of $\mathfrak{U}_1$ on a covering $\mathfrak{U}_3$—the $\mathfrak{D}_n(\mathfrak{U}_2\,, \mathfrak{U}_0\,; S, S)$ of Lemma 4.1—of norm less than $\mathfrak{U}_0$.

Now let γ^{n-r} be any C-cycle of S. Then $\tau^*\gamma^{n-r}(\mathfrak{U}_1) = \gamma_r(\mathfrak{U}_3)$ is a cocycle of $\mathfrak{U}_3$. We let $\gamma_r \frown \gamma^n = Z^{n-r}$. To show that $\gamma^{n-r} \sim \eta Z^{n-r}$, $\eta = \pm 1$, it is sufficient to show that $\gamma^{n-r}(\mathfrak{U}_0) \sim \eta Z^{n-r}(\mathfrak{U}_0)$; confer the proof of Theorem VI 3.1. In order to do this, consider the complex K obtained from $\mathfrak{U}_1$ together with a complex $\mathfrak{U}_1'$ which is isomorphic with $\mathfrak{U}_1$, and the deformation complex $\mathfrak{D}\mathfrak{U}_1$(V 6) associated with the mapping of $\mathfrak{U}_1$ into $\mathfrak{U}_1'$ which maps each element of $\mathfrak{U}_1$ into its corresponding element in $\mathfrak{U}_1'$. We call $\mathfrak{U}_1$ and $\mathfrak{U}_1'$ the base and top of K, respectively.

Denoting corresponding chains on $\mathfrak{U}_1$ and $\mathfrak{U}_1'$ by C and C' respectively, we recall that by Lemma V 6.7 if z^i is a cycle of $\mathfrak{U}_1$, then there exists a chain

[4]A homomorphism φ is linear if $a\varphi(x) = \varphi(ax)$, $x \in \mathfrak{F}$; cf. V 9.1.

$C^{i+1}(=\mathfrak{D}z^i)$ on K such that $\partial C^{i+1} = z^i - z'^i$. Let τ' be a chain-mapping of K into $\mathfrak{U}_1$ defined as follows: On $\mathfrak{U}_1$, τ' is the identity, and for any chain C'^i on the top of K, we let $\tau' C'^i = (-1)^k \pi_3^1(\tau^* C^i \frown \gamma^n(\mathfrak{U}_3))$ if i is an odd number $2k - 1$, or an even number $2k$. (The power of -1 in the coefficients is inserted to ensure that τ' satisfies the conditions previously imposed (VI 2.1) on chain-mappings. The symbol π_3^1 is used for typographical reasons to denote a projection from $\mathfrak{U}_3$ to $\mathfrak{U}_1$.) Then τ' is a partial realization of K on $\mathfrak{U}_1$ of norm less than $\mathfrak{U}_n^*(\mathfrak{U})$, and consequently τ' can be extended to a realization τ of K on $\mathfrak{U}_1 \cup \mathfrak{U}_0$.

Now, as noted above, there exists a chain C^{n-r+1} of K such that $\partial C^{n-r+1} = \gamma^{n-r}(\mathfrak{U}_1) - \gamma'^{n-r}(\mathfrak{U}_1')$. Then $\partial \tau C^{n-r+1} = \gamma^{n-r}(\mathfrak{U}_1) - (-1)^m \pi_3^1(\tau^* \gamma^{n-r}(\mathfrak{U}_1) \frown \gamma^n(\mathfrak{U}_3)) = \gamma^{n-r}(\mathfrak{U}_1) - \eta \pi_3^1 Z^{n-r}(\mathfrak{U}_3)$, $\eta = \pm 1$. As τC^{n-r+1} is on $\mathfrak{U}_1 \cup \mathfrak{U}_0$ and $\mathfrak{U}_1 > \mathfrak{U}_0$, we may let π be a projection of $\mathfrak{U}_1 \cup \mathfrak{U}_0$ into $\mathfrak{U}_0$ of the obvious type, and obtain $\pi\tau C^{n-r+1}$ on $\mathfrak{U}_0$ such that $\partial \pi\tau C^{n-r+1} = \pi_1^0 \gamma^{n-r}(\mathfrak{U}_1) - \eta \pi_1^0 \pi_3^1 Z^{n-r}(\mathfrak{U}_3) \sim \gamma^{n-r}(\mathfrak{U}_0) - \eta Z^{n-r}(\mathfrak{U}_0)$. Hence $\gamma^{n-r}(\mathfrak{U}_0) \sim \eta Z^{n-r}(\mathfrak{U}_0)$.

5. **The open n-gm.** For the open manifold, the Poincaré duality takes the form of an isomorphism between $\mathfrak{H}^r(S)$ and $h^{n-r}(S)$, in case either of these is of finite dimension, that is, between the r-dimensional infinite homology group and the $(n - r)$-dimensional compact homology group, defined in §2 above. In general, the isomorphism is between $h^{n-r}(S)$ and a subspace $\mathfrak{H}_f^r(S)$ of $\mathfrak{H}^r(S)$ which is defined below, and which for finite dimension is identical with $\mathfrak{H}^r(S)$. Our problem is chiefly to indicate what changes are necessary in the material of §§3 and 4 in order to adapt it to the proof of the new form of the duality. *We consider only the connected n-gm until* §5.20.

Considering first the preliminary material of §3, we must define what is meant by saying that an infinite cycle is on a proper closed subset of S. An infinite cycle, Γ^n, being an inverse limit, has a coordinate, Γ_ν^n, on every open set U_ν whose closure is compact. We make the convention that if this coordinate is 0 for some ν, then Γ^n is on a proper closed subset, $S - U_\nu$, of S. Condition D then takes the form:

D′. *If F is a proper closed subset of S, then every infinite n-cycle on F bounds on S.*

(For the general n-gm, we shall ask that D′ hold for each component.)

We can then prove:

5.1 Theorem. *If S is an open n-gm satisfying D′, then either all infinite n-cycles of S bound on S, or there exists exactly one lirh nonbounding infinite n-cycle.*

Proof. Suppose Γ_1^n, Γ_2^n are infinite cycles that are lirh on S. Let $x \in S$, and P, Q open sets such that $x \in Q \subset P$, $\overline{P}$ compact, and $p_n(x; P, Q) = 1$ (condition C of the definition of n-gm). Then there exists a relation

$$(5.1a) \qquad a_1 \Gamma_{1P}^n + a_2 \Gamma_{2P}^n \sim 0 \quad \mod S - Q,$$

where Γ_{1P}^n, Γ_{2P}^n are the respective "coordinates" of Γ_1^n, Γ_2^n on P. As S is n-

dimensional, we may suppose all cycles on a complete family of n-dimensional coverings, so that homology implies identity. But then (5.1a) implies that $a_1\Gamma_1^n + a_2\Gamma_2^n$ is on a proper closed subset of S, and hence bounds on S, contradicting the assumption that Γ_1^n and Γ_2^n are lirh on S. Hence S cannot carry more than one nonbounding infinite n-cycle.

5.2 As a consequence of Theorem 5.1, just as in the case of the n-gcm, we may assume that an orientable open n-gm has a unique nonbounding infinite cycle Γ^n, which we call its *fundamental cycle*. And if P is an open set such that $\overline{P}$ is compact, then by Theorem 2.10 there exists in P a compact cocycle γ_n such that $\Gamma^n \cdot \gamma_n = 1$, which we call a *fundamental cocycle in P*. And analogous to the case of the n-gcm, we have:

5.3 THEOREM. *All fundamental cocycles of an orientable open n-gm lie in the same compact cohomology class of S.*

As a consequence of Theorem 5.3 we may, whenever we have a relation $a_1\gamma_n^1 \backsim a_2\gamma_n^2$, where γ_n^1 and γ_n^2 are fundamental cocycles, assume that $a_1 = a_2 = 1$.

We may now prove the duality for open manifolds. We first note, however, in contrast to the case of the n-gcm, all of whose Betti groups are of finite dimension, that the Betti groups encountered in the study of the open manifold are generally infinite. Consequently, in order to obtain the particular type of duality which we need for our later purposes, we shall assume that the space S is the union of a countable number of compact sets. This will mean, then, that in later applications of the theory of open manifolds to the study of open subsets of the n-gcm, we shall have to assume that the open subsets of the n-gcm have the above property—in other words, that the n-gcm is a perfectly normal space (V 20.1).

We precede the proof of the duality for open manifolds with some lemmas.

5.4 LEMMA. *Let S be an orientable open n-gm with fundamental cycle Γ^n, and let γ^{n-r} be a compact cycle on a compact set L, which bounds on a compact set L_1. Then if Q, P are open sets containing L, L_1, respectively, there is a compact cocycle $\gamma_r = \tau^*\gamma^{n-r}$ in Q such that $\gamma_r \frown \Gamma^n \sim \gamma^{n-r}$ in Q and $\gamma_r \backsim 0$ in P.*

PROOF. We first show the existence of a γ_r satisfying the condition $\gamma_r \frown \Gamma^n \sim \gamma^{n-r}$ in Q. Let Q_1 and Q_2 be open sets such that $L \Subset Q_2 \Subset Q_1 \Subset Q$, and let $\mathfrak{U}$ be a covering of S such that $\mathrm{St}(\overline{Q}_2, \mathfrak{U}) \subset Q_1$. Also, let $\mathfrak{E} >^* \mathfrak{U}_n^*(\mathfrak{U}; \overline{Q}_2, Q_1)$—the covering introduced in Theorem VI 3.6—and such that St $(L, \mathfrak{E}) \subset Q_2$. Let $\mathfrak{U}_0$ be the covering introduced in Corollary VI 3.7, relative to the sets Q and $\overline{Q}_1$, and let $\mathfrak{U}_1$ be an n-dimensional refinement of $\mathfrak{U}_0$, such that $\mathfrak{U}_1 >^* \mathfrak{E}_n(\mathfrak{E}; L, Q_2)$, $\mathfrak{U}_1 > \mathfrak{B}_n^*(\mathfrak{U}, \mathfrak{U}_0; \overline{Q}_2, Q_1)$, $\mathfrak{U}_1 >^* \mathfrak{E}_n(\mathfrak{E}; \overline{Q}_2, Q_1)$ the latter being coverings defined in Lemma 4.1 and Theorem VI 3.6.

In each element of $\mathfrak{U}_1$ there is a fundamental cocycle of S, and we may assume these are all on a $\mathfrak{U}_2 > \mathfrak{U}_1$. Let τ'^* assign to each vertex of $\mathfrak{U}_1$ the corresponding fundamental cocycle on $\mathfrak{U}_2$. Then there exists a cochain realiza-

tion τ^* of $\mathfrak{U}_1 \wedge \bar{Q}_2$ of norm $< \mathfrak{E}$ on $\mathfrak{U}_3 \wedge Q_1$, where $\mathfrak{U}_3$ is a refinement of the $\mathfrak{D}_n(\mathfrak{U}_2, \mathfrak{E}; L, Q_2)$ and $\mathfrak{D}_n(\mathfrak{U}_2, \mathfrak{E}; \bar{Q}_2, Q_1)$ of Lemma 4.1 such that $\tau^*\gamma^{n-r}(\mathfrak{U}_1) = \gamma_r(\mathfrak{U}_3)$ is a cocycle in Q_2. Then $\gamma_r(\mathfrak{U}_3)$ is a compact cocycle of S.

To show that the cycle $Z^{n-r} = \gamma_r \frown \Gamma^n \sim \gamma^{n-r}$ in Q, we first form a complex K from $\mathfrak{U}_1 \wedge \bar{Q}_2$ just as the complex K in the proof of Theorem 4.2 was formed from $\mathfrak{U}_1$. And as in the proof just cited, we make a partial realization τ' of K on $\mathfrak{U}_1$ of norm less than $\mathfrak{U}_n^*(\mathfrak{U}; \bar{Q}_2, Q_1)$ which can be extended to a realization of K on $(\mathfrak{U}_1 \cup \mathfrak{U}_0) \wedge Q_1$. The chain τC^{n-r+1} will now be on $(\mathfrak{U}_1 \cup \mathfrak{U}_0) \wedge Q_1$, and $\gamma^{n-r}(\mathfrak{U}_0) \sim Z^{n-r}(\mathfrak{U}_0)$ on $\bar{Q}_1$. And by Corollary VI 3.7 this is sufficient in order that $\gamma^{n-r} \sim Z^{n-r}$ in Q.

Finally, if $\gamma^{n-r} \sim 0$ on L_1, then there is a chain C^{n-r+1} on $\mathfrak{U}_1 \wedge L_1$ such that $\partial C^{n-r+1} = \gamma^{n-r}(\mathfrak{U}_1)$. Then $\tau^* C^{n-r+1} = C^{r-1}$ is a chain on $\mathfrak{U}_3$ such that $\delta C^{r-1} = \delta\tau^* C^{n-r+1} = \tau^*\partial C^{n-r+1} = \tau^*\gamma^{n-r}(\mathfrak{U}_1) = \gamma_r(\mathfrak{U}_3)$. Hence $\gamma_r(\mathfrak{U}_3) \backsim 0$, and as C^{n-r+1} is on L_1, it is only necessary to choose the coverings employed above "sufficiently small" in order to make $\gamma_r(\mathfrak{U}_3) \backsim 0$ in P.

5.5 Definition. If P and Q are open sets such that $P \supset Q$, then by the symbol $h^r(S\colon Q; P)$ we denote the group of compact cycles in Q reduced modulo the subgroup of those that bound in P.

5.6. Lemma. *If P and Q are open subsets of an orientable n-gm S, such that $\bar{P}$ is compact and $Q \Subset P$, then there exists a linear homomorphism of $H_r(S; Q, 0; P, 0)$ onto $h^{n-r}(S\colon Q; P)$.*

Proof. Let γ_r be a compact cocycle in Q. Then $\varphi(\gamma_r) = \gamma_r \frown \Gamma^n = \gamma^{n-r}$, where Γ^n is the fundamental cycle of S, is a compact cycle of Q. And if γ_r cobounds on a compact subset of P, then $\varphi(\gamma_r)$ bounds on a compact subset of P. Hence φ induces a homomorphism Φ of $H_r(S; Q, 0; P, 0)$ into $h^{n-r}(S\colon Q; P)$.

Let γ^{n-r} be a compact cycle of Q. Then by Lemma 5.4, there exists a compact cocycle $\gamma_r = \tau^*\gamma^{n-r}$ in Q such that $\varphi(\gamma_r) \sim \gamma^{n-r}$ in Q.

5.7 Lemma. *With P and Q as in Lemma 5.6, there exist open sets P', Q' such that $P \Supset P' \Supset Q \Supset Q'$ and such that if A, B are closed sets such that $P \supset A \supset P'$, $Q \supset B \supset Q'$, then $h^{n-r}(S\colon Q; P)$, $H_{n-r}(S\colon S, S - A; S, S - B)$, $H^{n-r}(S\colon B, 0; A, 0)$ and $h^{n-r}(S\colon Q'; P')$ are all linearly isomorphic.*

Proof. Let $\{z_i^{n-r}\}$ be a collection of compact cycles in Q, finite in number, that form a basis for homologies in P. Consider any open set W such that $P \Supset W \Supset Q$. To form a basis for compact cycles in Q relative to homologies in W, we may add to $\{z_i^{n-r}\}$ a finite set $\{\gamma_s^{n-r}\}$ of compact cycles of Q. Now each $\gamma_s^{n-r} \sim \sum a_i z_i^{n-r}$ in P; let F_s be a compact subset of P on which this homology relation holds. Then we may select P' so as to contain $\bigcup F_s$ and so that $P \Supset P' \Supset W$. Evidently $h^{n-r}(S\colon Q; P) = h^{n-r}(S\colon Q; P') = h^{n-r}(S\colon Q; A)$, where A is any closed set such that $P \supset A \supset P'$, and $h^{n-r}(S\colon Q; A)$ is the group of compact cycles in Q reduced modulo the subgroup of those that have coordinates bounding on A.

Let B_i be a compact subset of Q that carries z_i^{n-r}, and let Q' be an open set such that $Q \supset Q' \supset \bigcup B_i$. Then $h^{n-r}(S: Q; P) = h^{n-r}(S: Q; A) = H^{n-r}(S: B, 0; A, 0) = h^{n-r}(S: Q; P') = h^{n-r}(S: Q'; P')$.

By Theorem V 18.30, $H_{n-r}(S: S, S - A; S, S - B) = H^{n-r}(S: B, 0; A, 0)$ and the lemma now follows from the above relations.

5.8 Lemma. *With P and Q as before, there exist P' and Q' such that $P \supset P' \supset Q \supset Q'$ and such that if A', B' are open sets such that $P \supset \bar{A}' \supset P'$, $Q \supset \bar{B}' \supset Q'$, then there exists a linear homomorphism of $H_{n-r}(S: S, S - \bar{A}'; S, S - \bar{B}')$ onto $H^r(S: S, S - P; S, S - Q)$ $[= H^r(S: S, S - P'; S, S - Q')]$.*

Proof. By Theorems V 18.31 and 1.1, $H^r(S: S, S - P; S, S - Q) = H_r(S: Q, 0; P, 0)$. Let $\{z_r^i\}$ be a finite collection of cocycles in Q forming a basis relative to cohomologies in P. By virtue of Lemma V 8.7 we may assume each z_r^i has a carrier whose closure lies in Q. Then there exists an open set Q' such that $Q \supset Q'$ and such that Q' contains a set of carriers of the cocycles z_r^i. If W is an open set such that $P \supset W \supset Q$, a basis of compact cocycles of Q relative to cohomologies in W may be formed by adding to $\{z_r^i\}$ a finite collection of cocycles $\{\gamma_r^s\}$ of Q. Each γ_r^s is cohomologous to a linear combination of the z_r^i's in an open set whose closure lies in P, and we may select $P' \supset W$ so as to contain such a subset of P for each of the γ_r^s. Then $H_r(S: Q, 0; P, 0) = H_r(S: Q', 0; P', 0) = H_r(S: \bar{B}', 0; A', 0)$ for any such open sets A', B' as in the statement of the lemma. Consequently by Theorem V 18.31, $H^r(S: S, S - P; S, S - Q) = H^r(S: S, S - P'; S, S - Q') = H^r(S: S, S - A'; S, S - B')$. We note, incidentally, that if $\{\Gamma_i^r\}$ forms a basis of cycles mod $S - P$ relative to homologies mod $S - Q$, then the Γ_i^r, considered as cycles mod $S - P'$, form a basis relative to homologies mod $S - Q'$.

With A', B' as above, consider a cocycle Z_{n-r} mod $S - \bar{A}'$. Let $\varphi(Z_{n-r}) = Z_{n-r} \frown \Gamma^n = \Gamma^r$, where Γ^n is the fundamental cycle of S. Inasmuch as $(\delta Z_{n-r}) \frown \Gamma^n$ is on $S - A'$, the chain Γ^r is a cycle mod $S - A'$. If $Z_{n-r} \backsim 0$ mod $S - \bar{B}'$, then there exists a chain C^{n-r-1} of some refinement $\mathfrak{B}$ of the covering carrying Z_{n-r}, such that $\delta C^{n-r-1} = Z_{n-r} + L_{n-r}$, where L_{n-r} is in $S - \bar{B}'$, and again it follows from the formula for the boundary of a cap product that $\partial(C^{n-r-1} \frown \Gamma^n(\mathfrak{B})) = (-1)^{r+1} Z_{n-r}(\mathfrak{B}) \frown \Gamma^n(\mathfrak{B})$ mod $S - \bar{B}'$, so that $\Gamma^r \sim 0$ mod $S - B'$. Hence φ induces a homomorphism of $H_{n-r}(S: S, S - \bar{A}'; S, S - \bar{B}')$ into $H^r(S: S, S - A'; S, S - B')$.

To show that this homomorphism is onto, we may consider φ as a homomorphism of $H_{n-r}(S: S, S - \bar{A}'; S, S - \bar{B}')$ into $H^r(S: S, S - P; S, S - Q)$. Let Γ^r be a cycle mod $S - P$. As $S - P$ and $\bar{A}'$ are nonintersecting closed sets, there is a covering $\mathfrak{U}$ such that no element of $\mathfrak{U}$ meets both $S - P$ and $\bar{A}'$. We now use the methods of the proof of Theorem 4.2. Let $\tau^*\Gamma^r = \gamma_{n-r}$, a chain of $\mathfrak{U}_3$. Since $\tau^*\partial\Gamma^r$ is on $S - \bar{A}'$, γ_{n-r} is a cocycle mod $S - \bar{A}'$. The chain $Z^r = \tau^*\Gamma^r \frown \Gamma^n$ is a cycle mod $S - A'$. That $Z^r \sim \Gamma^r$ mod $S - B'$ may be shown by a virtual paraphrase of the corresponding part of the proof of Theorem 4.2.

We can now state, as a result of Theorem V 18.31 and Lemmas 5.6-5.8:

5.9 THEOREM. *Let P and Q be open sets in an orientable n-gm S such that $P \Supset Q$ and $\bar{P}$ is compact. Then there exist open sets P', Q' such that* (1) *$P \Supset P' \Supset Q \Supset Q'$ and* (2) *if A and B are open sets such that $P \supset \bar{A} \supset P'$, $Q \supset \bar{B} \supset Q'$, then $H^r(S: S, S - A; S, S - B) = H^{n-r}(S: \bar{B}, 0; \bar{A}, 0)$.*

5.10 Now suppose a space S is the union of a countable set of open sets $Q_1, \cdots, Q_i, \cdots$ such that $\bar{Q}_i$ is compact and $\bar{Q}_i \subset Q_{i+1}$ for all i. Then by a *fundamental system of infinite r-cycles of S* we shall mean a collection decomposable into sets $\mathcal{P}_i^r$ of infinite r-cycles such that (1) $\mathcal{P}_i^r \subset \mathcal{P}_{i+1}^r$, (2) the elements of $\mathcal{P}_i^r$ form a base for infinite r-cycles of S relative to homologies mod $S - Q_i$. As we shall see immediately below, for an n-gm the sets $\mathcal{P}_i^r$ are finite, so that the collection $\bigcup \mathcal{P}_i^r$ is countable.

5.11 The subspace of $\mathfrak{H}^r(S)$ generated by cosets corresponding to a fundamental system of infinite r-cycles will be denoted by $\mathfrak{H}_f^r(S)$ and called the *rth infinite fundamental homology group of S over $\mathfrak{F}$.* For finite-dimensional $\mathfrak{H}^r(S)$, $\mathfrak{H}_f^r(S)$ and $\mathfrak{H}^r(S)$ are the same.

5.12 THEOREM. *If S is an n-gm that is the union of a countable collection of compact point sets, then $\mathfrak{H}_f^r(S)$ is of countable dimension for all r. Moreover, there exist* (1) *open sets $P_1, \cdots, P_k, \cdots$ whose union is S such that $\bar{P}_k$ is compact, $\bar{P}_k \subset P_{k+1}$, and* (2) *a fundamental system of infinite cycles $\Gamma_1^r, \cdots, \Gamma_i^r, \cdots$ such that for each k, $\Gamma_1^r, \cdots, \Gamma_{i(k)}^r$ form a base for infinite cycles of S relative to homologies* mod $S - P_k$.

PROOF. The existence of the sets P_k follows easily from the fact that S is the union of a countable collection of compact point sets.

By Theorems V 18.31 and 1.1, $p^r(S: S, S - P_2; S, S - P_1)$ is finite. In particular, only finitely many infinite cycles of S can be lirh mod $S - P_1$, and there exists a finite set of infinite cycles, $\Gamma_1^r, \cdots, \Gamma_{i(1)}^r$ forming a base for infinite cycles relative to homologies mod $S - P_1$. Passing to P_2, a base for infinite cycles relative to homologies mod $S - P_2$ may be formed by augmenting the collection $\Gamma_1^r, \cdots, \Gamma_{i(1)}^r$ by cycles $\Gamma_{i(1)+1}^r, \cdots, \Gamma_{i(2)}^r$, by virtue of Lemma V 18.26; and so on.

REMARK. Since an open set with compact closure must eventually lie in some P_k, if an infinite cycle Γ^r fails to bound on S, then there exists k such that $\Gamma^r \not\sim 0$ mod $S - P_k$. Hence $\Gamma^r \sim \sum_{i=1}^{i(k)} a^i \Gamma_i^r$, $a^i \in \mathfrak{F}$, mod $S - P_k$. This does not imply that $\Gamma^r \sim \sum_{i=1}^{i(k)} a^i \Gamma_i^r$ *on* S, however; for a homology $\Gamma^r \sim 0$ on S implies that $\Gamma^r \sim 0$ mod $S - P_k$ *for all* k.

5.13 THEOREM. *Under the hypothesis of Theorem 5.12, if $\mathfrak{H}_f^r(S)$ is of finite dimension, then $\mathfrak{H}_f^r(S) = \mathfrak{H}^r(S)$.*

PROOF. If $\mathfrak{H}_f^r(S)$ is of finite dimension, then the cycles Γ_i^r of Theorem 5.12 form a finite sequence $\Gamma_1^r, \cdots, \Gamma_{i(k)}^r$, and if Γ^r is any infinite cycle of S, $\Gamma^r \sim$

$\sum_{i=1}^{i(k)} a^i\Gamma_i^r$ mod $S - P_{k+t}$ for all natural numbers t. Since if U_ν is any open subset of S with compact closure there exists t such that $P_{k+t} \supset U_\nu$, it is clear that $\Gamma^r \sim \sum_{i=1}^{i(k)} a^i\Gamma_i^r$ mod $S - U_\nu$. The cycles $\Gamma_1^r, \cdots, \Gamma_{i(k)}^r$ are lirh mod $S - P_k$, hence lirh on S by definition. Thus they form a base for $\mathfrak{H}^r(S)$.

5.14 THEOREM. *Let S be an orientable n-gm which is the union of a countable collection of compact point sets. Then $\mathfrak{H}_f^r(S) = h^{n-r}(S)$ for $0 \leqq r \leqq n$.*

PROOF. As in the proof of Theorem 5.12, we may express S as $\bigcup_{k=1}^\infty P_k$ where (1) P_k is open, (2) $\bar{P}_k$ is compact, and (3) $\bar{P}_k \subset P_{k+1}$.

Now dim $h^{n-r}(S)$ cannot be an uncountable cardinal. For it it were, then, since F compact implies a k such that $F \subset P_k$, there would exist a k such that P_k contains coordinates of an uncountable set of lirh compact cycles of S. But by Corollary VI 3.8 at most a finite number of compact cycles of P_k are lirh in P_{k+1}.

Now suppose $h^{n-r}(S)$ is of finite dimension, m. Let $\gamma_1^{n-r}, \cdots, \gamma_m^{n-r}$ be lirh compact cycles forming an $(n - r)$-dimensional homology basis and let k be any integer large enough so that P_k contains carriers of all the cycles γ_i^{n-r}. Now $h^{n-r}(S\colon P_k ; P_{k+1})$ is of finite dimension, and since the cycles γ_i^{n-r} are a fortiori lirh in P_{k+1}, we may select cycles $z_1^{n-r}, \cdots, z_h^{n-r}$ on compact subsets of P_k in such a way that every compact cycle of P_k is related to the system $\gamma_1^{n-r}, \cdots, \gamma_m^{n-r}, z_1^{n-r}, \cdots, z_h^{n-r}$ by a homology on a compact subset of S. It follows that there exists an integer j such that only the cycles γ_i^{n-r} are lirh in P_{k+j} and consequently that $h^{n-r}(S\colon P_k ; P_{k+j})$ has dimension m. Then for all integers $s > j$, the dimensions of $H^r(S\colon S, S - P_{k+s} ; S, S - P_k)$ and $H^{n-r}(S\colon \bar{P}_k, 0; \bar{P}_{k+s}, 0)$ are m, by Theorem 5.9, and it follows that dim $\mathfrak{H}^r(S) \leqq m$, and consequently dim $\mathfrak{H}_f^r(S) \leqq m$.

Next suppose $\mathfrak{H}_f^r(S)$ is of finite dimension, h. Then by Theorem 5.13, dim $\mathfrak{H}^r(S) = h$. By Theorems 2.10 and V 18.16, $\mathfrak{H}^r(S) = h_r(S)$, so that dim $h_r(S) = h$. Let $\gamma_r^1, \cdots, \gamma_r^h$ be compact cocycles of S, linearly independent relative to cohomology in S. Let i be any integer large enough so that P_i contains carriers of all the cocycles $\gamma_r^1, \cdots, \gamma_r^h$. By reasoning similar to that employed in the preceding paragraph, there exists an integer t such that dim $H_r(S\colon P_i, 0; P_{i+t}, 0) = h$, and this holds for all integers $t' \geqq t$. By Lemma 5.6 there exists a homomorphism of $H_r(S\colon P_i, 0; P_{i+t}, 0)$ onto $h^{n-r}(S\colon P_i ; P_{i+t})$. It follows that dim $h^{n-r}(S\colon P_i ; P_{i+t}) \leqq h$. And because of the fact that i and t may be taken as large as we please, it follows that dim $h^{n-r}(S) \leqq h$.

Thus if either $h^{n-r}(S)$ or $\mathfrak{H}_f^r(S)$ is of finite dimension, then the other is of finite, but not greater, dimension. Hence if either is of finite dimension, their dimensions are equal. And since if both are of infinite dimension these dimensions are aleph-null, the required duality follows.

5.15 LEMMA. *Let S be an orientable n-gm which is the union of a countable collection of compact point sets. Then $\mathfrak{H}_f^r(S) = h_r(S)$.*

PROOF. We showed in the proof of Theorem 5.12 that dim $\mathfrak{H}_f^r(S)$ is at most

a denumerable cardinal. That dim $h_r(S)$ is at most denumerable follows as in the case of dim $h^r(S)$, except that the fact that S has property $(P, Q)_r$ (cf. Theorem 1.1) is used instead of Corollary VI 3.8.

Now if either $\mathfrak{H}_f^r(S)$ or $h_r(S)$ is of finite dimension, then $\mathfrak{H}_f^r(S) = h_r(S)$ by Theorems 5.13, 2.10 and V 18.16. The only remaining case, then, is that where both $\mathfrak{H}_f^r(S)$ and $h_r(S)$ are of dimension aleph-null.

As a corollary of Theorem 5.14 and Lemma 5.15, we now have:

5.16 LEMMA. *Under the same hypothesis as in Lemma* 5.15, $h_r(S) = h^{n-r}(S)$.

5.17 A *fundamental system of infinite r-cocycles* of S may be defined analogous to that for the infinite r-cycles. If $P_1, \cdots, P_i, \cdots$ are as in Theorem 5.12, we apply Corollary VI 3.9 to obtain a base $\Gamma_r^1, \cdots, \Gamma_r^{n(1)}$ of infinite r-cocycles relative to cohomology mod $S - \bar{P}_1$; augment this to obtain a base relative to cohomology mod $S - \bar{P}_2$; and so on. The corresponding space $\mathfrak{H}_r^f(S)$ is then defined as the subspace of $\mathfrak{H}_r(S)$ generated by the homology classes of the Γ's.

5.18 LEMMA. *Under the same hypothesis as in Lemma* 5.15, $h^r(S) = \mathfrak{H}_r^f(S)$.

PROOF. It was shown in the proof of Theorem 5.14 that dim $h^r(S)$ is at most denumerable. The lemma now follows easily from Theorem 2.5.

As a corollary of Theorem 5.14 and Lemma 5.18 we have:

5.19 LEMMA. *Under the same hypothesis as in Lemma* 5.15, $\mathfrak{H}_r^f(S) = \mathfrak{H}_f^{n-r}(S)$.

5.20 REMARK. In the applications, particularly to open subsets of an orientable n-gcm, the above results will continue to hold whenever the number of components is countable. This will be the case whenever the n-gcm is perfectly normal, as will be shown later in another connection (Theorem XI 2.17). It is necessary to show, however, that in such a case condition D′ holds for each component:

5.21 THEOREM. *If S is a perfectly normal orientable n-gcm, and M a closed subset of S, then every component of $S - M$ satisfies condition* D′.

PROOF. Let C be a component of $S - M$, and Z^n an infinite cycle of C that is on a closed (rel. C) proper subset F' of C. By Lemma 6.1 below we may consider Z^n to be a cycle mod $S - C = M'$. Let $C - F' = U_\nu$. By Lemma VII 2.3, there exists a minimal closed subset F of $S - U_\nu$ containing M' such that $Z^n \sim 0 \bmod F$. By Lemma VII 2.6, F is unique and a closed carrier of Z^n.

Suppose $F \cap C \neq 0$. Then there exists $x \in C \cap F$ such that x lp $C - F$. Let U and V be open subsets of C such that $x \in V \subset U$ and $p^n(x; U, V) = 1$. Then there exists a relation

$$aZ^n + b\gamma^n \sim 0 \qquad \text{mod } S - V, \qquad a, b \in \mathfrak{F}, \tag{5.20a}$$

where γ^n is the fundamental cycle of S and not both a and b are zero. Now

$a \neq 0$ by Lemma VII 3.6; and if $b = 0$, (5.20a) implies F is not minimal since (5.20a) can be considered an identity, inasmuch as S is n-dimensional; and with $a \neq 0 \neq b$, (5.20a) implies $\gamma^n = -a/bZ^n$ on V. This is impossible, since the set $(S - F) \cap V$ is not empty and S carries γ^n irreducibly. We conclude, then, that $Z^n = 0$ and D′ is satisfied.

5.22 REMARK. For the sake of completeness, it should be stated that hereafter the general open n-gm S is considered orientable only when each of its components satisfies D′ and S carries a nonbounding infinite cycle that is not homologous to a cycle on a closed proper subset of S.

6. The Alexander type of duality for a closed subset of an n-gcm. First proof. If P is an open subset of an n-gm such that $\overline{P}$ is compact, and P is an F_σ, then a group $H_f^r(S; S, S - P)$ may be defined relative to P in the same manner as $\mathfrak{H}_f^r(S)$ was defined relative to S, except that instead of a fundamental system of infinite cycles of S, we set up a fundamental system of cycles mod $S - P$. We first prove the following lemma:

6.1 LEMMA. *If P is an open subset of an n-gm such that $\overline{P}$ is compact and P is an F_σ then there exist natural isomorphisms $\mathfrak{H}^r(P) = H^r(S; S, S - P)$, and $\mathfrak{H}_f^r(P) = H_f^r(S; S, S - P)$.*

($\mathfrak{H}^r(P)$ and $\mathfrak{H}_f^r(P)$ are determined from P just as $\mathfrak{H}^r(S)$ and $\mathfrak{H}_f^r(S)$ are determined from S.)

PROOF. If Γ^r is a cycle mod $S - P$, then Γ^r determines an infinite cycle on P as follows: Consider any covering $\mathfrak{U}(P)$ of P by a finite number of open sets. As the elements of $\mathfrak{U}(P)$ are also open in S, and $F(P)$, the boundary of P, is compact, there exists an obvious covering $\mathfrak{U}$ of S by a finite number of open sets consisting of the elements of $\mathfrak{U}(P)$ and of a collection $\{U_i'\}$ such that each U_i' contains points of $S - P$ and does not cover any element of $\mathfrak{U}(P)$. The coordinate $\Gamma_\nu^r(\mathfrak{U})$ of an element $\Gamma_\nu^r(U_\nu)$ of an infinite cycle Γ^r of P on $\mathfrak{U}(P)$ may then be determined from the coordinate of the given Γ^r on $\mathfrak{U}$. It is easily shown that such a collection of coordinates yields an infinite cycle of P. And if the cycle Γ^r mod $S - P$ is homologous to zero mod $S - P$, the corresponding infinite cycle Γ^r of P will be a bounding cycle. There exists, then, an obvious homomorphism Φ of $H^r(S; S, S - P)$ into $\mathfrak{H}^r(P)$.

Consider an infinite cycle Γ^r of P. By the definition of infinite cycle, the element Γ_k^r of Γ^r on P_k is a cycle mod $P - P_k$ such that $\Gamma_k^r = \Gamma_{k+1}^r \bmod S - P_k$. If $\mathfrak{U}$ is a covering of S, the collection $\mathfrak{U} \cap P$ is a covering of P, and there exists an integer $k = k(\mathfrak{U})$ such that $\mathfrak{U} \cap P_{k(\mathfrak{U})} = \mathfrak{U} \cap P$ and hence such that $\Gamma_{k(\mathfrak{U})}^r(\mathfrak{U} \cap P) = \Gamma_{k(\mathfrak{U})+i}^r(\mathfrak{U} \cap P)$ for all $i > 0$. Let $\gamma^r(\mathfrak{U}) = \Gamma_{k(\mathfrak{U})}^r(\mathfrak{U} \cap P)$. Then $\gamma^r(\mathfrak{U})$ is a cycle mod $S - P$ on $\mathfrak{U}$, since $\partial\gamma^r(\mathfrak{U})$ lies on $S - P_{k(\mathfrak{U})}$ and hence on $S - P$. And if $\mathfrak{V} > \mathfrak{U}$, $\pi_{\mathfrak{U}\mathfrak{V}}\gamma^r(\mathfrak{V}) \sim \gamma^r(\mathfrak{U}) \bmod S - P$, since with $k > \max(k(\mathfrak{U}), k(\mathfrak{V}))$ and the fact that Γ_k^r is a cycle mod $S - P_k$, we have $\pi_{\mathfrak{U}\mathfrak{V}}\Gamma_k^r(\mathfrak{V}) \sim \Gamma_k^r(\mathfrak{U}) \bmod S - P_k$, which implies the former homology. Evidently if Γ_1^r, Γ_2^r are infinite

cycles in the same homology class, the corresponding cycles γ_1^r, γ_2^r mod $S - P$ will be homologous mod $S - P$.

The proof that $\mathfrak{H}_f^r(P) = H_f^r(S; S, S - P)$ is left to the reader.

6.2 Lemma. *Let S be a space such that $p_a^r(S) = p_a^{r+1}(S) = 0$. If M is any closed subset of S, then $H_a^r(M) = H_a^{r+1}(S; S, M)$.*

Proof. Consider a cycle (augmented) γ_1^r on M. As $p_a^r(S) = 0$, $\gamma_1^r \sim 0$ on S, and by Lemma VII 1.4 there exists a cycle Γ_1^{r+1} mod M such that

$$\partial\Gamma_1^{r+1} \sim \gamma_1^r \qquad \text{on } M. \tag{6.2a}$$

Now suppose that Γ_2^{r+1} is any other cycle mod M such that $\partial\Gamma_2^{r+1} \sim \gamma_1^r$ on M. Then since $\partial(\Gamma_1^{r+1} - \Gamma_2^{r+1}) \sim 0$ on M, there exists by Lemma VII 1.6 a cycle Γ^{r+1} such that $\Gamma^{r+1} \sim \Gamma_1^{r+1} - \Gamma_2^{r+1}$ mod M. By hypothesis, $\Gamma^{r+1} \sim 0$ on S, and hence $\Gamma_1^{r+1} - \Gamma_2^{r+1} \sim 0$ mod M. Therefore by the above process a unique homology class of $H_a^{r+1}(S; S, M)$ is determined, and denoting homology classes by parentheses, we define a mapping $\Phi : (\gamma_1^r) \to (\Gamma_1^{r+1})$. Evidently this defines a homomorphism $\Phi : H_a^r(M) \to H_a^{r+1}(S: S, M)$.

The mapping Φ is "onto." For let $(\Gamma^{r+1}) \in H_a^{r+1}(S; S, M)$. Then by Lemma VII 1.1, $\partial\Gamma^{r+1}$ is a cycle γ^r of M, and we can use Γ^{r+1} and $\partial\Gamma^{r+1}$ as the Γ_1^{r+1} and γ_1^r of relation (6.2a), giving that $\Phi(\partial\Gamma^{r+1}) = (\Gamma^{r+1})$.

Finally, Φ is (1-1). Suppose $\Phi(\gamma_1^r) = \Phi(\gamma_2^r)$. Take Γ_1^{r+1}, Γ_2^{r+1} such that $\partial\Gamma_i^{r+1} \sim \gamma_i^r$ on M, $i = 1, 2$, Γ_i^{r+1} being a cycle mod M. By hypothesis, $\Gamma_1^{r+1} \sim \Gamma_2^{r+1}$ mod M and, by Lemma VII 1.5, $\partial(\Gamma_1^{r+1} - \Gamma_2^{r+1}) \sim 0$ on M. Hence $\partial\Gamma_1^{r+1} \sim \partial\Gamma_2^{r+1}$ on M, implying that $\gamma_1^r \sim \gamma_2^r$ on M.

Remark. The augmented case of the "f-spaces" is again denoted by the addition of an index "a"; such as $H_{fa}^r(M)$, for instance.

6.3 Lemma. *If M is a compact G_δ and S is lc^r, then the isomorphism $\Phi : H_a^r(M) \to H_a^{r+1}(S; S, M)$ established in Lemma 6.2, when applied to $H_{fa}^r(M)$, is an isomorphism $\Phi : H_{fa}^r(M) \to H_{fa}^{r+1}(S; S, M)$.*

Proof. Let $\{\gamma_i^r\}$ be a fundamental system of (augmented) r-cycles of M determined relative to a system of open sets $\{U_k\}$ as in Theorem VI 5.1. For $i = 1, \cdots, n(1)$, let $\partial\Gamma_i^{r+1} = \gamma_i^r$ on S. The cycles Γ_i^{r+1}, mod M, are lirh mod $\overline{U}_1$. For if there exists a relation $\sum_{i=1}^{n(1)} a^i\Gamma_i^{r+1} \sim 0$ mod $\overline{U}_1$, then by Lemma VII 1.9, there exists a cycle Z^{r+1} mod M on $\overline{U}_1$ such that

$$\sum_{i=1}^{n(1)} a^i\Gamma_i^{r+1} \sim Z^{r+1} \qquad \text{mod } M. \tag{6.3a}$$

Relation (6.3a) implies (Lemma VII 1.2) that $\partial(\sum_{i=1}^{n(1)} a^i\Gamma_i^{r+1}) \sim \partial Z^{r+1}$ on M; i.e., $\partial Z^{r+1} \sim \sum_{i=1}^{n(1)} a^i\gamma_i^r$ on M. But since Z^{r+1} lies on $\overline{U}_1$, this implies that $\sum_{i=1}^{n(1)} a^i\gamma_i^r \sim 0$ on $\overline{U}_1$, contradicting the fact that the cycles γ_i^r are lirh on $\overline{U}_1$.

Now suppose Γ^{r+1} is any cycle mod M. Let $\gamma^r = \partial\Gamma^{r+1}$. Then $\gamma^r \sim \sum_{i=1}^{n(1)} b^i\gamma_i^r$ on U_1. Hence by Lemma VII 1.6, there exists a cycle Z^{r+1} such that $Z^{r+1} \sim \Gamma^{r+1}$

$-\sum_{i=1}^{n(1)} b^i\Gamma_i^{r+1} \bmod \overline{U}_1$. Since $p_a^{r+1}(S) = 0$, $Z^{r+1} \sim 0$ on S and therefore $\Gamma^{r+1} \sim \sum_{i=1}^{n(1)} b^i\Gamma_i^{r+1} \bmod \overline{U}_1$.

Hence the Γ_i^{r+1}, $i = 1, \cdots, n(1)$, form a base of $(r+1)$-cycles mod M relative to homologies mod $\overline{U}_1$. The remainder of the proof should be clear.

The duality theorem of Alexander type for an n-gcm may now be stated in the following form:

6.4 THEOREM. *Let S be a perfectly normal orientable n-gcm and r a non-negative integer $< n$ such that $p_a^r(S) = p_a^{r+1}(S) = 0$, if $r < n - 1$, and $p_a^{n-1}(S) = 0$ if $r = n - 1$ (or $p_a^1(S) = 0$; cf. Theorem 4.2). If M is a closed subset of S then*

$$H_{fa}^r(M) = h_a^{n-r-1}(S - M).$$

Case $r < n - 1$. By Lemma 6.3, $H_{fa}^r(M) = H_{fa}^{r+1}(S; S, M)$, and by Lemma 6.1, $H_{fa}^{r+1}(S; S, M) = \mathfrak{H}_{fa}^{r+1}(S - M)$. By Lemma VII 3.6, if γ^n is the fundamental cycle of S, $\gamma^n \nsim 0 \bmod M$ and hence by Lemma 6.1, there is a non-bounding infinite n-cycle on $S - M$. Thus $S - M$ is an orientable n-gm and by Theorem 5.14, $\mathfrak{H}_{fa}^{r+1}(S - M) = h_a^{n-r-1}(S - M)$.

Case $r = n - 1$. By Corollary 3.2 and condition C, the hypothesis of Theorem VII 5.10 is satisfied and consequently M is n-extendible (VII 5.1) at every point. Hence Theorem VII 5.9 applies, with $m = 1$, and $p^{n-1}(M) = k - 1$ (k may $= \infty$). By Theorem VII 5.7 and Corollary VII 5.8, $k = p_n(S; S - M, 0)$. That is, $\dim H_f^{n-1}(M) = \dim H_n(S; S - M, 0) - 1$, which by Theorem 2.12 $=$ $\dim h_n(S - M) - 1$; and the latter, by Lemma 5.16, $= \dim h^0(S - M) - 1 =$ $\dim h_a^0(S - M)$.

7. The Alexander type of duality for a closed subset of an n-gcm. Second proof. In this section we give a proof of the Alexander type of duality which brings out interrelationships of a type not discernible in the proof given above. For the new proof we need the following theorem:

7.1 THEOREM. *If S is a locally compact, lc^r space such that for some r, $p_r^a(S) = p_{r+1}^a(S) = 0$, M is a compact G_δ subset of S, and $\{\gamma_r^i\}$ is a base for cocycles mod $S - M$ relative to cohomologies mod $S - M$, determined as in Theorem* VI 5.8, *then the cocycles $\gamma_{r+1}^i = \delta\gamma_r^i$ form a base for compact $(r+1)$-cocycles of $S - M$ relative to cohomologies in $S - M$. Incidentally, then, $H_r^a(M) =$ $h_{r+1}^a(S - M)$.*

PROOF. With sets U_k as in Theorem VI 5.8, if γ_{r+1} is a compact cocycle of $S - M$, then there exists U_m such that γ_{r+1} is a compact cocycle of $S - \overline{U}_m$. Since $p_{r+1}(S) = 0$, there exists a covering $\mathfrak{U}$ and a chain Γ_r of $\mathfrak{U}$ such that $\delta\Gamma_r = \gamma_{r+1}(\mathfrak{U})$. Then Γ_r is a cocycle mod $S - \overline{U}_m$ and because of the manner in which the cocycles γ_r^i were determined in Theorem VI 5.8, there exists a cohomology

$$\Gamma_r \backsim \sum_{j=1}^{n(m)} a_j\gamma_r^j \qquad \bmod S - M. \tag{7.1a}$$

Relation (7.1a) implies the existence of a chain C_{r-1} such that

$$\delta C_{r-1} = \Gamma_r - \sum_{j=1}^{n(m)} a_j\gamma_r^j + C_r , \tag{7.1b}$$

where C_r is in $S - M$ (we leave out the symbols for coverings and projections for sake of brevity; we may assume all coverings employed to be of the type $\mathfrak{B}$ of Lemma V 8.7). Applying δ to both members of (7.1b), recalling $\delta^2 = 0$, we get

$$\delta C_r = \sum_{j=1}^{n(m)} a_j\gamma_{r+1}^j - \gamma_{r+1} . \tag{7.1c}$$

Relation (7.1c) implies that $\gamma_{r+1} \backsim \sum_{j=1}^{n(m)} a_j\gamma_{r+1}^j$ in $S - M$.

The cocycles γ_{r+1}^j are lircoh in $S - M$. For suppose not. Then there exists a relation $\delta C_r = \sum_{j=1}^{n(k)} a_j\gamma_{r+1}^j$ in $S - M$. But then $\sum_{j=1}^{n(k)} a_j\gamma_r^j - C_r$ is a cocycle of S and, since $p_r(S) = 0$, is a cobounding cocycle. But as C_r is in $S - M$, this would imply $\sum_{j=1}^{n(k)} a_j\gamma_r^j \backsim 0 \bmod S - M$, in contradiction to the fact that the cocycles γ_r^j are lircoh mod $S - M$.

7.2 The second proof of the duality proceeds as follows: In case $r < n - 1$, it follows from Theorem V 18.31, Theorem 1.1, and Lemmas 5.6-5.8 (with $P = S - M$, $Q = S - \overline{U}_m$), that $H_{r+1}(S\colon S - \overline{U}_m , 0; S - M, 0) = h^{n-r-1}(S\colon S - \overline{U}_m ; S - M)$, where the sets M, U_m are as above. We recall (cf. proof of Lemma 5.6) that this isomorphism is induced by a mapping $\gamma_{r+1} \rightarrow \gamma_{r+1} \frown \Gamma^n$, where Γ^n is the fundamental cycle of S. In particular, the cycles $\gamma_i^{n-r-1} = \gamma_{r+1}^i \frown \Gamma^n$, $i \leqq n(m)$, form a base for cycles of $S - \overline{U}_m$ relative to homologies in $S - M$. The collection $\{\gamma_i^{n-r-1}\}$ determined in this way for all U_m determines a base for $h^{n-r-1}(S - M)$. For (1) any compact cycle of $S - M$ lies in some $S - \overline{U}_m$, and (2) any homology relating a finite number of the cycles γ_i^{n-r-1} in $S - M$ would hold on some $S - \overline{U}_m$, and imply nonindependence of the corresponding cocycles γ_{r+1}^i in the isomorphism referred to above. The theorem now follows, for this case, from Theorem VI 5.8 and the above Theorem 7.1.

7.3 Now the chiefly noteworthy feature of the above proof is that it sets up the desired isomorphism in a natural way; to each γ_i^r of the base of cycles on M is made to correspond a cycle γ_i^{n-r-1} of $S - M$ in such a way that the cycles γ_i^{n-r-1} form a base for $(n - r - 1)$-cycles of $S - M$ relative to homologies in $S - M$. It would be desirable likewise to determine a base for the augmented 0-cycles of $S - M$ in an analogous manner. In order to accomplish this, we need an analogue of Theorem 7.1 for the case $r = n - 1$:

7.4 THEOREM. *If S is an orientable n-gm such that $p_{n-1}(S) = 0$, M is a compact G_δ subset of S, and $\{\gamma_{n-1}^i\}$ is a base for cocycles* mod $S - M$ *relative to cohomologies* mod $S - M$ *determined as in Theorem* VI 5.8, *then the cocycles $\gamma_n^i = \delta\gamma_{n-1}^i$, together with a fundamental cocycle Γ_n in $S - M$, form a base for*

compact n-cocycles of $S - M$ *relative to cohomologies in* $S - M$. *Incidentally,* $p_{n-1}(M) = \dim h_n(S - M) - 1$.

PROOF. Let γ_n be a compact cocycle of $S - M, \not\sim 0$ in $S - M$. There exists U_m as before such that γ_n and Γ_n are in $S - \overline{U}_m$. As Γ_n forms a base for cocycles of S relative to cohomology on S, $\gamma_n \sim a\Gamma_n$ on S. Accordingly, on some covering $\mathfrak{U}$, there exists a relation

$$\text{(7.2a)} \qquad \delta C_{n-1} = \gamma_n - a\Gamma_n .$$

Let L_{n-1} be the portion of C_{n-1} on $\overline{U}_m$. Then L_{n-1} is a cocycle mod $S - \overline{U}_m$, and therefore $L_{n-1} \sim \sum_{i=1}^{n(m)} a_i\gamma_{n-1}^i$ mod $S - M$. Hence on some $\mathfrak{B} > \mathfrak{U}$, there exists a relation

$$\text{(7.2b)} \qquad \delta C_{n-2} = \pi^*_{\mathfrak{U}\mathfrak{B}} L_{n-1} - \sum_{i=1}^{n(m)} a_i\gamma_{n-1}^i + Q_{n-1}, \qquad Q_{n-1} \text{ in } S - M.$$

Applying δ to both sides of (7.2b),

$$\text{(7.2c)} \qquad \pi^*_{\mathfrak{U}\mathfrak{B}}\delta L_{n-1} + \delta Q_{n-1} = \sum_{i=1}^{n(m)} a_i\delta\gamma_{n-1}^i .$$

From (7.2a) and (7.2c) it follows that

$$\text{(7.2d)} \qquad \delta(\pi^*_{\mathfrak{U}\mathfrak{B}}C_{n-1} - \pi^*_{\mathfrak{U}\mathfrak{B}}L_{n-1} - Q_{n-1}) = \pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_n - a\pi^*_{\mathfrak{U}\mathfrak{B}}\Gamma_n - \sum_{i=1}^{n(m)} a_i\gamma_n^i .$$

Relation (7.2d) implies that $\gamma_n \sim a\Gamma_n + \sum_{i=1}^{n(m)} a_i\gamma_n^i$ in $S - M$.

That the cocycles γ_n^i are lircoh in $S - M$ follows as in the last paragraph of the proof of Theorem 7.1. Now suppose that there exists a relation

$$\text{(7.2e)} \qquad \delta C^{n-1} = \sum a_i\gamma_n^i + a\Gamma_n \qquad \text{in } S - M.$$

Then not all the a_i are zero, since Γ_n cannot cobound in $S - M$. And $a \neq 0$, since the γ_n^i are lircoh in $S - M$. We may therefore rewrite (7.2e) in the form

$$\text{(7.2f)} \qquad \delta C^{n-1} = \sum a_i\gamma_n^i + \Gamma_n \qquad \text{in } S - M.$$

Then, since $\delta\gamma_{n-1}^i = \gamma_n^i$, we have from (7.2f) that $\delta(C^{n-1} - \sum a_i\gamma_{n-1}^i) = \Gamma_n$, contradicting the fact that Γ_n does not cobound on S.

7.5 *Case* $r = n - 1$. This may now be established by noting that the cycles $\gamma_i^0 - \gamma^0$, where $\gamma_i^0 = \gamma_n^i \frown \Gamma^n$, $\gamma^0 = \Gamma_n \frown \Gamma^n$ form a base for (augmented) cycles of $S - M$ relative to homology in $S - M$.

7.6 *Inverse form of the second proof.* We now raise the question: Can we not reverse the above procedure, starting with a base for cycles of $S - M$ and making correspond thereto in a natural manner a fundamental system for cycles of M? It is easy to see that the answer is affirmative. Indeed, in case $r < n - 1$, starting with a base $\gamma_1^{n-r-1}, \cdots, \gamma_{n(1)}^{n-r-1}$ for cycles of $S - \overline{U}_1$ relative to homology in $S - M$, we pass to the set $\gamma_{r+1}^i = \tau^*\gamma_i^{n-r-1}$, $i = 1, \cdots, n(1)$ which forms a base for $(r + 1)$-cocycles in $S - \overline{U}_1$ relative to cohomology in

$S - M$. Since $p_{r+1}(S, \mathfrak{F}) = 0$, there exist relations $\delta C_r^i = \gamma_{r+1}^i$ on S. The chains C_r^i are cocycles mod $S - \overline{U}_1$ and are easily proved to form a base for r-cocycles of $\overline{U}_1$ relative to cohomology mod $S - M$ by methods similar to those used above. Hence there exists by Theorem V 18.30 a base of cycles γ_j^r on M, $j = 1, \cdots, n(1)$, such that $C_r^i \cdot \gamma_j^r = \delta_i^j$. The cycles γ_j^r form a base for r-cycles of M relative to homology on $\overline{U}_1$.

In the case $r = n - 1$, a base for (augmented) 0-cycles of $S - \overline{U}_1$, lirh in $S - M$, consisting of $n(1)$ nontrivial 0-cycles, determines exactly $n(1) + 1$ components D_j, $j = 1, \cdots, n(1) + 1$, of $S - M$, that meet $S - \overline{U}_1$ (cf. Theorem V 11.10). Let $\Gamma_n^0, \Gamma_n^1, \cdots, \Gamma_n^{n(1)}$ be fundamental cocycles such that Γ_n^i lies in $D_i \cap (S - \overline{U}_1)$. Since $\Gamma_n^0 \backsim \Gamma_n^i$ on S, $i = 1, \cdots, n(1)$, there exist chains C_{n-1}^i such that $\delta C_{n-1}^i = \Gamma_n^0 - \Gamma_n^i$. The chains C_{n-1}^i are cocycles mod $S - \overline{U}_1$, lircoh mod $S - M$, and there exist cycles γ_i^{n-1} of M such that $\gamma_i^{n-1} \cdot C_{n-1}^j = \delta_i^j$. We leave further details of the proof to the reader.

8. **Linking theorems.** The significance of the second proof of the duality above becomes plainer if we consider a relationship of cycles which may be called "geometric linking." Here we consider only subsets of a gcm S and we assume that S is spherelike in the dimensions considered (otherwise we are in a position similar to that in which we would be if we discussed cut points of nonconnected sets!).

8.1 Definition. A cycle γ^r will be said to *link* a set of points M in S if γ^r is on a compact subset of $S - M$ and does not bound on a compact set therein. (Compare the use of the term "link" in the euclidean case, as in the proof of Theorem II 5.22.)

8.2 Definition. If γ^r and γ^{n-r-1} are cycles with disjoint closed carriers, and neither bounds in the complement of a closed carrier of the other, then we say that γ^r and γ^{n-r-1} are *linked* in S. Two homology classes $\{\gamma^r\}$, $\{\gamma^{n-r-1}\}$ of compact M, $S - M$, respectively, will be called linked in S if every element of the one is linked in S with every element of the other.

(In the case of both 8.1 and 8.2, we shall frequently leave out the phrase "in S" whenever S is the space under consideration.)

8.3 Theorem. *Let S and M satisfy the hypothesis of Theorem* 6.4. *Then in the isomorphism between $H_{fa}^r(M)$ and $h_a^{n-r-1}(S - M)$ established in the second proof (and its inverse) of Theorem* 6.4, *corresponding homology classes are linked in S.*

Proof. We may as well consider the classes corresponding to γ_1^r and γ_1^{n-r-1} in the notation used above. Let $\gamma^r \sim \gamma_1^r$ on M and $\gamma^{n-r-1} \sim \gamma_1^{n-r-1}$ on a closed subset of $S - M$. Let N be a carrier of γ^{n-r-1} in $S - M$. We shall show that γ^r and γ^{n-r-1} are linked in S. Suppose there is a closed carrier C of γ^{n-r-1} such that $\gamma^r \sim 0$ on a closed subset T of $S - C$; evidently we may suppose $C \cap N \subset S - \overline{U}_k$ for some k. Let Q be an open set such that $C \cap N \subset Q \Subset S - (\overline{U}_k \cup T)$.

By Lemma 5.4 there exists a cocycle $\gamma_{r+1} = \tau^*\gamma^{n-r-1}$ in Q such that $\gamma_{r+1} \frown \Gamma^n \sim \gamma^{n-r-1}$ in $S - M$. Now from the isomorphism between $H_{r+1}(S: S - \overline{U}_k, 0; S - M, 0)$ and $h^{n-r-1}(S: S - \overline{U}_k ; S - M)$ (7.2) as established above by products of type $\gamma_{r+1} \frown \Gamma^n$, we know that $\gamma^{n-r-1} \sim \gamma_1^{n-r-1}$ in $S - M$ implies $\gamma_{r+1} \backsim \gamma_{r+1}^1$ in $S - M$.

If $\mathfrak{U}$ is a covering on which γ_{r+1} , γ_{r+1}^1 and γ_r^1 have representatives, there exists a relation $\delta C_r(\mathfrak{B}) = \pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_{r+1} - \pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_{r+1}^1$ in $S - M$. This relation together with the relation $\delta\pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_r^1 = \pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_{r+1}^1$ gives $\delta[C_r(\mathfrak{B}) + \pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_r^1] = \pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_{r+1}$. Since γ_{r+1} is in $S - (\overline{U}_k \cup T)$, the portion of $C_r(\mathfrak{B}) + \pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_r^1$ on $\overline{U}_k \cup T$ is a cocycle Z_r mod $S - (\overline{U}_k \cup T)$. And evidently $Z_r \cdot \gamma_1^r = \gamma_r^1 \cdot \gamma_1^r = 1$. But $\gamma_1^r \sim \gamma^r \sim 0$ on $M \cup T$, so that $Z_r \cdot \gamma_1^r = 0$.

Suppose that there is a closed carrier K of γ^r such that $\gamma^{n-r-1} \sim 0$ on a closed set $J \subset S - K$. Let P be an open set such that $J \subset P \subset S - K$. This time we select U_k so that $J \cap N \subset S - \overline{U}_k$ and the cocycle $\gamma_{r+1} = \tau^*\gamma^{n-r-1}$ in $P \cap (S - \overline{U}_k)$. Note that $\gamma_{r+1} \backsim 0$ in P. (See Lemma 5.4.) Let us take $C_r(\mathfrak{B})$ as before, and let $\delta L_r(\mathfrak{B}) = \pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_{r+1}$ in P. Then $\pi^*_{\mathfrak{U}\mathfrak{B}}\gamma_r^1 + C_r(\mathfrak{B}) - L_r(\mathfrak{B})$ is a cocycle of S, and as $p_r(S, \mathfrak{F}) = 0$, there is a covering $\mathfrak{W}$ such that

$$\text{(8.3a)} \qquad \pi^*_{\mathfrak{U}\mathfrak{W}}\gamma_r^1 + \pi^*_{\mathfrak{B}\mathfrak{W}}C_r(\mathfrak{B}) - \pi^*_{\mathfrak{B}\mathfrak{W}}L_r(\mathfrak{B}) \backsim 0 \qquad \text{on } S.$$

Now since $\pi^*_{\mathfrak{B}\mathfrak{W}}C_r(\mathfrak{B})$ is in $S - M$, relation (8.3a) implies that $\gamma_r^1 - Z_r$, where Z_r is the portion of $L_r(\mathfrak{B})$ on M, is a cobounding cocycle mod $S - M$. Hence $(\gamma_r^1 - Z_r) \cdot \gamma^r = 0$. However, $Z_r \cdot \gamma^r = 0$, and therefore $(\gamma_r^1 - Z_r) \cdot \gamma^r = \gamma_r^1 \cdot \gamma^r = \gamma_r^1 \cdot \gamma_1^r = 1$.

We may state, then, as a result of the second proof of Theorem 6.4 and its inverse, that

8.4 Theorem. *If cycles γ_i^r , $i = 1, 2, \cdots$, form a fundamental system of r-cycles for the set M of Theorem 6.4, then a base Z_i^{n-r-1}, $i = 1, 2, \cdots$, for compact cycles of $S - M$ relative to homologies in $S - M$ can be selected in such a manner that the homology classes $\{\gamma_i^r\}$, $\{Z_i^{n-r-1}\}$ are linked for each value of i. Conversely, if cycles Z_i^{n-r-1}, $i = 1, 2, \cdots$, form a base for cycles of $S - M$ relative to homologies in $S - M$, there can be found a fundamental system of r-cycles, γ_i^r , $i = 1, 2, \cdots$, of M such that the homology classes $\{\gamma_i^r\}$, $\{Z_i^{n-r-1}\}$ are linked.*

8.5 Theorem. *Let M and S be as in Theorem 6.4. Then if γ_i^r , $i = 1, 2, \cdots$, form a fundamental system of r-cycles of M, every nonbounding $(n - r - 1)$-cycle of $S - M$ is linked with some cycle γ_i^r ; and if Z_i^{n-r-1}, $i = 1, 2, \cdots$, form a base for compact cycles of $S - M$ relative to homologies in $S - M$, then every nonbounding r-cycle of M is linked with some cycle Z_i^{n-r-1}.*

Proof. Given the base of cycles Z_i^{n-r-1}, let a fundamental system γ_i^r , $i = 1, 2, \cdots$, of r-cycles of M be determined as in the inverse form of the second proof of Theorem 6.4. Let γ^r be a nonbounding cycle of M. Then $\gamma^r \sim \sum_{i=1}^{\infty} a_i\gamma_i^r$ on M. Let a_m be the coefficient of smallest subscript that is not zero, and suppose $n(k - 1) < m \leqq n(k)$ in the notation used above. Then

with $\overline{U}_k$ as in the proof of Theorem 8.3 it may be shown that γ^r and Z_m^{n-r-1} are linked.

Similarly, given the fundamental system of cycles γ_i^r, we may determine the base of cycles Z_i^{n-r-1} as before, and if Z^{n-r-1} is a nonbounding cycle of $S - M$, then $Z^{n-r-1} \sim \sum_{i=1}^h a_i Z_i^{n-r-1}$ in $S - M$. And if a_m is a nonzero coefficient, then Z^{n-r-1} is linked with γ_m^r.

8.6 Corollary. *With M and S as in Theorem* 6.4, *if γ^r is a nonbounding cycle of M, then γ^r is linked with a compact cycle Z^{n-r-1} of $S - M$. And if Z^{n-r-1} is a compact cycle of $S - M$ that fails to bound in $S - M$, then Z^{n-r-1} is linked with a cycle γ^r of M.*

Evidently the above method of proof also shows:

8.7 Corollary. *If systems $\{\gamma_i^r\}$ and $\{Z_i^{n-r-1}\}$ are selected as in Theorem* 8.4, *then every finite linear combination of cycles of one system is linked with a cycle of the other system.*

Sometimes, in the sequel, we shall be concerned not with a complete fundamental system of cycles of a closed set, but with a set determined in the following manner: Let M be compact, and G^r a subgroup of the group of r-cycles of M. In the above proofs we may confine our attention to cycles of G^r in the following fashion: In Theorem VI 5.8, let $\gamma_1^r, \cdots, \gamma_{n(1)}^r$ be a set of cycles of G^r lirh on U_1 and such that if γ^r is a cycle of G^r, then γ^r is homologous on U_1 to a linear combination of the cycles $\gamma_1^r, \cdots, \gamma_{n(1)}^r$, etc. In the manner indicated there may be obtained what might be called a fundamental system of cycles of G^r relative to homologies on M.

8.8 Theorem. *If M and S are as in Theorem* 6.4, *and G^r is a group of r-cycles of M, then there exists a fundamental system, γ_i^r, $i = 1, 2, \cdots$, of cycles of G^r relative to homologies on M, and a system of lirh compact cycles Z_i^{n-r-1}, $i = 1, 2, \cdots$, of $S - M$ such that the homology classes $\{\gamma_i^r\}$, $\{Z_i^{n-r-1}\}$ are linked for each value of i. Moreover, every finite linear combination of cycles of the system $\{\gamma_i^r\}$ is linked with an element of $\{Z_i^{n-r-1}\}$ and conversely. Indeed, if Z^{n-r-1} is a compact cycle of $S - M$ such that $Z^{n-r-1} \sim \sum_{j=1}^m c^j Z_{i(j)}^{n-r-1}$ in $S - M$, where each $c^j \neq 0$, then Z^{n-r-1} is linked with each of the cycles $\gamma_{i(j)}^r$. Similarly, if $\gamma^r \sim \sum_{j=1}^m d^j \gamma_{i(j)}^r$ on M, $d^j \neq 0$, then γ^r is linked with each of the cycles $Z_{i(j)}^{n-r-1}$.*

Similarly, we have the following theorem:

8.9 Theorem. *If M and S are as before, and E is a subgroup of the set of all compact $(n - r - 1)$-cycles of $S - M$, then there exists a complete set $\{Z_i^{n-r-1}\}$ of cycles of E lirh relative to homologies in $S - M$ and a system $\{\gamma_i^r\}$ of cycles of M such that the respective homology classes $\{\gamma_i^r\}$, $\{Z_i^{n-r-1}\}$ in M and $S - M$ are linked for each value of i. More generally, if Z^{n-r-1} is a cycle of E such that $Z^{n-r-1} \sim \sum_{j=1}^m c^j Z_{i(j)}^{n-r-1}$ in $S - M$, where each $c^j \neq 0$, then Z^{n-r-1} is linked with each of the cycles $\gamma_{i(j)}^r$. Similarly, if $\gamma^r \sim \sum_{j=1}^m d^j \gamma_{i(j)}^r$ on M, $d^j \neq 0$, then γ^r is linked with each of the cycles $Z_{i(j)}^{n-r-1}$.*

Finally, if $\gamma_1^r, \cdots, \gamma_k^r$ are lirh cycles of M, finite in number, there exists (Theorem V 19.7) an open set U_1 containing M such that these cycles are lirh on U_1. Hence the cycles $\gamma_1^r, \cdots, \gamma_k^r$ may be included in a fundamental system of r-cycles for M. On the other hand, if $Z_1^{n-r-1}, \cdots, Z_k^{n-r-1}$ are lirh compact cycles of $S - M$, then for a choice of U_1 such that $S - \overline{U}_1$ contains compact carriers of all these $(n - r - 1)$-cycles, the latter are lirh in $S - \overline{U}_1$. We can state, then,

8.10 THEOREM. *If M and S are as before, and $\gamma_1^r, \cdots, \gamma_k^r$ are lirh cycles of M, then there exists a fundamental system of r-cycles for M of which the cycles $\gamma_1^r, \cdots, \gamma_k^r$ are elements. Consequently there exist compact cycles $Z_1^{n-r-1}, \cdots, Z_k^{n-r-1}$ in $S - M$ such that γ_i^r and Z_i^{n-r-1} are linked, $i = 1, \cdots, k$.* Conversely, *if $Z_1^{n-r-1}, \cdots, Z_k^{n-r-1}$ are lirh compact cycles of $S - M$, then there exists a base for compact cycles of $S - M$ having these cycles as elements, and consequently there exist cycles γ_i^r of M such that γ_i^r and Z_i^{n-r-1} are linked, $i = 1, \cdots, k$.*

9. **A duality for nonclosed sets.** The duality theorems of the last three sections relate only to the cycles of a *compact* subset of a manifold and its complement. For certain purposes it is convenient to have a duality relating compact cycles of an *arbitrary* subset of a manifold to compact cycles of its complement.

9.1 THEOREM. *Let M be an arbitrary subset of a perfectly normal orientable n-gcm S and r a nonnegative integer $< n$ such that if $r < n - 1$, $p_a^r(S) = p_a^{r+1}(S) = 0$ and otherwise $p_a^{n-1}(S) = 0$. If γ_i^r, $i = 1, \cdots, k$, are compact cycles of M that are linearly independent relative to unrestricted homologies on M, then there exist compact cycles Z_i^{n-r-1}, $i = 1, \cdots, k$, of $S - M$ such that γ_i^r and Z_i^{n-r-1} are linked.*

PROOF. It follows from Theorem VI 4.6 that there exists an open subset P of S containing M such that the cycles γ_i^r are linearly independent in P relative to homologies on compact subsets of P. Hence by Theorem 8.10, there exist compact cycles Z_i^r of $S - P \subset S - M$ of the type desired.

REMARK. Since M is arbitrary, there is obviously no need of a "converse" case (covering the case where the cycles Z_i^r are given instead of γ_i^r) as in Theorem 8.10.

BIBLIOGRAPHICAL NOTES

§1. The definition of n-gm as given here is that of Begle [b] with an unnecessary axiom (requiring the space to be lc^n) deleted. Earlier definitions given by Čech [b, c, e], Lefschetz [c], Wilder [n] and Alexandroff-Pontrjagin [i] did not utilize the (at the time nonexistent) theory of cocycles.

§2. Compare Begle [b].

§4. References to Poincaré's papers and the original form of the Poincaré duality may be found in Veblen [V]. The proof given here is based on Begle [a].

§§5, 6, 8. For proofs of Theorems 5.14, 6.4, and 8.3 in the case where the Betti numbers of S or M are all finite, see Begle [b]. The original statement and proof of the Alexander duality theorem will be found in Alexander [a]. Subsequent extensions, as for instance to closed subsets of a manifold, appeared in Alexandroff [c], Frankl [a], Lefschetz [b, L_2], and Pontrjagin [a, b]. If M is a classical manifold and K a homeomorph in M of some complex, then Pontrjagin showed in [d] that there exists an Alexander type of duality between the cycles of K that bound in M and the cycles of $M - K$ that bound in M. See also Alexandroff (k). For a discussion of an extension of his duality to spaces of infinite dimension, see Alexander [d]. The first proofs of dualities for generalized manifolds appeared in Čech [b] and Lefschetz [c]. See also Lefschetz [L; VI] and Alexandroff [j]. Euclidean analogues of linking theorems of the type given in §8 will be found in Pontrjagin [c].

§9. Theorem 9.1 was stated and proved for arbitrary subsets of S^n by S. Kaplan in his dissertation [a, b].

CHAPTER IX

FURTHER PROPERTIES OF n-GMS; REGULAR MANIFOLDS AND GENERALIZED CELLS

In this chapter we shall prove further "justification" theorems for the n-gcm, establish certain avoidability properties, and discuss some equivalent or alternate definitions.

1. **Case $n = 1$.** Since the S^1 is the only 1-dimensional closed manifold in the classical sense, we would like that the metric case of the 1-gcm also reduce to the S^1. Without the metric requirement, however, one may expect the 1-gcm's to present an infinite variety of topological categories, differing, for example, in their properties in the large, such as in the cardinality of a minimal base for open sets, as well as in their properties in the small, such as the cardinality of local bases ("character" at a point).

Turning then to the metric case, let M be a metric 1-gcm. By Theorem VIII 1.1, M is a Peano space. As such, it cannot be acyclic (III 3.31), since if it were it would have an end point p, and $p_1(M, p) = 0$, by the following lemma.

1.1 LEMMA. *If M is an acyclic Peano space, and p is an end point of M, then* $p_1(M, p) = 0$.

PROOF. Let $\epsilon > 0$ be arbitrary, and let x be a point of M such that $M - x = M_1 \cup M_2$ separate and $p \in M_1 \subset S(p, \epsilon)$. Let δ be any positive number $<\epsilon$ such that $M_2 \cap S(p, \delta) = 0$. Let Z^1 be any cycle mod $M - S(p, \epsilon)$.

Now Z^1 is a cycle mod $\overline{M_2}$ and consequently by Corollary VII 1.16 there exists a cycle γ^1 mod (x) on $\overline{M}_1$ such that $\gamma^1 \sim Z^1$ mod $\overline{M}_2$. Since $\partial\gamma^1$ is on x, hence ~ 0 on x, there exists by Lemma VII 1.6 a cycle Γ^1 on $\overline{M}_1$ such that $\Gamma^1 \sim \gamma^1$ mod $\overline{M}_2$ on $\overline{M}_1$. But $\Gamma^1 \sim 0$ on M, since M is acyclic, hence ~ 0 mod $\overline{M}_2$. The combination of these homologies gives $Z^1 \sim 0$ mod $\overline{M}_2$, and a fortiori mod $M - S(p, \delta)$. Hence $p^1(M, p) = p_1(M, p) = 0$.

Continuing with M as above, in the remarks preceding Lemma 1.1, we know by Lemma 1.1 that M contains a simple closed curve J. And by condition D of Chapter VIII, it follows at once that $M = J$. Thus we have:

1.2 THEOREM. *The metric case of the 1-gcm reduces to the 1-sphere.*

REMARKS. As a matter of fact, condition D is not needed in the 1-dimensional case if it be assumed that M is connected. For if $M \neq J$, let A be an arc of M having only one of its end points, x, in common with J. Given a neighborhood P of x, let A' be a subarc of A which has x as one end point, has an end point y on $F(P)$, and lies entirely in P except for the point y. Let B be an arc

of J containing x and lying, except for its end points x' and y', which are on $F(P)$, entirely in P. Let Q be an arbitrary neighborhood of x lying in P.

Then B carries a cycle Z_2^1 mod $M - P$ such that $Z_2^1 \not\sim 0$ mod $M - Q$ since M is 1-dimensional. Likewise, the arc consisting of the portion of B from x' to x, together with A', carries a similar cycle Z_1^1. The cycles Z_1^1 and Z_2^1 are lirh mod $S - Q$ because of the 1-dimensionality of M, and hence $p^1(M, p) \geqq 2$. Hence *a 1-dimensional metric continuum M such that* $p_1(M, x) = 1$ *at every point $x \in M$ must be an S^1* (cf. Corollary VI 6.12).

One may go even further than this, as a matter of fact, since, as Alexandroff has shown [f, p. 12, Corollary 2], a 1-dimensional metric continuum M such that $p^1(M, x)$ has the *same finite value* at each $x \in M$ is an S^1.

2. **Case** $n = 2$. For the orientable case of the 2-gcm the proof that the separability condition restricts it to the classical case is very simple, and derives immediately from the following lemma which, being fundamental in our later developments, may as well be introduced at this point. The nonorientable case requires more extended treatment and will be deferred to a later section.

2.1 LEMMA. *An orientable n-gcm is n-extendible at every point.*

PROOF. This is a direct consequence of Theorem VII 5.10.

2.2 COROLLARY. *Every point of an orientable n-gcm is a locally* $(n - 1)$-*avoidable point.*

PROOF. This is a consequence of Lemma VII 5.3 and the above lemma.

2.3 THEOREM. *The metric case of the orientable 2-gcm reduces to the classical closed 2-manifold.*

PROOF. This is a direct consequence of Theorem VII 4.23 and Corollary 2.2.

3. **Avoidability properties.** We saw above that the orientable n-gcm is locally $(n - 1)$-avoidable at every point. We next show that it is completely r-avoidable at every point for $r < n - 1$.

3.1 LEMMA. *Let S be a locally compact space and $x \in S$ such that* (1) *S is r-lc at x and* (2) $p^{r+1}(x) = 0$. *Then S is completely r-avoidable at x.*

PROOF. Let $x \in S$ and P an open set containing x. Then there exist open sets Q and R such that $x \in R \Subset Q \Subset P$ and such that (1) r-cycles on Q bound on P and (2) $p^{r+1}(x; Q, R) = 0$.

Let γ^r be a cycle of $F(Q)$. Then $\gamma^r \sim 0$ on a compact subset M of $\overline{P}$, and accordingly by Lemma VII 1.4, there exists a cycle Z^{r+1} mod $F(Q)$ on M such that

$$\partial Z^{r+1} \sim \gamma^r \qquad \text{on } F(Q). \tag{3.1a}$$

By (2) of the preceding paragraph, $Z^{r+1} \sim 0$ mod $S - R$, and therefore by

Corollary VII 1.17, there exists a cycle γ^{r+2} mod $M \cup (S - R)$ on $\bar{R}$ such that $\partial\gamma^{r+2} \sim Z^{r+1}$ mod $S - R$ on M. It follows easily that Z^{r+1} is homologous mod $F(Q)$ on P to a cycle γ^{r+1} mod $F(Q)$ which lies on $\bar{P} - R$, and by Lemma VII 1.2, $\partial Z^{r+1} \sim \partial\gamma^{r+1}$ on $F(Q)$. Hence, by relation (3.1a), $\gamma^r \sim 0$ on $\bar{P} - R$, and S is completely r-avoidable at x.

3.2 Corollary. *If S is an n-gm and $r < n - 1$, then S is completely r-avoidable at every point.*

And now to prepare for a complete characterization of the orientable n-gcm by means of avoidability properties, we introduce the following lemmas.

3.3 Lemma. *Let S be a locally compact space which is semi-r-connected and completely $(r - 1)$-avoidable at $x \in S$. Then $p_r(S, x) = 0$.*

Proof. Let U, V, and W be open sets such that $x \in W \Subset V \Subset U$, r-cycles of $\bar{U}$ bound on S, and $(r - 1)$-cycles of $F(V)$ bound on $\bar{U} - W$. Consider any cycle γ^r mod $S - U$. The portion of γ^r in $\bar{V}$ is a cycle γ_1^r such that $\partial\gamma_1^r$ is on $F(V)$. Since $\partial\gamma_1^r \sim 0$ on $\bar{U} - W$, there exists by Lemma VII 1.6 a cycle Γ^r on $\bar{U}$ such that $\Gamma^r \sim \gamma_1^r$ mod $S - W$ on $\bar{U}$. But $\Gamma^r \sim 0$ on S, hence $\gamma_1^r \sim 0$ mod $S - W$ and $\gamma^r \sim 0$ mod $S - W$.

Note that by virtue of Lemmas 3.1 and 3.3 we have:

3.4 Lemma. *If a locally compact space S is r-lc and semi-$(r + 1)$-connected at $x \in S$, then complete r-avoidability at x is equivalent to $p^{r+1}(x) = 0$.*

3.5 Lemma. *Let S be a compact space such that $p^r(S) = m$ finite, all r-cycles on proper closed subsets of S being bounding cycles of S. Then for any $x \in S$, r-extendibility at x is equivalent to $p^r(x) = m$.*

Proof. That the condition $p^r(x) = m$ implies that S is r-extendible has already been proved in Theorem VII 5.10. Now suppose, conversely, that S is r-extendible at x. By hypothesis there exist cycles Γ_i^r, $i = 1, \cdots, m$, that form a base for r-cycles relative to homologies on S, and by Lemma VII 3.7, these cycles are lirh mod $S - P$ for every nonempty open set P. Let U be any open set containing x, and V an open subset of U containing x such that if γ^r is a cycle mod $S - U$, then there exists a cycle Γ^r of S such that $\gamma^r \sim \Gamma^r$ mod $S - V$. Now the cycles Γ_i^r are lirh mod $S - V$, and if $p^r(x; U, V)$ were greater than m, there would exist by Lemma V 18.26 a cycle γ^r mod $S - U$ such that γ^r, $\Gamma_1^r, \cdots, \Gamma_m^r$ are lirh mod $S - V$. But with Γ^r as above, there would then exist a homology $\Gamma^r \sim \sum a^i\Gamma_i^r$, which would hold a fortiori mod $S - V$, implying $\gamma^r \sim \sum a^i\Gamma_i^r$ mod $S - V$. It follows that $p^r(x) = m$.

As a consequence of the above lemmas and Corollary VI 6.12 we have:

3.6 Theorem. *In order that an n-dimensional compact space S should be an orientable n-gcm, the following conditions are necessary and sufficient:*

(1) *$p^n(S) = 1$ and all n-cycles on closed proper subsets of S bound on S.*

(2) *S is semi-r-connected for all $r \leqq n - 1$.*

(3) *S is completely r-avoidable at all points for all $r \leqq n - 2$.*
(4) *S is n-extendible at all points.*

4. **Characterization by means of local linking.** Another of the equivalent forms by which the orientable n-gcm may be defined depends on the notion of a simple local duality.

4.1 DEFINITION. A space S will be said to "satisfy a simple local duality at $p \in S$ rel. n" if for arbitrary open set U containing p there exists an open set V such that $x \in V \subset U$ and (1) if M is a closed subset of V and Z^r $(0 < r < n - 1)$ is a compact cycle of $V - M$ which links M in U, then there exists a cycle γ^{n-r-1} on M such that Z^r and γ^{n-r-1} are linked in U; and (2) if γ^{n-r-1} is a cycle on M that fails to bound on M, then there exists a compact cycle Z^r in $V - M$ such that Z^r and γ^{n-r-1} are linked in U.

4.2 THEOREM. *Every perfectly normal orientable n-gcm S satifies a simple local duality at $x \in S$ rel. n for every point x of S.*

INDICATION OF PROOF. Given U, an open set containing $x \in S$, we select V so that cycles in V bound in[1] U and cocycles in V of dimension $<n$ cobound in U. Let $M \subset V$ and let Z^r, $0 < r < n - 1$, be a compact cycle of $V - M$ that fails to bound in $U - M$. We may now proceed exactly as in the second proof of the Alexander Duality in Chapter VIII to obtain a cycle γ^{n-r-1} of the type desired. A similar remark applies to the proof of condition (2) of Definition 4.1.

4.3 LEMMA. *If the compact space M is n-extendible at every point, semi-$(n - 1)$-connected, and such that $p^n(M) > 0$ but every n-cycle on a proper closed subset of M bounds on M, then M is 0-lc, locally $(n - 1)$-avoidable and completely 0-avoidable.*

PROOF. That M is 0-lc follows from Theorem VII 5.4. As M is n-extendible and semi-$(n - 1)$-connected, it is locally $(n - 1)$-avoidable by Lemma VII 5.3. Hence, given $p \in M$ and open set U containing p, there exist open sets V and W such that $x \in W \Subset V \Subset U$ and such that $(n - 1)$-cycles on $F(V)$ bound on $M - W$. Since M is 0-lc, we may assume U connected.

Now suppose $U - p = A \cup B$ separate. Then $A \cap W \neq 0 \neq B \cap W$. And if Γ^n is a nonbounding cycle of M, the portion of Γ^n on $A \cap V$ is a cycle Z^n mod $[F(V) \cup p]$ whose boundary ∂Z^n bounds on $(M - W) \cup p$. Hence by Lemma VII 1.6, there exists a cycle γ^n on $(\overline{A \cap W}) \cup (M - W)$ such that $\gamma^n \sim Z^n$ mod $(M - W) \cup p$. But γ^n is on $M - B \cap W$ and consequently bounds on M. But this implies $Z^n \sim 0$ mod $(M - W) \cup p$ which in turn implies $\Gamma^n \sim 0$ mod $M - A \cap W$; this is impossible by Lemma VII 3.6. Hence by

[1]Hereafter, unless otherwise specified, when we speak of a cycle *in* an open set U we shall mean a cycle on a compact subset of U; and by bounding in U will be meant bounding on a compact subset of U.

Theorems IV 3.1 and VII 6.15, p must be a local non-0-cut point of M, and that M is completely 0-avoidable at p will follow from the following lemma.

REMARK. In view of Lemma 4.3, the case $r = 0$ in (3) of Theorem 3.6 may be deleted.

4.4 LEMMA. *If a locally compact space S is lc^r and p is a local non-r-cut point* [VII 6.11] *of S, then S is completely r-avoidable at p.*

PROOF. Let U be an open set containing p such that $\overline{U}$ is compact, and $V \Subset U$ an open set containing x such that r-cycles on $F(V)$ bound in $U - p$. Let $W \Subset V$ be another open set containing x. By Corollary VI 3.8, there exists a finite set of cycles Z_i^r, $i = 1, \cdots, m$, forming a base for r-cycles of $F(V)$ rel. homologies in $U - \overline{W}$. Each Z_i^r bounds on a compact subset of $U - p$, and it follows that there exists an open set R containing x and lying in W such that every r-cycle on $F(V)$ bounds in $U - \overline{R}$.

The following theorem may now be proved:

4.5 THEOREM. *In order that a perfectly normal, n-dimensional compact space S should be an orientable n-gcm, it is necessary and sufficient that, in addition to satisfying conditions* (1) *and* (4) *of Theorem* 3.6, *it be semi-$(n - 1)$-connected and satisfy a simple local duality at all points rel. n.*

To prove the sufficiency of the conditions, note that the condition that S satisfy a simple local duality rel. n implies that S is r-lc as well as that every point of S is a local non-r-cut point, for $r = 1, \cdots, n - 2$. By Theorem VII 5.4, S is 0-lc, and by Lemma 4.3, S is completely 0-avoidable. By Lemma 4.4, S is completely r-avoidable for $r = 1, \cdots, n - 2$. Thus the sufficiency of the theorem follows from Theorem 3.6. The necessity follows from Theorem 4.2.

5. The case $n = 2$ without the orientability condition. It was remarked above in §2 that for the nonoriented case a more extended proof was needed to show that in the metric case the 2-gm is locally euclidean. In view of the fact that, for the general n, the nonoriented case seems to require (for local dualities, etc.) the imposition of local orientability (this term will be defined below), it is remarkable that in the case $n = 2$ this is unnecessary. An incidental result is a noteworthy characterization of the classical 2-manifold as well as of the euclidean plane in terms of the local numbers $p^r(M, x)$.

5.1 The following example is instructive: In a euclidean space S, let K be a 2-cell (in the sense of I 11.16) and let x be a nonboundary point of K. Let $P_1, \cdots, P_n, \cdots$ be a sequence of projective planes in $S - K$ having in successive pairs a single common point, but otherwise disjoint, and such that if $p_n \in P_n$, then x is the sequential limit point of the sequence $\{p_n\}$. Let $M = \bigcup P_n \cup K$. Then with the integers mod 3 as the field $\mathfrak{F}$, $p^2(M, x) = 1$, since no P_n carries a nonbounding 2-cycle. The set M will be seen to lack the property which we mentioned above by the name "local orientability". Note,

however, that $p^1(M, x) \neq 0$, since if ax is an arc from some point a of one of the sets P_n, to x, lying except for x in $\bigcup P_n$, and bx is an arc of K, then the arc $ab = ax \cup bx$ carries a 1-cycle Z^1 mod $a \cup b$ which fails to be homologous to zero mod $M - Q$ on M for all sufficiently small open sets Q containing x.

We shall need the following lemma:

5.2 LEMMA. *In a metric space S, let M be an irreducible membrane relative to the fundamental 1-cycle Z^1 of a 1-sphere J. Then M is a continuum and $M - J$ is connected.*

PROOF. By Lemma VII 1.4, there exists a cycle Z^2 mod J on M such that $\partial Z^2 \sim Z^1$ on J; we may as well assume that $\partial Z^2 = Z^1$. Were $M = A \cup B$ separate, then J would lie in one of the sets A, B, say A, and consequently, denoting by Z_1^2 the portion of Z^2 on B, we would have $\partial Z_1^2 = 0$ and hence $\partial(Z^2 - Z_1^2) = Z^1$ on A, violating the irreducibility of M.

Suppose $M - J = M_1 \cup M_2$ separate. As before, there exists a cycle Z^2 mod J such that

$$\text{(5.2a)} \qquad \partial Z^2 \sim Z^1 \qquad \text{on } J.$$

The portion of Z^2 on M_i, $i = 1, 2$, is a cycle Z_i^2 such that ∂Z_i^2 is on J. If for either value of i, $\partial Z_i^2 \nsim 0$ on J, then there exists a homology $c\partial Z_i^2 \sim Z^1$, $c \in \mathfrak{F}$, on J, and hence $Z^1 \sim 0$ on $J \cup M_i$, again contradicting the fact that M is an irreducible membrane relative to Z^1. On the other hand, if $\partial Z_i^2 \sim 0$ on J, then there exists a cycle γ_i^2 on $M_i \cup J$ such that

$$\text{(5.2b)} \qquad \gamma_i^2 \sim Z_i^2 \qquad \text{mod } J \text{ on } M_i \cup J.$$

Because of the way in which Z_i^2 was chosen, we also have

$$\text{(5.2c)} \qquad Z^2 \sim Z_i^2 \qquad \text{mod } M - M_i \text{ on } M.$$

Relations (5.2b) and (5.2c) give $Z^2 \sim \gamma_i^2$ mod $M - M_i$ on M and hence, by Lemma VII 1.2, $\partial Z^2 \sim \partial\gamma_i^2$ on $M - M_i$. But $\partial\gamma_i^2 = 0$ and therefore

$$\text{(5.2d)} \qquad \partial Z^2 \sim 0 \qquad \text{on } M - M_i\,.$$

Relation (5.2d) together with relation (5.2a) gives $Z^1 \sim 0$ on $M - M_i$, contradicting the fact that M is an irreducible membrane rel. Z^1. We must conclude, then, that $M - J$ is connected.

5.3 DEFINITION. If M is a locally connected, locally compact metric space, and $p \in M$ disconnects some domain of M, then p is called a *local separating point* of M.

A point which is not a local separating point of such a set M is easily seen to be a local non-0-cut point of M, and conversely (cf. Theorems VII 6.15 and IV 3.1).

5.4 LEMMA. *If M is a 0-lc, locally compact, connected metric space without a local separating point, and $x, y \in M$, $x \neq y$, then there exist three arcs of M having x and y as end points, and which are otherwise disjoint.*

(It will be noticed that the method of proof used will show, by induction,

that "three" can be replaced by "n", where n is any positive integer. As a matter of fact, Whyburn has shown that "c" may be used, where c is the cardinal number of the set of all real numbers. Cf. G. T. Whyburn [i].)

PROOF. As M has no local separating point, it has no cut point. Hence by Corollary III 3.32a, there exist two arcs A and B having x and y as end points, and otherwise disjoint.

On the arc A let x_1 and y_1 be points such that in the order from x to y, $x < x_1 < y_1 < y$. Hereafter we denote subarcs of A by their end points—for instance, x_1y_1 denotes the subarc of A having x_1 and y_1 as end points. On xx_1 let $x_2, x_3, \cdots, x_n, \cdots$ be points such that for each n, $x < x_{n+1} < x_n < x_1$, and having x as sequential limit point. And on y_1y let $y_2, y_3, \cdots, y_n, \cdots$ be a sequence of points such that for each n, $y_1 < y_n < y_{n+1} < y$, and having y as sequential limit point.

Let η_1 be a positive number < 1 and $< \rho(x_1y_1, B)$. If $p \in x_1y_1$, let $U(p)$ denote a 0-ulc, connected open subset of M of diameter $< \eta_1$ that contains p and whose closure is compact; such a set exists by Theorem III 3.3. Let $U_1, \cdots, U_k$ be a simple chain of the sets $U(p)$ from x_1 to y_1 (Theorem I 12.3). By Theorem III 3.6 the sets $\overline{U}_i$ are 0-lc, hence their union, which we denote by C_1, is a Peano continuum.

Let η_2 be a positive number $< 1/2$ and $< \rho(x_1x_2, B)$, and if $p \in x_1x_2$ let $V(p)$ denote a 0-ulc, connected open subset of M of diameter $< \eta_2$ that contains p and has a compact closure. Let C_2 be the closure of a union of a simple chain of the sets $V(p)$ from x_1 to x_2. Let K_2 be a similar Peano continuum covering y_1y_2. And in general, having obtained a set C_n, $n > 1$, covering $x_{n-1}x_n$, let η_{n+1} be a positive number $< 1/(n+1)$ and $< \rho(x_nx_{n+1}, B)$, and cover x_nx_{n+1} by sets analogous to the sets $V(p)$ above of diameter $< \eta_{n+1}$, and from these obtain the continuum C_{n+1}. In similar fashion a continuum K_{n+1} is obtained covering y_ny_{n+1}.

The set $C = x \cup y \cup \bigcup C_n \cup \bigcup K_n$ is a Peano continuum containing x and y but no other point of the arc B. Also, since no point of M is a local separating point, it is easy to see that C has no cut point. Consequently C contains two arcs having x and y as end points and otherwise disjoint.

5.5 THEOREM. *Let M be a 2-dimensional, locally compact metric space such that for each $x \in M$, $p_0(M, x) = p_1(M, x) = 0$ and $p_2(M, x) = 1$. Then each point of M has a neighborhood homeomorphic with the euclidean plane.*

PROOF. By Theorem VI 7.9, M is lc^2.

Let $x \in M$. There exist open sets U and V containing x such that $\overline{U}$ is compact, $x \in V \subset U$, 2-cycles of $\overline{U}$ bound on M, $p^2(x; U, V) = 1$, and V lies in one component, L', of U. Let $L = \overline{L'}$. Throughout we shall suppose that all coverings used are 2-dimensional.

Since $p^2(x; U, V) = 1$, there exists a cycle Z^2 mod $M - U$ lirh mod $M - V$. Since L' is open, by Corollary VII 1.16 we may suppose that L is a carrier of

Z^2. The chain ∂Z^2 is a nonzero cycle on $L - L'$. Let K' be an irreducible carrier of the homology $\partial Z^2 \sim 0$ on L (in the sense of Lemma VII 2.8), and continue to use Z^2 to denote the chain realizing this homology on K'. The set K' is unique. For if $\partial C^2 = \partial Z^2$, C^2 on L, then $C^2 - Z^2 \sim 0$ on M, hence ~ 0 mod $M - U$, and as M is 2-dimensional this implies that $C^2 = Z^2$ mod $M - U$. Let $K' \cap U = K$.

The set K is 0-lc. For suppose K is not 0-lc at $y \in K$. Let P and Q be open sets such that $y \in Q \subset P \Subset L'$, and infinitely many components of $K \cap \bar{P}$ meet Q, but otherwise arbitrary except that $p^2(y; P, Q) = 1$. There exists a decomposition $K \cap \bar{P} = A \cup B$ separate, where $A \cap Q \neq 0 \neq B \cap Q$. The portions of Z^2 on A and B respectively form cycles Z_1^2 and Z_2^2 mod $M - P$, that cannot be lirh mod $M - Q$. Hence $aZ_1^2 \sim bZ_2^2$ mod $M - Q$ on M, a, $b \in \mathfrak{F}$. As M is 2-dimensional, $aZ_1^2 = bZ_2^2$ mod $M - Q$, so that either a or b must be zero. If $a = 0$, however, then $Z_2^2 = 0$ and K' is not an irreducible carrier of the homology $Z^2 \sim 0$.

Since K is 0-lc, we may henceforth suppose K is connected. The set K has no cut point. For suppose $p \in K$ such that $K - p = K_1 \cup K_2$ separate. Then by considering the portions of Z^2 on $K_1 \cup p$ and $K_2 \cup p$, it can readily be shown that $p^2(M, p) \geqq 2$, contrary to hypothesis. As a consequence, K is cyclicly connected by Corollary III 3.32a. And by the same type of reasoning it may be shown that K has no local separating point.

No point of K is a limit point of $U - K$. For suppose the contrary. Then some component, C, of $U - K$, has at least one boundary point, q, in K. Let tq be an arc from a point t of C to the point q, which we may suppose meets K only in the latter point. Let sq be an arc of K. Then for a small enough neighborhood P' of q, there exists on the continuum $st = sq \cup tq$ a 1-cycle Z^1 mod $M - P'$ lirh on st mod $S - Q'$ for all open sets Q' such that $q \in Q' \subset P'$. Now as $p_1(M, q) = 0$, there exists a Q' such that $p^1(q; P', Q') = 0$. Hence $Z^1 \sim 0$ mod $M - Q'$; i.e., there exists a cycle γ^2 mod $st \cup F(Q')$ such that $\partial\gamma^2 \sim Z^1$ mod $F(Q')$. We can assert that q is not the only limit point of $U - K$ in $Q' \cap K$. For suppose it were; then the portion of γ^2 in $Q' \cap (U - K)$ would be a cycle γ_1^2 mod $st \cup F(Q')$ whose boundary would consist of a chain Z_1^1 on tq and a chain on $F(Q')$, where $Z_1^1 \sim Z^1$ on tq mod $F(Q')$. But this is impossible, since the portion of Z^1 on tq is not a cycle mod $F(Q')$. The same type of argument shows, incidentally, that M itself has no local separating point.

Now suppose Q' small enough that absolute 1-cycles of $M \cap Q'$ bound in P'. Let R be an open set such that $q \in R \subset Q'$ and such that (1) R lies in one component, M', of Q', (2) $K \cap R$ lies in one component K'' of $K \cap Q'$. Let s' and t' be points of the components of sq and tq determined by q in R. By Corollary III 3.32a, there exist 2 arcs from s' to t' in M' having only s' and t' in common, and hence there exists an arc ab of M which lies in $C \cap Q'$ except for its end points a and b which lie on K''. And as K'' has no local separating points, by Lemma 5.4 there exist in K'' three arcs H_i, $i = 1, 2, 3$, from a to b which meet in pairs only at a and b. Each simple closed curve $J_i = ab \cup H_i$

carries a cycle γ_i^1 nonbounding on J_i, and by the choice of Q' there exist by Theorem VII 2.22 irreducible membranes M_i rel. γ_i^1 in P'. By Lemma 5.2, each set M_i is a continuum, and the set $N_i = M_i - J_i$ is connected.

Each continuum M_i meets K only in the arc H_i. For otherwise there would exist a point $x' \in K$, limit point of $M_i - K$, such that $p^2(M, x') \geqq 2$. Also, the continua M_i have only the arc ab in common. For suppose $M_1 \cap M_2 \not\subset ab$, for instance. Then $N_1 \cap N_2 \neq 0$. And since H_1 contains limit points of N_1, $N_1 - N_1 \cap N_2 \neq 0$. Of the two sets $N_1 \cap N_2$ and $N_1 - N_1 \cap N_2$, the former is closed rel. N_1 and hence contains a limit point x_1 of the latter. It now easily follows that $p^2(M, x_1) \geqq 2$, since M is 2-dimensional and each of the membranes M_1, M_2 contains the point x_1. We conclude, then, that the continua M_i meet only in ab.

But consider a point $x_2 \in ab - a - b$. Let U_1 and V_1 be open sets such that $x_2 \in V_1 \subset U_1$, $J_i \cap U_1 \subset ab - a - b$, $i = 1, 2, 3$, $p^2(x_2 ; U_1, V_1) = 1$, and if a_1b_1 is that subarc of ab containing x_2 which lies in U_1 except that its end points lie on $F(U_1)$, then $V_1 \cap ab \subset a_1b_1$. The three cycles γ_i^1 are homologous on a_1b_1, mod $a_1b_1 - V_1$, since a_1b_1 is an arc. It can now be shown that $p^2(x_2 ; U_1, V_1) \geqq 2$, contrary to the choice of U_1 and V_1.

The assumption that K contains a limit point of $U - K$ has thus led to a contradiction, and we therefore conclude that K is open. And since M is 1-lc and 2-lc, we now know that the same holds for K.

It can now be shown that K satisfies the three conditions of Theorem III 6.2. Let F be a compact subset of K and $p' \in F$. Let U_2 and V_2 be open sets such that $p' \in V_2 \subset U_2 \subset K$, and 1-cycles of V_2 bound in U_2. Let J be a simple closed curve of F in V_2 and Z^1 the fundamental 1-cycle of J. Let M_1 be an irreducible membrane rel. Z^1 in U_2. Then the sets $M_1 - J$ and $K - M_1$ are separated; otherwise, there would exist a point $m \in M_1 - J$ such that $p^2(M, m) \geqq 2$ (recall that K' is an irreducible carrier of the homology $\partial Z^2 \sim 0$ on L). It follows that there exists a number $\epsilon(F)$, as in the theorem just cited, satisfying condition 6.2b of that theorem. As K has no local separating points, it contains simple closed curves of diameter $< \epsilon(F)$.

Suppose H is an arc of K such that $K - H = K_1 \cup K_2$ separate. The portions of Z^2 on K_1 and K_2 are relative cycles Z_1^2 and Z_2^2 whose boundaries are on $H \cup F(U)$. Consider the portion of ∂Z_1^2 on H; it is an absolute cycle γ^1, and $\gamma^1 \sim 0$ on H. Therefore there exists a cycle γ_1^2 mod $F(U)$ on $K_1 \cup H$ such that $\gamma_1^2 \sim Z_1^2$ mod $H \cup F(U)$. Let γ_2^2 be a similar relative cycle on $K_2 \cup H$. Let r be a non-end point of H, limit point of both K_1 and K_2 (such a point exists since K has no local separating point), and W_1, W_2 open sets containing r such that $W_1 \subset K$ and $p^2(r; W_1, W_2) = 1$. Then there exists a homology $a\gamma_1^2 + b\gamma_2^2 \sim 0$ mod $M - W_2$, $a, b \in \mathfrak{F}$, where not both a and b are zero. Suppose $a \neq 0$. Then $\gamma_1^2 \sim 0$ mod $(M - W_2) \cup K_2 \cup H$, so that $Z_1^2 \sim 0$ mod $(M - W_2) \cup K_2 \cup H$, implying that $Z^2 = 0$ mod $(M - W_2) \cup K_2 \cup H$. But then K is not an irreducible membrane rel. ∂Z^2. We can therefore conclude that no arc of K separates K.

Thus K is an infinite 2-dimensional manifold in the classical sense, and the conclusion of the theorem follows.

In contrast to the many characterizations of the various types of 2-manifolds that either require or directly imply the local connectedness of the space under consideration, and which make no use of the dimensionality invariant (use is made of the dimensionality condition by Vaughan [a], however, in his characterizations of various 2-dimensional surfaces), we can now give the following theorems. In each case the theorem is either an obvious specialization of Theorem 5.5, or follows therefrom by a short argument employing previous theorems (such as Corollary VI 6.12) and well known classifications of the classical 2-manifolds.

5.6 THEOREM. *In order that a 2-dimensional metric continuum M should be a closed 2-manifold in the classical sense, it is necessary and sufficient that* $p_1(M, x) = 0$ *and* $p_2(M, x) = 1$ *for every* $x \in M$.

5.7 THEOREM. *In order that a 2-dimensional metric continuum M should be a 2-sphere, it is necessary and sufficient that* $p^1(M) = 0$, $p^2(M) > 0$, *and for every* $x \in M$, $p_1(M, x) = 0$, $p_2(M, x) = 1$.

5.8 THEOREM. *In order that a 2-dimensional, connected, locally compact separable metric space M should be a 2-dimensional infinite manifold in the classical sense, it is necessary and sufficient that* $p_1(M, x) = 0$ *and* $p_2(M, x) = 1$ *for every* $x \in M$.

5.9 THEOREM. *In order that a noncompact, 2-dimensional, connected, locally compact separable metric space M should be a euclidean plane, it is necessary and sufficient that* $p^1(M) = 0$ *and that for every* $x \in M$, $p^1(M, x) = 0$ and $p^2(M, x) = 1$.

5.10 REMARK. The analogue of Theorem 5.5 for the case $n = 1$ states that *if M is a 1-dimensional, locally compact metric space such that for every* $x \in M$, $p_0(M, x) = 0$ *and* $p_1(M, x) = 1$, *then each point of M has a neighborhood homeomorphic with the open real number interval* $\langle 0, 1\rangle$. If such a set M is compact, or contains a simple closed curve, then the proof of this statement is essentially as in §1 above. If M is acyclic and $x \in M$, then as a consequence of Lemma 1.1, there exists an arc pq in M having x as a non-end point, and it is easily shown that $M - pq$ does not have x as a limit point. As a corollary, *the homeomorph of the unbounded real number continuum may be characterized as a noncompact, connected, locally compact 1-dimensional, separable metric space M such that for each* $x \in M, p_1(M, x) = 1$. (Actually, the condition on $p_1(M, x)$ may be weakened; cf. the remarks at the close of §1 above.)

6. **The general non-orientable case.** In order to extend the results of §§3 and 4 above to the general nonorientable case, it seems (we have no example to offer in order to prove the necessity) to be necessary to augment the conditions A, B and C defining an n-gm by the following:

6.1 Definition. An n-gm S will be called *locally orientable* if every point of S has a neighborhood which is an orientable n-gm (i.e., a neighborhood P which carries a nonbounding infinite cycle Γ^n, and such that if F is a proper closed, rel. P, subset of P, every infinite n-cycle on F bounds on P). Obviously every orientable manifold is locally orientable.

Remark. As shown in §5, condition 6.1 is unnecessary in the 2-dimensional separable metric case, and in view of Theorems 5.6-5.9, it is trivial that the various types of locally orientable 2-gms reduce to the classical types. One might investigate, to be sure, whether the general (i.e., not locally separable metric) case of the 2-gm is necessarily locally orientable.

An alternative definition embodies a condition analogous to condition D′ of Chapter VIII, §5:

D″. *If* $x \in S$, *there exists an open set* P *containing* x, *and an infinite* n*-cycle* $\Gamma^n \bmod S - P$, *such that* $\Gamma^n \nsim 0 \bmod S - U$ *for every open set* $U \subset P$.

One can prove:

6.2 Theorem. *In order that an* n*-gm should be locally orientable at every point, it is necessary and sufficient that it satisfy condition* D″ *at every point.*

[The necessity is quite obvious, and the sufficiency follows easily by use of Lemma VII 2.4.]

That a locally orientable n*-gm which is perfectly normal satisfies a simple local duality at every point rel.* n*. may be shown by the methods of Chapter* VIII.

In order to complete the category of avoidability properties found essential above to the characterization of the orientable n-gcm, it is necessary to extend the notion of local r-avoidability and r-extendability in the following manner:

6.3 Definition. A space S will be called *locally* r*-avoidable at* $p \in S$ *in the relative sense* if there exists an open set P containing p such that for every open set U containing p and lying in P, there exist open sets V and W such that $p \in W \Subset V \Subset U$ and such that if γ^r is a cycle on $F(V)$, then $\gamma^r \sim 0 \bmod S - P$ on $S - W$.

6.4 Definition. A space M will be called r*-extendible at* $p \in M$ *in the relative sense* if there exists an open set P containing p such that if U is any open set containing p and lying in P, then there exists an open set V such that $p \in V \Subset U$ and such that if γ^r is any cycle mod $M - U$, then there exists a cycle $Z^r \bmod M - P$ such that $\gamma^r \sim Z^r \bmod M - V$.

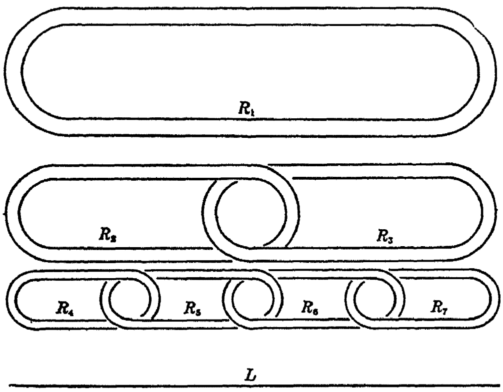
6.5 As in Chapter VII (Lemmas 5.2 and 5.3) we may show that (1) *local* $(r - 1)$*-avoidability at* $p \in S$ *in the relative sense implies* r*-extendability at* $p \in S$ *in the relative sense,* and (2) *if* S *is* r*-extendible at* $p \in S$ *in the relative sense,* $r > 0$, *and there exists a neighborhood of* p *such that all* $(r - 1)$*-cycles in that neighborhood bound on* S*, then* S *is locally* $(r - 1)$*-avoidable at* p *in the relative sense.* And as in Lemma 2.1 and Corollary 2.2 above, it may be shown that

a locally orientable n-gm is n-extendible at every point in the relative sense and hence locally $(n - 1)$*-avoidable at every point in the relative sense.*

6.6 One can now characterize a locally orientable n-gm as *an n-dimensional, locally compact space S such that* $p_n(S, x) = 1$ *at every* $x \in S$ *and satisfying conditions* (2) *and* (3) *of Theorem* 3.6 *as well as condition* D″. A characterization more closely analogous to Theorem 3.4, perhaps, may be obtained by employing, instead of the condition on $p_n(S, x)$, the n-extendibility in the relative sense together with a strengthening of condition D″ which implies the existence of a *single* Γ^n in the neighborhood of each point—the latter condition being the local analogue of condition (1) of Theorem 3.6.

7. Comparison of the case $n > 2$ with the classical case; regular manifolds and generalized n-cells. Having shown the relations to the classical types of manifolds in the cases $n = 1, 2$ above, it is natural next to inquire just how closely the cases $n \geqq 3$ conform to the classical types of manifolds. Here we encounter for the first time the actual magnitude of the generality obtained in the introduction of the n-gm concept.

7.1 The example of van Kampen, of a 3-gcm incapable of subdivision into



3-cells of the classical type, has already been given in §1 of Chapter VIII. However, in this example every point has neighborhoods whose boundaries are 2-spheres, and it is natural to ask whether this is necessarily the case in every 3-gcm. An example to show that the answer is negative may be constructed as follows: First, in ordinary euclidean 3-space E^3, consider (see the figure) a straight line interval L, which is the limit superior of a sequence of anchor rings $R_1, R_2, \cdots, R_n, \cdots$, interlinked in finite sets as shown in the accompanying figure. Let us delete the interiors, I_n, of the rings R_n from the space E^3, supplanting each I_n by a space M_n of type M such as was used in Chapter VIII, §1, in the construction of a Poincaré space. This is done in such a way that the knotted equatorial curve on the torus boundary of the space of type

M goes into a meridional curve on the boundary of R_n, just as in the above-mentioned construction of the Poincaré space. The metric is so adjusted that L is still the limit superior of the sets M_n. The resulting space is a 3-gm, but points on L do not have arbitrarily small neighborhoods with 2-sphere boundaries—nor even boundaries that are closed 2-manifolds. (It would be interesting to know if all small enough neighborhoods of such points will of necessity have boundaries B such that $p^1(B)$ is infinite.)

7.2 The above example suggests investigation of the properties of a type of generalized manifold more closely analogous to some of the "generalized manifolds" that have been introduced within the framework of the classical combinatorial topology (as for instance by van Kampen [b], or see Lefschetz [L_2]). The chief characteristic of these configurations is their neighborhoods formed by the join of a point with a spherelike (in the sense of homology) manifold of dimension $(n - 1)$. Such characterizations allow of inductive formulation, and may be simulated within the compass of the present investigations by such a definition as the following (only compact cycles and homologies on compact sets are employed throughout this discussion):

(1) A *regular* 0-*manifold* is a pair of points. *A regular closed* 0-*manifold* is a pair of points, and the nontrivial 0-cycle carried by it is called its *fundamental cycle.*

(2) A *regular n-manifold, $n > 0$,* is an n-dimensional, locally compact connected space S which is semi-r-connected, $r \leqq n$ (in the sense of compact cycles), and such that if $p \in S$ and U is a neighborhood of p, then there exists a spherelike[2] regular closed $(n - 1)$-manifold K such that $S - K = A \cup B$ separate, where $p \in A \subset U$ and the fundamental cycle Γ^{n-1} of K bounds on $\bar{A}$ as well as on $\bar{B}$ mod $S - U$. If Γ^{n-1} bounds on $\bar{B}$, then S is called a *regular closed n-manifold.*

We shall show that regular n-manifolds are special cases of the locally orientable n-gms, and that the regular closed n-manifolds are special cases of the orientable n-gcms.

7.3 Lemma. *If S is a locally compact, connected space and x is a point of S such that for every open set U containing x there exist two points p and q such that $S - (p \cup q) = A \cup B$ separate, where $x \in A \subset U$, then S is 0-lc at x.*

Proof. Let U be an open set containing x such that $\bar{U}$ is compact. By hypothesis, there exist two points p and q in U such that $S - (p \cup q) = A \cup B$ separate, where $x \in A \subset U$. Consider any component C of A. Since $\bar{C}$ is compact, C has p or q as a limit point by Theorem IV 1.10. Hence $\bar{A}$ is composed of at most two components, containing p and q respectively. That component which contains x forms, when p and q are deleted, an open set containing x which lies in one constituant of U.

[2]As in the case of the n-gcm, this means that the homology groups are isomorphic with the corresponding groups of the n-sphere.

7.4 Lemma. *In a locally compact, 0-lc space S, let K be a 0-lc continuum such that $S - K = A \cup B$ separate, where $\bar{A}$ is compact. Then $K \cup A$* is 0-lc.

Proof. That $K \cup A$ is 0-lc at points of A is trivial. Consider $x \in K$ and an open set P containing x. There exists an open set Q such that $x \in Q \subset P$ and such that 0-cycles of $K \cap Q$ bound in $K \cap P$; and there exists an open set R such that $x \in R \subset Q$ and 0-cycles of R bound in Q. Hence if Z^0 is a 0-cycle in $R \cap (K \cup A)$, there exists a cycle Z^1 mod $K \cup A$ in Q such that

$$\partial Z^1 \sim Z^0 \qquad \text{in } Q \cap (K \cup A), \tag{7.3a}$$

and ∂Z^1 is on $K \cap Q$. By the choice of Q,

$$\partial Z^1 \sim 0 \qquad \text{in } K \cap P. \tag{7.3b}$$

Relations (7.3a) and (7.3b) imply that $Z^0 \sim 0$ in $P \cap (K \cup A)$.

Analogous to Lemma 5.2, and proved in a similar manner, we have:

7.5 Lemma. *In a compact space S, let M be an irreducible membrance relative to a cycle Z^n, where Z^n is the single lirh n-cycle of a compact subset J of M. Then M is a continuum and $M - J$ is connected.*

7.6 Theorem. *A regular n-manifold is a locally orientable n-gm, and a regular closed n-manifold is an orientable n-gcm.*

Proof. The theorem is trivial for $n = 0$ and we use mathematical induction for the proof, assuming the theorem holds for the dimension $n - 1$.

Let S be a regular n-manifold. Then, n being always > 0, S is 0-lc. For the case $n = 1$ this follows from Lemma 7.3. In the case $n > 1$, consider a $p \in S$ and U an open set that contains p and has compact closure. Then there exists a regular closed $(n - 1)$-manifold K such that $S - K = A \cup B$ separate, where $p \in A \subset U$. As S and K are connected, $A \cup K$ is connected (Theorem I 9.8), and it follows that S is 0-lc. Moreover, $K \cup A = \bar{A}$ is a 0-lc continuum by Lemma 7.4. (To obtain the last-mentioned property in the case $n = 1$, we may use the additional property that the fundamental 0-cycle of K bounds on $\bar{A}$, together with the fact established in the proof of Lemma 7.3 that all components of A have boundary points in K, to establish that $K \cup A$ is a continuum. That it is 0-lc at each of the two points of K is an easy corollary of the 0-lc property of S itself—compare the proof of Lemma 7.4.)

We next show that for every $x \in S$, $p^n(S, x) = 1$. Given $x \in S$, let U be an open set containing x, such that n-cycles of $\bar{U}$ bound on S, and let us confine ourselves to n-dimensional coverings throughout. There exists in U a sphere-like $(n - 1)$-gcm K such that $S - K = A \cup B$ separate, $x \in A \subset U$, and the fundamental cycle Γ^{n-1} of K bounds on $\bar{A}$ as well as on $\bar{B}$ mod $S - U$. Let A' be an irreducible membrane (Theorem III 2.22) on $\bar{A}$ relative to Γ^{n-1}; by Lemma 7.5, A' is connected. Let γ^n be a cycle mod K on A' such that $\partial\gamma^n \sim \Gamma^{n-1}$ on K (Lemma VII 1.4).

Since $\Gamma^{n-1} \sim 0$ on $\bar{B}$ mod $S - U$, there exists by Lemma VII 1.13 a cycle Z^{n-1} on $F(U)$ such that $\Gamma^{n-1} \sim Z^{n-1}$ on $\bar{U} \cap \bar{B}$. That is, $\Gamma^{n-1} \sim 0$ on $\overline{U \cap B}$

mod $\overline{U} \cap \overline{B} - U$. By Lemma VII 2.4, with $D = \overline{U} \cap \overline{B}$, there exists a maximal open subset P of $\overline{U} \cap \overline{B}$ containing $U \cap \overline{B}$ such that $\Gamma^{n-1} \sim 0$ on $\overline{U} \cap \overline{B}$ mod $\overline{U} \cap \overline{B} - P$. Let $F = \overline{U} \cap \overline{B} - P$; then F is a minimal closed subset of $F(U)$ such that $\Gamma^{n-1} \sim 0$ on $\overline{U} \cap \overline{B}$ mod F. Let γ^{n-1} be a cycle on F such that $\Gamma^{n-1} \sim \gamma^{n-1}$ on $\overline{U} \cap \overline{B}$, and let H be a minimal carrier of this homology on $\overline{U} \cap \overline{B}$. Then the set $A' \cup H$ is connected. For suppose $A' \cup H = A_1 \cup A_2$ separate. We may assume $A' \subset A_1$ since A' is connected, and hence $K \subset A_1$. Then clearly we may write $H = H_1 \cup H_2$ separate where $K \subset H_1$. And since $\Gamma^{n-1} \sim \gamma^{n-1}$ on H, we may write $\gamma^{n-1} = \gamma_1^{n-1} + \gamma_2^{n-1}$ where γ_i^{n-1} is on H_i, $i = 1, 2$. But then $\Gamma^{n-1} \sim \gamma_1^{n-1}$ on H_1, and consequently $\Gamma^{n-1} \sim 0$ mod $F \cap H_1$ on $\overline{U} \cap \overline{B}$. But as $F \cap H_2 \neq 0$ and F was minimal, this is impossible. Hence $A' \cup H$ must be connected. Furthermore, $A' \cup H$ is a minimal carrier of the homology $\gamma^{n-1} \sim 0$. For if $\gamma^{n-1} \sim 0$ on H, then there exists an absolute cycle Γ^n on $\overline{U}$ such that $\Gamma^n \sim \gamma^n$ on $\overline{A}$ mod K, and as $\Gamma^n \sim 0$ on S we would have $\gamma^n \sim 0$ mod K, hence $\partial\gamma^n \sim 0 \sim \Gamma^{n-1}$ on K. Hence if L is an irreducible membrane relative to the homology $\gamma^{n-1} \sim 0$ on $A' \cup H$, there must exist points of L in $A' - K$, and it easily follows that A' would have to be a subset of L, and consequently that $\gamma^{n-1} \sim \Gamma^{n-1}$ on a proper subset of H, unless $L \supset H$.

Now suppose $A' \neq \overline{A}$. As $\overline{A}$ is connected and 0-lc, and A' is closed, some component C of $\overline{A} - A'$ has a boundary point, q, in A'. Let P be an open set containing q such that $C - P \neq 0$ and $P \subset U$. There exists in P an $(n-1)$-gcm K' such that $S - K' = A' \cup B'$ separate and $q \in A' \subset U$. But since $A' \cup H$ is an irreducible membrane relative to γ^{n-1}, it follows readily that $K' \subset A' \cup H$. This is impossible, since $K' \cap C \neq 0$ and $C \cap (A' \cup H) = C \cap (A' \cup B) \subset (C \cap A') \cup (C \cap B) = 0$.

It follows that $p^n(S, x) \geqq 1$. Now if $p^n(S, x) \geqq 2$, there would exist an open set U containing x such that $p^n(x; U, V) \geqq 2$ for all open sets $V \subset U$. However, with a V such as A above, and cycles Z_1^n, Z_2^n mod $S - U$, the boundaries, on K, of the portions of Z_1^n, Z_2^n in A would be homologous on K since $\underline{K}$ is spherelike. Consequently there would exist an absolute n-cycle Γ^n on $\overline{A}$ such that $\Gamma^n \sim aZ_1^n + bZ_2^n$, $a, b \in \mathfrak{F}$, mod K on $\overline{A}$, implying, since $\Gamma^n \sim 0$ on S, that $aZ_1^n \sim -bZ_2^n$ mod $S - A$. Thus $p^n(S, x)$ must be 1.

That $p^r(S, x) = 0$ for $r < n$ is shown as follows: With U small enough so that r-cycles of U bound on S, take K and A as above. Then if Z^r is a cycle mod $S - U$, the portion of Z^r in A is a cycle γ^r mod K. As $\partial\gamma^r$ is on K and is of dimension $< n - 1$, and K is spherelike, it follows that $\partial\gamma^r \sim 0$ on $\underline{K}$. Hence there exists an absolute cycle Γ^r on $\overline{A}$ such that $\Gamma^r \sim \gamma^r$ mod K on $\overline{A}$. But $\Gamma^r \sim 0$ on S, hence $\gamma^r \sim 0$ mod $S - A$, implying that $Z^r \sim 0$ mod $S - A$.

Thus S is an n-gm. That it is locally orientable follows easily from the properties of A and K above.

If S is a regular closed n-manifold, so that, with K and A as above, $\Gamma^{n-1} \sim 0$ on B, then it may be shown, by methods similar to those used in studying A above, that $\overline{B}$ is an irreducible membrane relative to Γ^{n-1}, and that no proper closed subset of S carries a nonbounding n-cycle. In this case, then, S is an orientable n-gcm and carries a single nonbounding n-cycle.

7.7 Since in the metric separable case the 1- and 2-dimensional generalized manifolds reduce to the classical manifolds, one may state obvious corollaries of Theorem 7.6 for this case. However, in the case $n = 1$, the conditions imposed are much stronger than needed, since it is known (see K. Menger [a, Satz XXIV, p. 303]) that a compact metric connected space having the property that every point can be ϵ-separated by a pair of points is an S^1. In the case $n = 2$ the characterizations obtained are closely analogous to those of Miss Gawehn [a], whose conditions, however, include the Jordan Curve Theorem locally or in the large, in place of the condition placed on Γ^{n-1} in the definition of regular n-manifold above.

7.8 Of course the problem arises as to how much closer one gets to the locally euclidean manifolds by the use of regular manifolds instead of n-gms. The van Kampen example of Chapter VIII, §1, is a regular 3-manifold, so that apparently the regular manifolds occupy a position between the n-gms and the classical cases. No attempt will be made in the present work to investigate their general properties. Some of these properties will emerge as corollaries of properties of the n-gm, however. For example, as a consequence of the positional properties of $(n - 1)$-gms in orientable n-gcms which will be established in Chapter X, together with the fact that the set $A' - K$ discussed in the proof of Theorem 7.6 is by Lemma 7.5 a domain, it will appear that *the regular closed n-manifolds have bases of open sets which are r-ulc for all dimensions* r (see Definition X 1.6). It will be recalled that we showed in Theorem III 3.3 that in the metric case, the locally compact, connected, "lc" (=0-lc) spaces have bases of "ulc" (=0-ulc) open sets. For the regular closed manifolds, even in the nonmetric case, this property still holds, as well as an analogous property for higher dimensions.

As will become evident in Chapter X, however, for the points of a locally orientable n-gm to possess arbitrarily small neighborhoods which have the r-ulc properties mentioned above is *equivalent* to the property of each point having arbitrarily small neighborhoods bounded by $(n - 1)$-gcms. Plainly, for dimensions $r > 0$, a hierarchy of "homology" local connectedness properties of a locally compact space S suggest themselves (cf. VI 8), as for instance:

(1) The r-lc already defined above in terms of compact cycles.

(2) Given $x \in S$ and an open set U containing x, there exists a simply r-connected open set V such that $x \in V \subset U$. [I.e., $p^r(V) = 0$ in terms of compact cycles.]

(3) Given $x \in S$ and an open set U containing x, there exists an r-ulc (in the sense of Definition X 1.6) open set V such that $x \in V \subset U$.

(4) Given $x \in S$ and an open set U containing x, there exists a simply r-connected, r-ulc open set V such that $x \in V \subset U$.

(5), (6), (7)—these are respectively the same as (2), (3), (4) with the additional condition in each case that the properties stated hold for all $r \leqq$ some fixed integer n.

The above list of properties serves to bring out the extent of the difference between the n-gm and the regular n-manifold, since the n-gm has in general, presumably, only local connectedness of type (1), whereas, as will be proved in Chapter X, the regular n-manifolds have local connectedness of type (7). Whether, and in what cases, some of these types of local connectedness coalesce, we have not investigated.

Another general observation should be made here, as a result of the investigations of this chapter. As the reader will have observed, the types of generalized manifolds which we study are what one might call "homology manifolds." Even the strongest type of these, the regular manifolds, do not merge with the classical cases when $n > 2$. A closer approach to the classical case in higher dimensions would probably be afforded by "homotopy manifolds." Such a manifold could be obtained from the n-gm, for instance, by the imposition of local connectedness in the sense of homotopy. We recall that if S is a space, K the n-sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$ in E^{n+1} and A the set $x_1^2 + \cdots + x_{n+1}^2 < 1$, then a continuous mapping $f : K \to S$ is called *homotopic to zero in* S if there exists a continuous mapping $g : K \cup A \to S$ such that for $x \in K, f(x) = g(x)$. Then, for example, analogous to type (1) local connectedness we have:

7.9 Definition. A metric separable space S is *r-lc in the sense of homotopy at $x \in S$* if for arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that every continuous mapping of the r-sphere into $S(x, \delta)$ is homotopic to zero in $S(x, \epsilon)$. This property may be denoted by the symbol r-LC.

Similar modifications are obvious for the other types of local connectedness.

Whether the investigation of such "homotopy manifolds" would be worthwhile we are not prepared to say. Apparently the existing state of the literature on homotopy might necessitate restriction to the metric separable case. However, such a restriction seems not to be inevitable if the generalized concepts we have been discussing are employed. For example, homotopy to zero of the 0-sphere S^0 in a space M could conceivably be considered as an extension to S of the mapping of the end points, into S^0, of a compact space S which, except for its two end points, satisfies the condition that $p_1(S, x) = 1$ at every $x \in S$ (see Remark 5.10). More generally, one might replace, in the definition of homotopy, the n-sphere by a spherelike n-gcm and the closed n-cell by a generalized closed n-cell defined as follows:

7.10 Definition. A *generalized n-cell* is a noncompact, orientable n-gm which is cell-like in the sense that its compact homology groups of dimension $r < n$ reduce to the identity. A *generalized closed n-cell* is a compact space S consisting of a spherelike $(n - 1)$-gcm K and a generalized n-cell A such that (1) $K \cap A = 0$, (2) if Γ^{n-1} is the fundamental cycle of K, then $\Gamma^{n-1} \sim 0$ on $K \cup A$, and $K \cup A$ is an irreducible membrane relative to Γ^{n-1}; and (3) $p_r(K \cup A, x) = 0$, $r \leqq n$, for all $x \in K$.

7.11 In the separable metric case the generalized 2-cell is an ordinary 2-

cell (Theorem 5.9) and a generalized closed 2-cell is the ordinary closed 2-cell.[3] This may be established in the following manner: *Let A be a 2-cell and K an S^1 constituting a compact metric space such that* (1) $K \cap A = 0$, (2) *if* Γ^1 *is the fundamental cycle of* K, *then* $\Gamma^1 \sim 0$ *on* $K \cup A$ *and* $K \cup A$ *is an irreducible membrane relative to* Γ^1, *and* (3) $p_r(K \cup A, x) = 0$ *for* $r = 1, 2$ *at all* $x \in K$. *Then* $K \cup A$ *is a closed* 2-*cell.* To prove this, note first that $K \cup A$ is 0-lc by Corollary VI 6.12 and Theorem VI 7.9. Then let M be an arc, with end points a and b, spanning K. We shall show that the set $(K \cup A) - M$ is not connected.

Suppose that $(K \cup A) - M$ is connected. As A is a 2-cell, the set $A - M = A_1 \cup A_2$ separate, where A_1 and A_2 are 2-cells. We shall show that the sets $\overline{A}_1 \cap K$ and $\overline{A}_2 \cap K$ are connected. As A_1 is a 2-cell, it is homeomorphic with the domain $\rho < 1$ of the (ρ, θ)-plane under a homeomorphism ϕ. Denote the simple closed curve in A_1 that corresponds under ϕ to the circle $\rho = 1 - 1/n$ by K_n, $n = 1, 2, \cdots$, and let $K' = M \cup (\overline{A}_1 \cap K)$. Evidently lim sup $K_n = K'$. Let $x, y \in K'$, and $x_n, y_n \in K_n$ such that $x = \lim x_n$ and $y = \lim y_n$. By application of Theorem IV 1.14, it readily follows that some component of lim sup K_n meets both x and y. Hence K' is connected. Then if $\overline{A}_1 \cap K$ is not connected, it must consist of two (possibly degenerate) arcs of K, meeting M in a and b respectively. Consequently $p^1(K') = 0$.

Now there exists on $K \cup A$ a 2-cycle Z^2 mod K such that $\partial Z^2 \sim \Gamma^1$ on K. The portion of Z^2 in A_1 is a cycle Z_1^2 mod K', and as $p^1(K') = 0$, $\partial Z_1^2 \sim 0$ on K'. Hence there exists a 2-cycle Γ^2 on $\overline{A}_1$ such that $\Gamma^2 \sim Z_1^2$ mod K'. But as $K \cup A$ is 2-dimensional, we may restrict ourselves to 2-dimensional coverings and therefore assume that $\Gamma^2 = Z_1^2$ mod K'. This also implies that $Z^2 = \Gamma^2$ mod $(K \cup A) - A_1$. But $\partial(Z^2 - \Gamma^2) = \partial Z^2$, since Γ^2 is an absolute cycle, and $Z^2 - \Gamma^2$ is a chain on $(K \cup A) - A_1$ such that $\partial(Z^2 - \Gamma^2) \sim \Gamma^1$ on K—implying $\Gamma^1 \sim 0$ on $(K \cup A) - A_1$. This contradicts the assumption that $K \cup A$ is an irreducible membrane relative to Γ^1. We conclude that $\overline{A}_1 \cap K$ and $\overline{A}_2 \cap K$ are connected.

Let $B_i = \overline{A}_i \cap (K - a - b)$, $i = 1, 2$. From (2) it follows that $B_1 \cup B_2 = K - a - b$. Then $(K \cup A) - M = (A_1 \cup B_1) \cup (A_2 \cup B_2)$, and since this set is supposed to be connected, and A_1 and A_2 are separated, the sets B_1, B_2 must have a common point p. Then B_1 and B_2 have an arc in common. For if K_1 and K_2 are the two components of $K - (a \cup b)$, where K_1 contains p, and $(B_1 \cap K_1) \cap (B_2 \cap K_1) = p$, then $K_2 \subset B_1 \cap B_2$ since $\overline{A}_1 \cap K$ and $\overline{A}_2 \cap K$ both contain $a \cup b$.

---

[3]The necessity for condition (3) even in this case is shown by the following example: Let A be the open region in the (ρ, θ)-plane consisting of all points for which $\rho < 1$, and let K' denote the circle $\rho = 1$. Let the configuration $K' \cup A$ be deformed as follows: Fold the arc of K' between $\theta = -\pi/3$ and $\theta = 0$ over, through an angle of 180°, onto the arc between $\theta = 0$ and $\theta = \pi/3$—in such a way that $(1, 0)$ remains fixed but $(1, -\pi/3)$ comes to coincide with $(1, \pi/3)$ and the deformed K' is still an S^1, which we denote by K. However, during the course of this deformation, let A also be slightly deformed so that after conclusion of the deformation, no pair of the points of A coincide; i.e., A is still a 2-cell with boundary K. Then at the point $x = (1, \pi/6)$, $p^2(K \cup A, x) > 0$.

Let xy be an arc of $B_1 \cap B_2$ and $q \in xy - x - y$. Then with Z_1^2 as above and Z_2^2 the portion of Z^2 in A_2, there exists a homology

$$c\partial Z_1^2 + d\partial Z_2^2 \sim 0 \bmod (K \cup M) - \langle xy \rangle, \qquad c, d \in \mathfrak{F}. \tag{7.11a}$$

From the homology (7.11a) it easily follows that $p^2(K \cup A, q) > 0$, and a contradiction of (3) results.

We conclude, then, that $K \cup A$ is disconnected by every arc that spans K, and hence, by Theorem VII 9.5, $K \cup A$ is a closed 2-cell. Thus Definition 7.10 gives a definition of the closed 2-cell principally in terms of compact cycles and their homologies (compare H. E. Vaughan [a], Principal Theorem B):

7.12 Theorem. *If M is a 2-dimensional compact metric space containing a simple closed curve K such that* (1) *M is an irreducible membrane relative to the fundamental cycle of K,* (2) $p^1(M - K) = 0$ (in the sense that all compact 1-cycles of $M - K$ bound on compact subsets of $M - K$) *and* (3) $p^1(M, x) = 0$ *or all $x \in M$ and $p^2(M, x) = 0$ or 1 according as $x \in K$ or $x \in M - K$, then M is a closed 2-cell with the boundary K.*

[Note that by Lemma 5.2, $M - K$ is connected and hence is a 2-cell by Theorem 5.9.]

It also follows from the various results of this chapter, particularly Theorem 7.6, that:

7.13 Theorem. *Every point of a regular n-manifold has arbitrarily small generalized n-cell neighborhoods.*

However, returning to the discussion of Definition 7.10, there is evidently a weakness due to the fact that for the nonseparable case the corresponding configuration is evidently not topologically unique, and uniqueness would be virtually a *sine qua non* for a generalization of homotopy. The problem arises: *Determine, for the nonseparable case, conditions which, when added to the conditions defining the spherelike n-gm and the generalized n-cell, yield topologically unique configurations.*

For $n = 1, 2$, solutions of this problem would probably not be formidable. (For example, topological homogenity and equivalent cardinality of bases suggest themselves as conditions to be considered.) For $n > 2$, it would seem to be of the same order of difficulty as the problem of characterizing the 3-sphere among the Peano spaces. However, aside from the question of homotopy, the generalized n-cell deserves recognition as a special type of configuration of the "homology" category to whose study we return in the next chapter.

Bibliographical comment

§3. The characterization embodied in Theorem 3.6 was given in Wilder [n], except that local n-avoidability was used instead of local n-extendibility.

§6. It seems probable that the local orientability property of an n-gm is equivalent to Axiom 7° of Čech [f; p. 686].

§7. Results concerning regular manifolds were abstracted in Wilder [A_5]. Regarding r-LC spaces, see Lefschetz [L_1].

CHAPTER X

SUBMANIFOLDS OF A MANIFOLD; DECOMPOSITION INTO CELLS

We shall now turn to the study of positional invariants. We have already discussed the subject in a general way in I 6.

1. Positional invariants. Topological invariants are a special case of a more general type of invariant which we call *positional topological invariant,* or simply positional invariant.

1.1 DEFINITION. If a space M is imbedded in a space S, then by a *positional invariant* of M in S we mean a property $P(M, S)$ which remains invariant under all topological transformations of M in S.

A positional invariant of M in S may be merely a topological invariant of M; for a topological invariant of M is always a positional invariant, no matter what M and S may be. That a positional invariant of M may be topological is frequently suggested by the intrinsic nature of the property. For example, if one proved the duality of Poincaré type for an n-gcm M by the device of imbedding it in some S^k (as a matter of fact, this is easily done if M can be imbedded in an S^{n+1}; see Wilder [n]), one would have the duality only as a positional invariant of M and S^k, but would certainly suspect it to be a topological invariant. Properties that are not topological invariants may be positional invariants, however, such as, for instance, certain metric properties of the complement of an S^{n-1} in S^n. If a domain complementary to an S^{n-1} in S^n is subjected to topological transformations, these metric properties (such as the ulc property; see Theorem II 5.35) are not preserved. But as S^{n-1} is subjected to topological transformations, within S^n, these properties remain invariant, and are therefore positional invariants of S^{n-1} in S^n. The Jordan Curve Theorem shows that the number of domains of the complement of an S^1 in S^2 is a positional invariant of the S^1 in S^2. More generally, Theorem VIII 6.4 shows that if M is any closed subset of S^n, then $h_a^r(S - M)$ is a positional invariant of M and S.

In the present chapter, M will usually be a closed subset of an n-gm S. And since we shall wish to apply the duality theorem of Alexander type, we shall assume that all n-gms are perfectly normal and locally orientable. Also, unless statement to the contrary is made, if U is an open subset of a compact space, then by $p^r(U)$ will be understood the dimension of $h^r(U)$. And if Z^r is a compact cycle of an open set U, then by $Z^r \sim 0$ *in* U is meant that Z^r bounds on a compact subset of U.

1.2 We shall find useful the notion of *Betti number around a point.* Specifically, if M is a closed subset of a space S, and $x \in M$, then we may define numbers $p^r(S - M, x)$, $p_r(S - M, x)$ in the same way that we defined the numbers $p^r(x)$, $p_r(x)$ for a point x of a space S in VI 6, except that P and Q are replaced by $P - M$ and $Q - M$, respectively. And we may likewise prove:

1.3 THEOREM. *If $x \in M \subset S$, where M is closed, then $p^r(S - M, x) = p_r(S - M, x)$.*

1.4 THEOREM. *If $p_r(S, x) = p_{r+1}(S, x) = 0$, where $x \in M \subset S$ and M is closed, then $p_r(M, x) = p_{r+1}(S - M, x)$.*

PROOF. Let U be an open set containing x and k an integer such that $p_r(x; U \cap M, P \cap M) \geqq k$ for all P, and let P be such that $p_r(x; U, P) = 0$. Let Q be an open subset of S such that $x \in Q \subset P$. Let $Z_r^1, \cdots, Z_r^k$ be cocycles of M in Q that are lircoh on M in U (i.e., lircoh mod $S - M$ in U). Then δZ_r^i, $i = 1, \cdots, k$, is a cocycle in $Q - M$. If the cocycles δZ_r^i are not lircoh in $P - M$, there exists a relation $\delta C_r = \sum_{i=1}^k a_i \pi_{\mathfrak{U}\mathfrak{B}}^* \delta Z_r^i$ in $P - M$, $\mathfrak{U}$ being a covering on which the cocycles Z_r^i have coordinates. Then $\sum_{i=1}^k \pi_{\mathfrak{U}\mathfrak{B}}^* a_i Z_r^i - C_r$ is a cocycle in P, and there exists a relation

$$\delta C_{r-1} = \sum_{i=1}^{k} \pi_{\mathfrak{U}\mathfrak{W}}^* a_i Z_r^i - \pi_{\mathfrak{B}\mathfrak{W}}^* C_r \qquad \text{in } U. \tag{1.4a}$$

But since $\pi_{\mathfrak{B}\mathfrak{W}}^* C_r$ is in $P - M$, (1.4a) implies that the cocycles Z_r^i are not lircoh in U. We must conclude, then, that the cocycles δZ_r^i are lircoh in $P - M$. It follows that $p_r(M, x) \leqq p_{r+1}(S - M, x)$.

For U, k so that $p_{r+1}(x; U - M, P - M) \geqq k$ for all P, let P, Q be open sets such that $x \in Q \subset P \subset U$ and $p_{r+1}(x; P, Q) = 0$. Let γ_{r+1}^i, $i = 1, \cdots, k$, be cocycles in $Q - M$ that are lircoh in $U - M$. There exist relations

$$\delta C_r^i(\mathfrak{B}) = \pi_{\mathfrak{U}\mathfrak{B}}^* \gamma_{r+1}^i \qquad \text{in } P, \qquad i = 1, \cdots, k. \tag{1.4b}$$

If $Z_r^i(\mathfrak{B})$ is the portion of $C_r^i(\mathfrak{B})$ on M, then $Z_r^i(\mathfrak{B})$ is a cocycle of M in P. If the cocycles $Z_r^i(\mathfrak{B})$ are not lircoh on M in U, there exists a relation

$$\delta C_{r-1}(\mathfrak{W}) = \pi_{\mathfrak{B}\mathfrak{W}}^* \sum a^i Z_r^i(\mathfrak{B}) - L_r(\mathfrak{W}), \tag{1.4c}$$

where $C_{r-1}(\mathfrak{W})$ is on M in U and $L_r(\mathfrak{W})$ in $U - M$. Then, applying δ to both sides of (1.4c), we get $\pi_{\mathfrak{B}\mathfrak{W}}^* \sum a^i \delta Z_r^i(\mathfrak{B}) = \delta L_r(\mathfrak{W})$; and, utilizing (1.4b), and the fact that $\delta Z_r^i(\mathfrak{B})$ and $\pi_{\mathfrak{U}\mathfrak{B}}^* \gamma_{r+1}^i$ are cohomologous in $P - M$, we get a chain $L_r'(\mathfrak{W})$ in $U - M$ and a relation $\pi_{\mathfrak{U}\mathfrak{W}}^* \sum a^i \gamma_{r+1}^i = \delta L_r'(\mathfrak{W})$, contradicting the fact that the cocycles γ_{r+1}^i are lircoh in $U - M$. Hence the cocycles $Z_r^i(\mathfrak{B})$ are lircoh on M in U, and it follows that $p_r(M, x) \geqq p_{r+1}(S - M, x)$.

Analogous to Theorem 1.4, we may also prove a "localization" of the Alexander Duality Theorem in the following manner: With x, M, S as before, let P, Q be open sets such that $x \in Q \subset P$, and denote by $q^r(x; P - M, Q - M)$ the maximum number of compact r-cycles in $Q - M$ that are lirh in $P - M$.

Using these numbers, we may define a number $q^r(S - M, x)$ by the usual limiting process (Chap. VI, §6). Then we may state:

1.5 THEOREM. *Let M be a closed subset of an n-gm S. Then if $x \in M$, $p^r_\alpha(M, x) = q^{n-r-1}_\alpha(S - M, x)$ for all $r \leqq n - 1$.*

The proof utilizes the same machinery that has been used above and in the proofs of the duality theorems in Chapter VIII, and will be left to the reader. We call attention to the fact that any neighborhood of x contains open sets P, Q such that $x \in Q \subset P$, $p_r(x; P, Q) = p_{r+1}(x; P, Q) = 0$, $p_n(x; P, Q) = 1$, and $P - M$ and $Q - M$ are orientable n-gms.

In contrast to the property of r-Culc defined in VI 2.5, but in analogy to the r-ulc property defined in II 5.31, we define:

1.6 DEFINITION. An open subset P of a space S will be called *r-ulc* if for arbitrary covering $\mathfrak{E}$ of S there exists a covering $\mathfrak{D}$ of S such that if γ^r is a compact cycle (augmented) of P of diameter $< \mathfrak{D}$, then $\gamma^r \sim 0$ on a compact subset of P of diameter $< \mathfrak{E}$ (IV 3.5).

[We shall understand a compact γ^r to be of diameter $< \mathfrak{D}$ if it has a carrier in some element of $\mathfrak{D}$.]

We can then state the following corollary of Theorem 1.5:

1.7 THEOREM. *Let M be a closed subset of an n-gcm S. If $p^r(M, x) = 0$ for all $x \in M$, $r \leqq n - 1$, then $S - M$ is $(n - r - 1)$-ulc.*

PROOF. Let $\mathfrak{E}$ be any covering of S. For each $x \in M$ there exist open sets D_x, E_x such that $x \in D_x \subset E_x$, E_x lies in an element of $\mathfrak{E}$, and (1) $E_x \subset M$ if $x \notin F(M)$, (2) if $x \in F(M)$, then (using Theorem 1.5) $q^{n-r-1}_\alpha(x; E_x - M, D_x - M) = 0$. For $x \in S - M$ we let $D_x \subset E_x$ be open sets containing x with $\overline{E}_x$ lying both in $S - M$ and an element of $\mathfrak{E}$ and such that (using Corollary VI 6.15) $g^r(x; E_x, D_x) = 0$. Let $\mathfrak{D}$ be a finite number of the sets D_x that cover S.

1.8 COROLLARY. *If M is a k-gcm which is a subset of an n-gcm S, then $S - M$ is r-ulc for $r = n - k, n - k + 1, \cdots, n - 1$.*

(Compare Theorem II 5.35.)

2. Uniform local co-connectedness; duality of r-ulc and $(n - r)$-coulc. Analogous to Definition 1.6 we have the following (which may be shown equivalent to Definition VI 6.3 in case $D = S$):

2.1 DEFINITION. An open subset D of a space S will be called *uniformly locally co-connected in dimension r* ($=$ *r-coulc*) if for arbitrary covering $\mathfrak{E}$ of S there exists a covering $\mathfrak{D} = \mathfrak{D}_r(D; \mathfrak{E})$ such that if Z_r is a cocycle of D of diameter $< \mathfrak{D}$, then $Z_r \backsim 0$ in D on a set of diameter $< \mathfrak{E}$.

For locally compact spaces and $\overline{D} - D$ compact, Definition 2.1 is readily shown to be equivalent to the following:

2.2 Definition. An open subset D of a space S will be called r-coulc if for arbitrary covering $\mathfrak{E}$ of S there exists a covering $\mathfrak{D}$ of S such that if Z_r is a compact cocycle of D of diameter $< \mathfrak{D}$ (i.e., Z_r has a carrier in some element of $\mathfrak{D}$), then $Z_r \backsim 0$ in an open set whose closure lies in D and which is of diameter $< \mathfrak{E}$.

For open subsets of an n-gcm a close relationship exists between the ulc and coulc properties, which we exhibit below. First, however, if U is an open subset of a space S, let us extend the notion of Betti number around a point $x \in S - U$ so as to include points of U; in other words, we do not restrict the position of x in the previous definitions, and hence $p_r(U, x) = p_r(S, x)$ if $x \in U$; etc.[1]

2.3 Lemma. *If U is an r-ulc open subset of a regular space S, then for every $x \in S$, $q^r(U, x) = 0$.*

Proof. Let P be any open set containing x. Let $\mathfrak{E}$ be a covering of S consisting of P and an open set P' such that $x \notin \bar{P}'$. Since U is r-ulc, there exists $\mathfrak{D} > \mathfrak{E}$ such that if Z^r is a compact cycle of $U \cap D$, $D \in \mathfrak{D}$, then $Z^r \sim 0$ in some $U \cap E$, $E \in \mathfrak{E}$. Select a $D \in \mathfrak{D}$ such that $x \in D$; evidently D must be a subset of P. Let $Q = D - \bar{P}'$; then $x \in Q \subset P$.

Consider a compact cycle Z^r of $U \cap Q$. Since Z^r is also in $U \cap D$, it bounds on a compact subset of $U \cap E$, $E \in \mathfrak{E}$. As Z^r does not lie in P', evidently $E = P$ and $Z^r \sim 0$ in $U \cap P$. It follows that $q^r(U, x) = 0$.

2.4 Corollary. *Let U be an r-ulc open subset of an n-gm S, $r \leqq n - 1$. Then if $x \in S - U$, $p^{n-r-1}(S - U, x) = 0$.*

Corollary 2.4 follows from Lemma 2.3 and Theorem 1.5.

2.5 Theorem. *A necessary and sufficient condition that an open subset U of a compact space S be r-ulc is that for all $x \in S$, $q^r(U, x) = 0$.*

(The sufficiency follows from an argument similar to that employed in proving Theorem 1.7.)

2.6 Corollary. *A necessary and sufficient condition that an open subset U of an n-gcm S be r-ulc, $r \leqq n - 1$, is that for all $x \in S - U$, $p^{n-r-1}(S - U, x) = 0$; or, alternatively, that $q^r(U, x) = 0$.*

2.7 Lemma. *If U is an r-coulc open subset of a regular space S, then for every $x \in S$, $p_r(U, x) = 0$.*

The proof of Lemma 2.7 is similar to that of Lemma 2.3.

2.8 Theorem. *A necessary and sufficient condition that an open subset U of a compact space S be r-coulc is that for all $x \in S$, $p_r(U, x) = 0$.*

[1]Henceforth we understand that only augmented cycles are employed unless specific statement is made to the contrary. We therefore usually delete the subscript "a".

PROOF. The necessity follows from Lemma 2.7, and the sufficiency follows from an argument similar to that used in proving Theorem 1.7.

2.9 COROLLARY. *A necessary and sufficient condition that an open subset U of an n-gcm S be r-coulc, $r \leqq n - 1$, is that for all $x \in S - U$, $p_r(U, x) = 0$.*

2.10 COROLLARY. *A necessary and sufficient condition that an open subset U of an n-gcm S be $(r + 1)$-coulc, $r < n - 1$, is that for all $x \in S - U$, $p_r(S - U, x) = 0$.*

2.11 LEMMA. *If S is an n-gm, and M a closed subset of S, then for every $x \in M$ and $0 < r < n$, $p_{n-r}(S - M, x) = q^r(S - M, x)$.*

PROOF. Since $0 < r$, $n - r$ is less than n and therefore $p_{n-r-1}(S, x) = p_{n-r}(S, x) = 0$. Hence by Theorem 1.4, $p_{n-r-1}(M, x) = p_{n-r}(S - M, x)$.

Now $p_{n-r-1}(M, x) = p^{n-r-1}(M, x)$, and since $r < n$, $p^{n-r-1}(M, x) = q^r(S - M, x)$ by Theorem 1.5. Hence $p_{n-r}(S - M, x) = q^r(S - M, x)$.

By virtue of Lemma 2.11 and corollaries 2.6, 2.9, we can now state:

2.12 MAIN THEOREM. *For open subsets of an n-gcm, and $0 < r < n$, the properties r-ulc and $(n - r)$-coulc are equivalent.*

2.13 THEOREM. *If S is an orientable n-gcm and M is a closed subset of S having only a finite number of components, none of which is degenerate, then $S - M$ is both $(n - 1)$-ulc and 1-coulc.*

PROOF. By Theorem VI 6.9, $p^0_a(M, x) = 0$ for all $x \in M$. Hence by Corollary 2.6, $S - M$ is $(n - 1)$-ulc and by Theorem 2.12 is 1-coulc.

3. The Jordan-Brouwer type of separation theorem in an n-gcm, and its converse. Suppose M is an orientable $(n - 1)$-gcm in an orientable n-gcm S such that $p^{n-1}(S) = 0$. Then by Theorem VIII 6.4, $p^0_a(S - M) = 1$, so that by Theorem V 11.10, $S - M$ has exactly two components A and B. The set M is the common boundary of A and B. For suppose $x \in M$ and P is an open set containing x. Then $p^{n-1}(M - P) = 0$ by conditions A and D of the definition of $(n - 1)$-gcm and, by Theorem VIII 6.4, $p^0_a[S - (M - P)] = 0$. Consequently $A \cup B \cup (M \cap P)$ is a connected point set. As A and B are separated and $M \cap P$ is closed relative to $A \cup B \cup (M \cap P)$, it follows that both A and B have limit points in $M \cap P$. We have therefore proved:

3.1 JORDAN-BROUWER TYPE OF SEPARATION THEOREM FOR AN n-GCM. *If M is an orientable $(n - 1)$-gcm in an orientable n-gcm S such that $p^{n-1}(S) = 0$, then $S - M$ is the union of two disjoint domains A and B that have M as their common boundary.*

Theorem 3.1 contains, of course, the theorem (Brouwer) on the separation of the euclidean S^n by the S^{n-1}, or more generally by the orientable closed $(n - 1)$-manifold in the classical sense, as a special case.

Continuing with M as in Theorem 3.1, $p_r(M, x) = 0$ for $r = 0, 1, \cdots, n - 2,$

and by Corollary 2.10, $S - M$ is r-coulc for $r = 1, 2, \cdots, n - 1$. It follows that A and B are respectively r-coulc for $r = 1, 2, \cdots, n - 1$, and, by Theorem 2.12, that both A and B are r-ulc for the same range of values of r. We shall show that A and B are also 0-ulc.

Let $x \in M$. Then $p_{n-1}(M, x) = 1$ and therefore, by Theorem 1.5, $q^0(S - M, x) = 1$. Hence if P is a small enough open set containing x, there exists an open set Q such that $x \in Q \subset P$ and $q_a^0(x; P - M, Q - M) = 1$. This implies that exactly one (augmented) 0-cycle of $Q - M$ is lirh in $P - M$, which in turn implies (Theorem V 11.10) that exactly two components of $P - M$ meet Q. Since A and B both meet Q, these components must lie in A and B respectively. In particular, exactly one component of $A \cap P$ meets Q and it follows that A is 0-ulc (and B is 0-ulc).

REMARK. Note that the argument just given also shows: *If the complement of an $(n - 1)$-gcm M in an n-gcm contains two domains A and B having M as common boundary, then both A and B are r-ulc for $r = 0, 1, \cdots, n - 1$.*

3.2 THEOREM. *The domains A and B of Theorem* 3.1 *are r-ulc for $r = 0$ $1, \cdots, n - 1$.*

(Compare Theorem II 5.35.)

Now suppose, conversely, that M is an $(n - 1)$-dimensional[2] closed subset of an orientable n-gcm S such that $p^{n-1}(S) = 0$, and such that $S - M$ is the union of exactly two r-ulc ($r = 0, 1, \cdots, n - 2$) domains A and B of which M is the common boundary. We shall show that M is an orientable $(n - 1)$-gcm.

In the first place, $p^{n-1}(M) = 1$ by Theorem VIII 6.4, and hence if M is an $(n - 1)$-gcm it is orientable. And that condition D of the definition of $(n - 1)$-gcm is satisfied follows from the fact that M is the common boundary of A and B. We may henceforth assume $n > 1$.

Since A and B are r-ulc for $r = 1, \cdots, n - 2$, and $(n - 1)$-ulc by Theorem 2.13, it follows that $A \cup B = S - M$ has the same property. Hence by Theorem 2.12, $S - M$ is r-coulc for the same range of values of r, and by Corollary 2.10, $p_r(M, x) = 0$ for all $x \in M$ and $r = 0, 1, \cdots, n - 2$. Thus condition B of the definition of $(n - 1)$-gcm is satisfied.

It remains to prove condition C of the definition of $(n - 1)$-gcm; i.e., that $p_{n-1}(M, x) = 1$ for every $x \in M$. Since A and B are 0-ulc, there exists for arbitrary covering $\mathfrak{E}$ of S a covering $\mathfrak{D}$, such that if Z^0 is a compact cycle of A (B) of diameter $< \mathfrak{D}$, then Z^0 bounds on a compact subset of A (B) of diameter $< \mathfrak{E}$. Let $x \in M$ and P be any open set containing x. Let P' be an open set covering $S - P$ and such that $x \notin \overline{P'}$, and denote by $\mathfrak{E}$ the covering of S

[2]Since the spaces with which we are dealing are perfectly normal, it seems likely that a frontier set in an n-gm is of necessity $(n - 1)$-dimensional if it forms a local barrier. However, this seems not to have been shown in the literature on dimension theory, and in the lack of such a demonstration we place this dimension theoretic restriction in the hypothesis of this theorem. It is supplied only in order to give condition A of the definition of an $(n - 1)$-gm.

consisting of P and P'. Let $\mathfrak{D}$ be a refinement of $\mathfrak{C}$ such as just described above, and consider any open set Q such that $x \in Q \subset P$ and such that Q is a subset of some set $D - \bar{P}'$, where $D \in \mathfrak{D}$. That $q^0(x; P - M, Q - M) \geqq 1$ follows from the fact that M is the common boundary of A and B. On the other hand, if Z^0 is a compact cycle of A (B) in Q, then Z^0, being also in D, bounds in a set $A \cap E$ $(B \cap E)$ for some $E \in \mathfrak{C}$. Such a set E must lie in P, and it follows that $A \cap P$ $(B \cap P)$ has at most one component which meets Q. Thus $q^0(x; P - M, Q - M) \leqq 1$. We conclude that $q^0(S - M, x) = 1$, and, from Theorem 1.5, that $p^{n-1}(M, x) = 1$. We have then proved:

3.3 Converse of the Jordan-Brouwer separation theorem for an orientable n-gcm. *If M is an $(n - 1)$-dimensional closed subset of an orientable n-gcm S such that $p^{n-1}(S) = 0$ and $S - M$ is the union of exactly two r-ulc $(r = 0, 1, \cdots, n - 2)$ domains of which M is the common boundary, then M is an orientable $(n - 1)$-gcm.*

4. **Generalization.** It might be expected that Theorems 3.1-3.3, on the positional status of an $(n - 1)$-gcm in an n-gcm, are the $(n - 1)$-dimensional cases of theorems on the positional properties of a k-gcm in an n-gcm. It is the purpose of this section to show that this is the case.

4.1 Definition. A compact cycle Z^r of the complement of a closed subset M of an orientable n-gcm S is said to *link M irreducibly* if it links M but does not link any proper closed subset of M.

4.2 Lemma. *Let Z^r be a compact cycle of the complement of a closed subset M of an orientable n-gcm S, which links M irreducibly. Then if $x \in M$ and P is an open set containing x, there exists in $P - M$ a compact cycle γ^r such that $Z^r \sim \gamma^r$ on a compact subset of $S - M$.*

Proof. Let K denote a carrier of Z^r in $S - M$, and let Q be an open set such that $x \in Q \Subset P - K$. Then Z^r bounds on a compact subset L of $S - (M - Q)$, and by Lemma VII 1.4 there is a cycle γ^{r+1} mod K on L such that $\partial\gamma^{r+1} \sim Z^r$ on K. Denote $L - Q$ by K'. Then γ^{r+1} is a cycle mod K' on L, and by Corollary VII 1.16 there exists a cycle γ_1^{r+1} mod $F(Q)$ on $L \cap \bar{Q}$ such that $\partial\gamma^{r+1} \sim \partial\gamma_1^{r+1}$ on K'. The cycle $\partial\gamma_1^{r+1}$ is in $P - M$ and $\partial\gamma_1^{r+1} \sim Z^r$ on $K' \subset S - M$.

4.3 Definition. If G is a special class of compact cycles, then an open subset P of a space S will be called *r-ulc relative to G* if for arbitrary covering $\mathfrak{C}$ of S there exists a covering $\mathfrak{D}$ of S such that if $\gamma^r \in G$ has a carrier of diameter $< \mathfrak{D}$, then $\gamma^r \sim 0$ on a compact subset of P of diameter $< \mathfrak{C}$.

Regarding the property of being r-ulc relative to a special class G, theorems like those proved above for ordinary r-ulc may be proved; in particular, we want the following lemma. By $q^r(U, x; G)$ we shall denote a number determined exactly like $q^r(U, x)$, except that only cycles of G are taken into consideration.

4.4 Lemma. *A necessary and sufficient condition that an open subset U of an n-gcm S be r-ulc relative to a class G of compact cycles is that the number $q^r(U, x; G) = 0$ for all $x \in S - U$.*

The proof of the necessity of Lemma 4.4 is like that of Lemma 2.3, and the proof of the sufficiency like that of Theorem 1.7.

Now Theorems 3.1-3.3 are all derivable from the following general theorem:

4.5 Theorem. *In order that a k-dimensional closed subset M of an orientable n-gcm S for which $p^k(S) = p^{k+1}(S) = 0$ if $k < n - 1$, and $p^{n-1}(S) = 0$ if $k = n - 1$, should be an orientable k-gcm, $k < n$, it is necessary and sufficient that (1) $p^{n-k-1}(S - M) = 1$, but that $p^{n-k-1}(S - F) = 0$ if F is a proper closed subset of M, (2) in case $k > 1$, $S - M$ be r-ulc for $r = n - k, \cdots, n - 2$, and (3) $S - M$ be $(n - k - 1)$-ulc relative to the group G of bounding compact cycles of $S - M$.*

Proof of necessity. Condition (1) follows from Corollary VIII 3.2 and Theorem VIII 6.4. Condition (2) follows from Corollary 1.8.

To prove (3), note that by Theorem VIII 8.4 there exists a base for $(n - k - 1)$-cycles of $S - M$ relative to homologies in $S - M$ consisting of a single compact cycle γ^{n-k-1}. The cycle γ^{n-k-1} links M irreducibly. By Theorem 1.5, if $x \in M$ there exist open sets P and Q such that $x \in Q \subset P$ and such that $q^{n-k-1}(x; P - M, Q - M) = 1$. Then since by Lemma 4.2, $Q - M$ contains a cycle z^{n-k-1} which is in the same homology class of $S - M$ as γ^{n-k-1} and hence nonbounding in $P - M$, we may choose z^{n-k-1} as a base for cycles of $Q - M$ relative to homologies in $P - M$. Let w^{n-k-1} be a bounding cycle of $S - M$ that lies in $Q - M$. Then $w^{n-k-1} \sim 0$ in $P - M$. For otherwise there exists a homology $cw^{n-k-1} \sim z^{n-k-1}$ in $P - M$. But then $cw^{n-k-1} \sim \gamma^{n-k-1}$ in $S - M$, and as $w^{n-k-1} \sim 0$ in $S - M$, it would follow that $\gamma^{n-k-1} \sim 0$ in $S - M$. Consequently $q^{n-k-1}(S - M, x; G) = 0$ for all $x \in M$ and condition (3) follows from Lemma 4.4. (It should be pointed out that $S - M$ is obviously r-ulc for all $r < n - k - 1$ also, but this is not needed to give a characterization of the orientable k-gcm by positional invariants.)

Proof of sufficiency. Let M be given satisfying conditions (1)-(3) of the theorem. Condition (1) will yield the orientability as well as condition D of the definition of k-gcm. And when $k > 1$, condition (2) yields condition B of the definition of k-gm by Corollary 2.4 and Lemma 2.13. It remains to prove condition C, namely that for each $x \in M$, $p_k(M, x) = 1$.

Let $x \in M$, P an open set containing x, and Q an open set such that $x \in Q \subset P$ and a bounding compact $(n - k - 1)$-cycle of $S - M$ which lies in Q must bound in $P - M$. Let γ^{n-k-1} be a base for compact cycles relative to homologies in $S - M$—such a cycle exists by condition (1). By Lemma 4.2, there exists a compact cycle z^{n-k-1} in $Q - M$ such that $\gamma^{n-k-1} \sim z^{n-k-1}$ in $S - M$. Consider any compact cycle w^{n-k-1} of $Q - M$. If $w^{n-k-1} \sim 0$ in $S - M$, then $w^{n-k-1} \sim 0$ in $P - M$ because of the choice of Q. If $w^{n-k-1} \nsim 0$ in $S - M$,

PROOF. Suppose the open subset U of an n-gcm S is 0-ulc and that $\mathfrak{E}$ is any covering of S. Let $\mathfrak{E}' >^* \mathfrak{E}$, and $\mathfrak{D}$ a covering such that if Z^0 is an augmented compact cycle of U of diameter $< \mathfrak{D}$, then Z^0 bounds on a compact subset of U of diameter $< \mathfrak{E}'$. Then suppose that D is an open subset of S of diameter $< \mathfrak{D}$ that meets U. Let $D' \in \mathfrak{D}$ contain D. Then the union of all elements of $\mathfrak{E}'$ that contain D' lies in some $E \in \mathfrak{E}$, and if p and q are points of $U \cap D$, then a nontrivial 0-cycle on $p \cup q$ bounds in $U \cap E$. It follows that $U \cap D$ lies in one component of $U \cap E$. This component is open in S since S is 0-lc, and hence by Theorem VII 5.7, $p_n(S\colon U \cap D, 0; U \cap E, 0) \leqq 1$. That $p_n(S\colon U \cap D, 0; U \cap E, 0) \geqq 1$ follows from the fact that $D \cap U \neq 0$.

Conversely, suppose $p_n(U, x)$ is uniformly $= 1$ over U, and let $\mathfrak{E}$ be any covering of S. Select $\mathfrak{D}$ as in Definition 5.1, and suppose $D \in \mathfrak{D}$ such that $D \cap U \neq 0$. There exists $E \in \mathfrak{E}$ containing D such that $p_n(S\colon U \cap D, 0; U \cap E, 0) = 1$, and, the union of the components of $U \cap E$ that meet D being open, it follows from Theorem VII 5.7 and its proof that the number of these components is 1.

5.3 LEMMA. *If U is an r-coulc open subset of a compact space S, $P \supset Q$ are arbitrary open sets in S and $\mathfrak{E}$ a covering of S, then there exists a covering $\mathfrak{D}_r' = \mathfrak{D}_r'(U\colon \mathfrak{E}; P, Q) > \mathfrak{E}$ such that if $Z_r(\mathfrak{U})$ is a compact cocycle in $Q \cap U$ of diameter $< \mathfrak{D}_r'$, then $Z_r(\mathfrak{U}) \backsim 0$ in an open subset of $P \cap U$ of diameter $< \mathfrak{E}$.*

PROOF. Select a covering $\mathfrak{E}' > \mathfrak{E}$ such that $\mathrm{St}(Q, \mathfrak{E}') \subset P$, and let $\mathfrak{D}_r' > \mathfrak{E}'$ be a covering such that a compact r-cocycle of U of diameter $< \mathfrak{D}_r'$ cobounds in an open subset of U of diameter $< \mathfrak{E}'$.

Similarly, we have:

5.4 LEMMA. *If U is an open subset of a compact space S such that $p_r(U, x)$ is uniformly $=1$ over U, $P \supset Q$ are arbitrary open subsets of S, and $\mathfrak{E}$ is a covering of S, then there exists a covering $\mathfrak{D}_r'' = \mathfrak{D}_r''(U\colon \mathfrak{E}; P, Q) > \mathfrak{E}$ such that if $Z_r(\mathfrak{U})$ and $\gamma_r(\mathfrak{U})$ are compact cocycles in $Q \cap U$ which lie together in a set of diameter $< \mathfrak{D}_r''$, then $aZ_r(\mathfrak{U}) \backsim b\gamma_r(\mathfrak{U})$, $a, b \in \mathfrak{F}$, in $P \cap U$ in a set of diameter $< \mathfrak{E}$.*

PROOF. Select a covering $\mathfrak{E}' > \mathfrak{E}$ such that $\mathrm{St}(Q, \mathfrak{E}') \subset P$, and let $\mathfrak{D}_r'' > \mathfrak{E}'$ be a covering such that if D is an open set of diameter $< \mathfrak{D}_r''$ that meets U, then there exists $E' \in \mathfrak{E}'$ containing D such that $p_r(S\colon U \cap D, 0; U \cap E', 0) = 1$.

5.5 DEFINITION. Hereafter, if a set is r-ulc for $r = m, m + 1, \cdots, k$, we denote this fact by the symbol ulc_m^k, with the exception that the symbol ulc_0^k will be abbreviated to ulc^k. Similar symbols may be introduced for the coulc properties.

5.6 LEMMA. *If U is a ulc^k open subset of an orientable n-gcm S, $P \supset Q$ are arbitrary open sets in S and $\mathfrak{E}$ a covering of S, then there exists a covering $\mathfrak{D}_k^* = \mathfrak{D}_k^*(U\colon \mathfrak{E}; P, Q) > \mathfrak{E}$ such that if $\tau^{*\prime}$ is a partial co-realization of an at most $(k + 1)$-dimensional complex K in $Q \cap U$ of norm $< \mathfrak{D}_k^*$ on some covering $\mathfrak{U}$, then there exists a co-realization τ^* of K in $P \cap U$ of norm $< \mathfrak{E}$ on a $\mathfrak{V} > \mathfrak{U}$ such that $\tau^* C^r = \pi_{\mathfrak{U}\mathfrak{V}}^* \tau^{*\prime} C^r$ wherever the latter is defined for chains C^r of K.*

from $\mathfrak{U}_1$. And as in the latter proof, we may make a partial realization τ' of K on $\mathfrak{U}_1$ of norm $< \mathfrak{U}_n^*$ on $\mathfrak{U}_1 \wedge \overline{Q}_2$, which can be extended to a realization of K on $(\mathfrak{U}_1 \cup \mathfrak{U}_0) \wedge \dot{Q}_1$. The proof now concludes as in the corresponding part of the proof of Theorem VIII 5.4.

5.8 Theorem. *If U is a ulc^k open subset of an orientable n-gcm S, then $\overline{U}$ is lc^k.*

Proof. Consider $x \in F(U)$ and P any open set containing x. Since by Lemma 2.3, $q^r(U, x) = 0, r \leqq k$, there exists an open set Q such that $x \in Q \subset P$ and $q^r(x; P \cap U, Q \cap U) = 0$. Let γ^r be a cycle of $\overline{U}$ in Q, and let $\mathfrak{U}$ be any covering of S. Denote $\overline{Q} \cap \overline{U}$ by A. Then by Lemma V 8.7, there exists a covering $\mathfrak{V} > \mathfrak{U}$ and an open set R containing A such that a cell of $\mathfrak{V}$ that meets R also meets A. By Lemma 5.7 there is a compact cycle Z^r in $R \cap Q \cap U$ such that $\gamma^r \sim Z^r$ in $R \cap Q$. In particular, $\gamma^r(\mathfrak{V}) \sim Z^r(\mathfrak{V})$ on R and there exists a chain $C^{r+1}(\mathfrak{V})$ on R such that

$$\partial C^{r+1}(\mathfrak{V}) = \gamma^r(\mathfrak{V}) - Z^r(\mathfrak{V}). \tag{5.8a}$$

Now the relation (5.8a) being on R, must also be on A, so that $\partial \pi_{\mathfrak{U}\mathfrak{V}} C^{r+1}(\mathfrak{V}) = \pi_{\mathfrak{U}\mathfrak{V}}\gamma^r(\mathfrak{V}) - \pi_{\mathfrak{U}\mathfrak{V}}Z^r(\mathfrak{V})$ holds on A. Since γ^r and Z^r are on A, this implies that $\gamma^r(\mathfrak{U}) \sim Z^r(\mathfrak{U})$ on $\underline{A}$. By the choice of Q, $Z^r \sim 0$ in $P \cap U$. It follows that $\gamma^r(\mathfrak{U}) \sim 0$ in $P \cap \overline{U}$, and $\overline{U}$ is r-lc at x.

We can now prove a theorem which strengthens the result of Lemma 5.7 for $r \leqq k$:

5.9 Theorem. *If U is a ulc^k open subset of an orientable n-gcm S, γ^r is a cycle on a closed subset L of $\overline{U}$, $r \leqq k$, and P any open set containing L, then there exists in $P \cap U$ a compact cycle Z^r such that $\gamma^r \sim Z^r$ on $P \cap \overline{U}$.*

Proof. Let R be an open set such that $P \supseteq \overline{R} \supset L$. Since $\overline{U}$ is lc^k by Theorem 5.8, there exists by Corollary VI 3.7 a covering $\mathfrak{U}_0$ of S such that if γ^r and Z^r are two cycles on $\overline{R} \cap \overline{U}$ which are homologous on $\mathfrak{U}_0 \wedge \overline{R} \cap \overline{U}$ then γ^r and Z^r are homologous on $P \cap \overline{U}$. Let $\mathfrak{V} > \mathfrak{U}_0$ be a covering and Q an open set such that $R \supset Q \supset L$ and such that if a cell of $\mathfrak{V}$ meets Q, then it meets L.

By Lemma 5.7 there exists a compact cycle Z^r in $Q \cap U$ such that $\gamma^r \sim Z^r$ in Q. Hence there exists a chain $C^{r+1}(\mathfrak{V})$ such that $\partial C^{r+1}(\mathfrak{V}) = \gamma^r(\mathfrak{V}) - Z^r(\mathfrak{V})$ on Q, hence on L. Consequently $\pi_{\mathfrak{U}_0\mathfrak{V}}\gamma^r(\mathfrak{V}) \sim \pi_{\mathfrak{U}_0\mathfrak{V}}Z^r(\mathfrak{V})$ on L and a fortiori on $Q \cap \overline{U}$, so that $\gamma^r(\mathfrak{U}_0) \sim Z^r(\mathfrak{U}_0)$ on $Q \cap \overline{U}$ and a fortiori on $\mathfrak{U}_0 \wedge \overline{R} \cap \overline{U}$. Therefore $\gamma^r \sim Z^r$ on $P \cap \overline{U}$.

By the same methods used in proving Lemma 5.7 and Theorem 5.9 we may prove the following lemmas:

5.10 Lemma. *If U is a ulc^k open subset of an orientable n-gcm S, K and M closed sets lying in U and $\overline{U}$ respectively, γ^r a cycle* mod K *on M, $r \leqq k + 1$, and P any open set containing M, then there exists in $P \cap U$ a compact set M' carrying a cycle Z^r* mod K *such that $\gamma^r \sim Z^r$* mod K *in P.*

to the dimension $n - 1$. (I.e., B satisfies the same duality relative to the numbers $p^k(B)$ as an orientable $(n - 1)$-gcm.)

PROOF. By Theorem 5.13, the following relations hold:

$$p^k(U) = p^{n-k-1}(S - \overline{U}), \qquad p^k(S - \overline{U}) = p^{n-k-1}(U). \tag{5.15a}$$

Adding the relations (5.15a) gives $p^k(S - B) = p^{n-k-1}(S - B)$, and both of these numbers are finite since the numbers involved in relations (5.15a) are finite by Theorem 5.13. Hence by the Alexander duality $p^k(B) = p^{n-k-1}(B)$.

5.16 LEMMA. *If U is an open subset of an orientable n-gcm S, K and M are closed subsets of $\overline{U}$, $K \subset M$, P and P' open subsets of S containing M and K, respectively, and γ^r a cycle* mod K *on M, $r \leqq k + 1$, then there exists a compact set M' in $P \cap U$ carrying a cycle Z^r* mod P' *such that $\gamma^r \sim Z^r$* mod P' *in P, and $\partial\gamma^r \sim \partial Z^r$ on $P' \cap \overline{U}$.*

Lemma 5.16 follows easily from Lemmas 5.9 and 5.10: We may suppose $P' \subset P$. By Lemma 5.9 there exists a cycle Z^{r-1} on a compact subset K' of $P' \cap U$ such that $\partial\gamma^r \sim Z^{r-1}$ on $P' \cap \overline{U}$. Let $L \supset K \cup K'$ be a compact subset of $P' \cap \overline{U}$ carrying this homology. Then there exists a cycle C^r mod K' on $L \cup M$ such that $C^r \sim \gamma^r$ mod P' on $L \cup M$. By Lemma 5.10 there exists in $P \cap U$ a compact set M' carrying a cycle Z^r mod K' such that $C^r \sim Z^r$ mod K' in P. Then $\gamma^r \sim Z^r$ mod P' in P and $\partial\gamma^r \sim \partial Z^r$ on $P' \cap \overline{U}$.

Similarly we have, analogous to Lemma 5.11:

5.17 LEMMA. *Under the same hypothesis as in Lemma 5.16, but with $r \leqq k$, there exists Z^r as before, but with $\gamma^r \sim Z^r$ mod P' on $P \cap \overline{U}$.*

5.18 LEMMA. *If U is a ulc^k open subset of an orientable n-gcm S, then $\overline{U}$ is completely r-avoidable at every $x \in \overline{U}$, $r \leqq k$, $r < n - 1$.*

PROOF. The lemma is trivial for $x \in U$, since S is an n-gcm (Corollary IX 3.2). Let $x \in \overline{U} - U$, and P, Q, R open sets such that $x \in R \Subset Q \Subset P$, and such that r-cycles of $\overline{U} \cap \overline{Q}$ bound on $P \cap \overline{U}$. By Theorem 5.8 and Corollary VI 3.8 there exists a finite base γ_i^r, $i = 1, \cdots, m$, of r-cycles of $\overline{U} \cap F(Q)$ relative to homologies in $\overline{U} \cap (P - \overline{R})$. Each of the cycles γ_i^r bounds on $P \cap \overline{U}$, and hence there exists a cycle C_i^{r+1} mod $F(Q)$ on $P \cap \overline{U}$ such that $\partial C_i^{r+1} \sim \gamma_i^r$ on $\overline{U} \cap F(Q)$. By Lemma 5.16, there exists a compact set M_i' in $P \cap U$ carrying a cycle Z_i^{r+1} mod $P - \overline{R}$ such that $\partial C_i^{r+1} \sim \partial Z_i^{r+1}$ in $(P - \overline{R}) \cap \overline{U}$ and $C_i^{r+1} \sim Z_i^{r+1}$ mod $P - \overline{R}$ in P. Then $\gamma_i^r \sim \partial Z_i^{r+1}$ in $(P - \overline{R}) \cap \overline{U}$, and hence $\gamma_i^r \sim 0$ on a closed subset L_i of P such that $L_i \cap R \subset U$. We may now select an open set R' such that $x \in R' \subset R - \bigcup L_i$. Then if γ^r is an arbitrary cycle of $\overline{U} \cap F(Q)$, there exists a set of elements $a_i \in \mathfrak{F}$ such that $\gamma^r \sim \sum a_i\gamma_i^r \sim 0$ in $(P - R') \cap \overline{U}$.

5.19 THEOREM. *If U is a ulc^k open subset of an orientable n-gcm S, $k \leqq n - 1$, then $S - \overline{U}$ is ulc$^{n-1}_{n-k-1}$.*

Proof. Let r be an integer such that $0 \leqq r \leqq k$. To prove that $S - \overline{U}$ is $(n - r - 1)$-ulc, it is sufficient to show that $p^r(\overline{U}, x) = 0$ for all $x \in \overline{U}$, by Theorem 1.7. Let $x \in \overline{U}$ and P an arbitrary open set containing x. Then by Lemmas 5.8 and 5.18, there exist open sets Q and R such that $x \in R \Subset Q \Subset P$, r-cycles of $P \cap \overline{U}$ bound on $\overline{U}$, and $(r - 1)$-cycles of $\overline{U} \cap F(Q)$ bound in $(P - \overline{R}) \cap \overline{U}$. Consider a cycle γ^r of $\overline{U}$ mod $(S - P)$. Then γ^r is also a cycle mod $S - Q$, and by Lemma VII 1.16 there exists a cycle Z^r mod $S - Q$ on $\overline{Q} \cap \overline{U}$ such that

$$Z^r \sim \gamma^r \quad \text{mod } S - Q \text{ on } \overline{U}. \tag{5.19a}$$

Since $\partial Z^r \sim 0$ on $(P - \overline{R}) \cap \overline{U}$, there exists a cycle C^r mod K, where K is a carrier of ∂Z^r on $F(Q)$, on $(P - \overline{R}) \cap \overline{U}$, such that $\partial C^r \sim \partial Z^r$ on K. By Lemma VII 1.6, there exists a cycle Γ^r on $P \cap \overline{U}$ such that $\Gamma^r \sim Z^r - C^r$ mod K. By the choice of P, $\Gamma^r \sim 0$ on $\overline{U}$. Hence $Z^r - C^r \sim 0$ mod $S - R$ on $\overline{U}$. Combining this homology with (5.19a), we have that $\gamma^r \sim 0$ mod $S - R$ on $\overline{U}$.

6. The boundary of a ulc^{n-2} domain in a manifold. Before continuing with the general open subset of an n-gcm and its ulc properties, we consider the case of a single domain. Here we may obtain a remarkable generalization of Theorem 3.3, in that we show that not only is it sufficient to impose the ulc^{n-2} condition on only one of the domains in question, but it is unnecessary to make any assumption regarding the number of domains complementary to the given closed set.

6.1 Lemma. *If U is a 0-ulc open subset of a space S, and C is a component of the boundary of U, then C is on the boundary of one and only one component of U.*

Proof. Let $x \in C$. Since U is 0-ulc, there exists an open set P containing x such that every 0-cycle of $P \cap U$ bounds in U. Hence x is on the boundary of only one component of U.

Let x and y be arbitrary points of C, and let $\mathfrak{E}$ be any covering of S. Let $\mathfrak{D} > \mathfrak{E}$ such that a compact 0-cycle of U of diameter $< \mathfrak{D}$ bounds on a compact subset of U of diameter $< \mathfrak{E}$, and let $\mathfrak{U} >^* \mathfrak{D}$. As C is connected, there exists a simple chain $U_1, \cdots, U_i, \cdots, U_m$ of elements of $\mathfrak{U}$ from x to y, each U_i containing points of C. Let $x_i \in U \cap U_i$. Then a nontrivial 0-cycle on $x_i \cup x_{i+1}$, $i = 1, \cdots, m - 1$, bounds on a compact subset of U, and it follows from Corollary V 11.11 that x_1 and x_m lie in the same component of U. The conclusion of the lemma now follows easily.

6.2 Lemma. *If x is a point of an orientable n-gcm S and P an open set containing x, then there exists an open subset Q of P containing x and a cycle γ^{n-1} on $F(Q)$ such that if $x' \in Q$ and $y \in S - \overline{P}$, then $\gamma^{n-1} \nsim 0$ in $S - x' - y$.*

Proof. Let $Q \Subset P$ be an open set containing x such that every $(n - 1)$-cycle on $F(Q)$ bounds in P. Then if Γ^n is the fundamental n-cycle of S, Γ^n

is a cycle mod $S - Q$, and by Lemma VII 1.16 there exists a cycle Z^n mod $F(Q)$ on $\overline{Q}$ such that $\Gamma^n \sim Z^n$ mod $S - Q$ and $\partial Z^n \sim \partial \Gamma^n$ on $S - Q$.

Suppose $x' \in Q$, $y \in S - \overline{P}$, and that $\partial Z^n \sim 0$ on $S - x - y$. Let R_1 and R_2 be open sets containing x' and y respectively, such that $R_1 \subset Q$, $R_2 \subset S - \overline{P}$, and $\partial Z^n \sim 0$ on $S - R$, where $R = R_1 \cup R_2$. Then Z^n is a cycle mod $S - R$ on $S - R_2$ such that $\partial Z^n \sim 0$ on $S - R$, and by Lemma VII 1.6 there exists a cycle γ^n on $S - R_2$ such that $\gamma^n \sim Z^n$ mod $S - R$.

Now γ^n must bound on S, since it lies on a closed proper subset of S, and hence $Z^n \sim 0$ mod $S - R$. Then a fortiori $Z^n \sim 0$ mod $S - R_1$, and as $\Gamma^n \sim Z^n$ mod $S - Q$, it follows that $\Gamma^n \sim Z^n \sim 0$ mod $S - R_1$. But this is impossible by Lemma VII 3.6.

6.3 Lemma. *If U is a ulc^{n-1} open subset of an orientable n-gcm S, then every point of the boundary of U is a limit point of $S - \overline{U}$.*

Proof. Suppose x is a boundary point of U that is not a limit point of $S - \overline{U}$. Then there exists an open set P containing x such that $P \subset \overline{U}$ and $S - \overline{P} \neq 0$. Let Q be an open subset of P containing x such that $q^{n-1}(x; P \cap U, Q \cap U) = 0$ (Theorem 2.5). Let $y \in S - \overline{P}$. By Lemma 6.2, there exists an open subset R of Q containing x and a cycle γ^{n-1} on $F(R)$ which is not homologous to zero in $S - x - y$. By Lemma 5.7, there exists a compact cycle Z^{n-1} in $(Q - x) \cap U$ such that $\gamma^{n-1} \sim Z^{n-1}$ in $Q - x$. But by the choice of R, $Z^{n-1} \sim 0$ in $P \cap U \subset S - x - y$, and therefore $\gamma^{n-1} \sim 0$ in $S - x - y$.

6.4 Lemma. *Let S be a space that is normal, connected and lc, and such that all its compact 1-cycles bound. Then if a compact set B is the boundary of a domain D in S, no component of $S - \overline{D}$ has boundary points in more than one component of B.*

Proof. Suppose E is a component of $S - \overline{D}$ that has limit points in two components, C_1 and C_2, of B. There exists a decomposition $B = B_1 \cup B_2$ separate, where $B_i \supset C_i$, $i = 1, 2$, by Theorem IV 1.1. Since $p^1(S) = 0$, S has property V of Chapter II (see Corollary VII 9.3 and the remark following), and there exists a closed, connected set $K \subset S - B$ which separates C_1 and C_2 in S. As E has limit points in both C_1 and C_2, $E \cap K \neq 0$, and it follows that $K \subset E$. But for the same reason $K \subset D$. This is impossible since $D \cap E = 0$.

6.5 Lemma. *If D is a ulc^{n-1} domain of an orientable n-gcm S, and $p^1(S) = 0$, then each component C of the boundary of D is the common boundary of two ulc^{n-1} domains E_1 and E_2 such that $S = C \cup E_1 \cup E_2$.*

Proof. By Lemma 6.3, B is the boundary of the nonempty open set $S - \overline{D}$, and by Theorem 5.19, $S - \overline{D}$ is ulc^{n-1}. If C is a component of B, then C is on the boundary of only one component, E_1, of $S - \overline{D}$, by Lemma 6.1. Furthermore, C is the complete boundary of E_1 by Lemma 6.4. The set E_1 is ulc^{n-1}. For if $x \in C$, there exists an open set P containing x such that $P \cap (S - \overline{D}) \subset P \cap E_1$, and consequently $q^r(E_1, x) = q^r(S - \overline{D}, x) = 0$, $r \leqq n - 1$. By

Theorem 5.19, $S - \bar{E}_1$ is ulc^{n-1} and consequently is connected by Lemmas 6.1 and 6.3.

REMARK. The necessity of the condition $p^1(S) = 0$ in Lemmas 6.4 and 6.5 may be seen from the example of the torus. In the case of Lemma 6.4, if S is a torus and B_1, B_2 are two disjoint equatorial circles on T, then $B = B_1 \cup B_2$ is the boundary of a domain D of S such that $S - \bar{D}$ is a domain with boundary points in both B_1 and B_2. And for the case of Lemma 6.5, the component B_1 of B is not a common boundary of two domains of S.

By the same kind of argument as was used to prove Lemma 6.2 we may also prove:

6.6 LEMMA. *If S is an orientable n-gcm, M a closed subset of S, and A and B are different components of $S - M$, then M carries an $(n - 1)$-cycle γ^{n-1} which is not homologous to zero in $S - x - y$ no matter what the choice of points $x \in A$, $y \in B$ may be.*

6.7 LEMMA. *Let S be an orientable n-gcm, such that $p^1(S) = 0$, and D a domain in S such that $p^{n-1}(D) = 0$. Then the boundary, B, of D is connected.*

PROOF. Suppose $B = B_1 \cup B_2$ separate. Then by Corollary VII 9.3 there would exist a continuum K in $S - B$ separating points x and y of B in S. By Lemma 6.6, K would carry an $(n - 1)$-cycle nonbounding in $S - x - y$. But evidently $K \subset D$, and the existence of such a cycle would contradict the hypothesis that $p^{n-1}(D) = 0$.

6.8 THEOREM. *Let S be an orientable n-gcm such that $p^1(S) = 0$, and D a ulc^{n-2} domain in S such that $p^{n-1}(D) = 0$. Then the boundary of D, if nondegenerate and not n-dimensional, is an orientable $(n - 1)$-gcm.*

PROOF. The boundary, B, of D is connected by Lemma 6.7.

By Theorem 2.13, $S - B$ is $(n - 1)$-ulc. Consequently D is $(n - 1)$-ulc and by Lemma 6.5, B is the common boundary of two ulc^{n-1} domains D and D' such that $S = B \cup D \cup D'$. Since $p^{n-1}(S) = 0$ by Theorem VIII 4.2, Theorem VII 5.9 may be applied to show that $p^{n-1}(B) = 1$ and $p^{n-1}(B') = 0$ for every closed proper subset B' of B.

The remainder of the proof is exactly like the latter part of the proof of Theorem 3.3 above.

As a corollary of Theorem 6.7 we can state much stronger theorems than the converse of the Jordan-Brouwer separation theorem:

6.9 THEOREM. *If S is an orientable n-gcm such that $p^1(S) = 0$, and M is an $(n - 1)$-dimensional set which is the common boundary of (at least) two distinct domains A and B of $S - M$, one of which, say A, is ulc^{n-2}, then the set M is an orientable $(n - 1)$-gcm.*

PROOF. Since by Corollary VII 9.3, S has Property IV of Chapter II, M is connected. Hence by Theorem VIII 6.4, $p^{n-1}(S - M) = 0$, and it follows that $p^{n-1}(A) = 0$ and Theorem 6.8 applies.

6.10 THEOREM. *If S is an orientable n-gcm such that $p^1(S) = 0$, and U a ulc^{n-1} open subset of S, then each component of the boundary of U [if not n-dimensional]*[3] *is an orientable $(n - 1)$-gcm.*

PROOF. Suppose first that U is connected. Let C be a [not n-dimensional] component of the boundary of U. By Lemma 6.5, C is the common boundary of two ulc^{n-1} domains, and Theorem 6.9 applies.

In the general case, since U is a 0-ulc open subset of a compact space S, it has only a finite number of components U_i, $i = 1, \cdots, m$. By Lemma 6.1 each component of the boundary, B, of U is on the boundary of one and only one U_i. Hence the continua $\overline{U}_i$ are disjoint, and the sets U_i themselves are ulc^{n-1}. Then each component of the boundary of U is also a component of the boundary of a ulc^{n-1} component of U and is consequently, as just shown above, an orientable $(n - 1)$-gcm.

6.11 THEOREM. *If S is an orientable n-gcm such that $p^1(S) = 0$, and D is a ulc^{n-2} domain of S whose boundary, B, is a [not n-dimensional] continuum, then B is an orientable $(n - 1)$-gcm.*

PROOF. It follows from Theorem 2.13 that D is $(n - 1)$-ulc, and hence Theorem 6.10 applies.

The following special cases have an intrinsic interest which justify separate statement:

6.12 THEOREM (R. L. MOORE). *If D is a 0-ulc, simply 1-connected domain in the sphere S^2, then the boundary of D, if nondegenerate, is an S^1.*

6.13 THEOREM. *If D is a ulc^1, simply 2-connected domain in the sphere S^3, then the boundary (nondegenerate), B, of D is an orientable closed 2-dimensional manifold in the classical sense. In particular, if $p^1(D) = 0$, then B is a 2-sphere.*

From the standpoint of existence, the following theorem is of interest:

6.14 THEOREM. *If a metric space S is an orientable 3-gcm such that $p^1(S) = 0$, and S contains a ulc^2 domain [with 2-dimensional boundary], then S contains at least one orientable closed 2-dimensional manifold in the classical sense.*

REMARK. Instead of the hypothesis of Theorem 6.13, one may of course suppose that S contains a ulc^1 domain which is simply 2-connected and has a nonempty 2-dimensional boundary (cf. Theorem 6.8).

7. Additional converses of the Jordan-Brouwer separation theorem.

In Theorems 6.8 and 6.11, the conditions on the complement of the $(n - 1)$-gcm

[3]Here, and in a number of later theorems, we place a dimensionality restriction in brackets. The reason for this is that while it is well known to be unnecessary in the case of the n-sphere or classical n-manifold, we do not know whether it is necessary in the case of an n-gcm. Since $p^{n-1}(M) > 0$ implies M *at least* $(n - 1)$-dimensional, we need not specify exactly "$(n - 1)$-dimensional", so that the question reduces to whether a common boundary of two domains

were placed entirely on one domain. In Theorem 6.9, although the principal (ulc) conditions were placed on one domain, use was made of the fact that the manifold is a common boundary of (at least) two domains. In the present section, stronger use is made of the latter fact, and the ulc conditions are distributed over the two domains—somewhat in the manner in which this was done in Theorem 3.3 except that in the latter case the ulc conditions imposed were much stronger than necessary, as we have already observed above. In the first place, we may state the following theorem:

7.1 THEOREM. *Let M be a [not n-dimensional] common boundary of (at least) two domains D_1 and D_2 in an orientable n-gcm S such that $p^1(S) = 0$. Also, suppose that D_k is ulc^{n_k}, $k = 1, 2$; $n_1 + n_2 = n - 2$. Then M is an orientable $(n - 1)$-gcm.*

PROOF. By Theorem 5.19, $S - \overline{D}_2$ is $\text{ulc}^{n-1}_{n-n_2-1}$, and as $n_1 + 1 = n - n_2 - 1$, D_1 is therefore ulc^{n-1}. Theorem 6.9 then applies.

7.2 THEOREM. *Let M be a [not n-dimensional] common boundary of (at least) two domains D_1 and D_2 in an orientable n-gcm S such that $p^1(S) = p^{n_1}(S) = p^{n_1+1}(S) = 0$, and D_k is ulc^{n_k}, $k = 1, 2$, where $n_1 + n_2 = n - 3$. Then if there exists a covering $\mathfrak{C}$ of S such that the $(n_2 + 1)$-cycles of D_2 of diameter $< \mathfrak{C}$ bound in D_2, the set M is an orientable $(n - 1)$-gcm.*

PROOF. By Theorem 5.19, $S - \overline{D}_1$ is $\text{ulc}^{n-1}_{n-n_1-1}$, hence D_2 is likewise. And as we already know that D_2 is ulc^{n_2} and $n_2 = n - n_1 - 3$, we need only show that D_2 is $(n_2 + 1)$-ulc; the theorem then follows from Theorem 6.9, for instance.

Denote $n_2 + 1$ by r. To show that D_2 is r-ulc, it is sufficient to show that $q^r(D_2, x) = 0$ for all $x \in S - D_2$ by Corollary 2.6. As this is obvious for the case $x \in S - \overline{D}_2$, let $x \in M$. It follows from the hypothesis that there exists an open set P containing x such that all compact cycles of D_2 in P bound in D_2. By Lemma 5.18, $\overline{D}_1$ is completely n_1-avoidable. Let Q and R be open sets such that $x \in R \Subset Q \Subset P$ and n_1-cycles of $\overline{D}_1$ on $F(Q)$ bound in $\overline{D}_1 \cap (P - \overline{R})$. Let V be an open set such that $x \in V \subset R$ and such that cycles of S in V bound in R. Let γ^r be a cycle on a closed subset K of $V \cap D_2$. Then γ^r bounds on a compact subset K_1 of D_2 as well as on a compact subset K_2 of R. If we denote $M \cap \overline{Q}$ by A_1 and $(M - Q) \cup F(Q)$ by A_2, then the hypothesis of Theorem VII 9.1 is satisfied, and γ^r will bound in $S - [M \cup F(Q)]$ if it can be shown that every $(r + 1)$-cycle of $K_1 \cup K_2$ bounds in $S - A_1 \cap A_2$. Now $A_1 \cap A_2 \subset M \cap F(Q)$, and therefore an $(r + 1)$-cycle of $K_1 \cup K_2$ that is nonbounding in $S - A_1 \cap A_2$ is linked, by Corollary VIII 8.6, with an n_1-cycle of $M \cap F(Q)$. But $M \subset \overline{D}_1$, and from the way in which Q was chosen, such a cycle bounds on $\overline{D}_1 \cap (P - \overline{R}) \subset S - K_1 \cup K_2$.

7.3 THEOREM. *With S and M as in Theorem 7.2, and again D_k ulc^{n_k}, $n_1 + n_2 = n - 3$, suppose that $p^{n_2+1}(D_2)$ is finite. Then M is an orientable $(n - 1)$-gcm.*

PROOF. Let $\{Z_i^{n_2+1}\}$, $i = 1, \cdots, p^{n_2+1}(D_2)$, constitute a base for compact

$(n_2 + 1)$-cycles of D_2 relative to homologies in D_2. The cycles $Z_i^{n_2+1}$ are lirh in $S - \overline{D}_1$, and consequently there exists by Theorem VIII 8.10 a collection of cycles $\{\gamma_i^{n_1+1}\}$ of $\overline{D}_1$ such that every linear combination of the cycles $Z_i^{n_2+1}$ is linked with at least one cycle $\gamma_i^{n_1+1}$. There exists an open set V containing $\overline{D}_1$ such that the cycles $\gamma_i^{n_1+1}$ are lirh in V (cf. proofs of Theorem VIII 6.3), and such that the cycles $Z_i^{n_2+1}$ all lie in $S - \overline{V}$. By Lemma 5.7, there exist compact cycles $Z_i^{n_1+1}$ in D_1 such that $Z_i^{n_1+1} \sim \gamma_i^{n_1+1}$ in V.

Let K be a closed subset of D_1 carrying all the cycles $Z_i^{n_1+1}$. Let $x \in M$ and P an open set containing x such that $K \cap \overline{P} = 0$. Let Q be an open set such that $x \in Q \subset P$ and such that compact $(n_2 + 1)$-cycles of Q bound in P. We assert that $q^{n_2+1}(x; D_2, Q \cap D_2) = 0$. For let Z be a compact $(n_2 + 1)$-cycle of $Q \cap D_2$, and suppose that $Z \not\sim 0$ in D_2. Then $Z \sim \sum a_i Z_i^{n_2+1}$ (where $a_i \in \mathfrak{F}$ and not all $a_i = 0$) on a compact subset L of D_2. Also, $Z \sim 0$ in P. Hence $\sum a_i Z_i^{n_2+1} \sim 0$ on $L \cup \overline{P} \subset S - K$. But this contradicts the fact that every linear combination of the cycles $Z_i^{n_2+1}$ is linked with a cycle $Z_i^{n_1+1}$. Hence $Z \sim 0$ in D_2.

The theorem is now easily obtained from Theorem 7.2.

7.4 COROLLARY. *If S is a spherelike n-gcm and M is a [not n-dimensional] common boundary of (at least) two domains D_k such that D_k is ulc^{n_k}, $k = 1, 2$, $n_1 + n_2 = n - 3$, and one of the numbers $p^{n_k+1}(D_k)$ is finite. Then M is an orientable $(n - 1)$-gcm.*

7.5 COROLLARY. *If M is a common boundary of (at least) two 0-ulc domains in S^3, and the number $p^1(D)$ is finite for at least one of these domains D, then M is a closed 2-dimensional manifold.*

8. **The general ulc^{n-2} open subset of an n-gcm.** The general ulc^{n-2} open subset of an n-gcm presents many interesting properties that are worth mention, although we shall not go into exhaustive detail here, and will leave to the reader the task of supplying some of the proofs. Information regarding the positional properties of such sets has already been obtained above, particularly in Theorem 5.15, which states that if B is the boundary of such a set U in a spherelike n-gcm, then the numbers $p^k(B)$, $k = 1, \cdots, n-2$, are all finite and satisfy the Poincaré duality relative to $n - 1$; and Lemma 6.1, which states that if C is a component of B, then C is on the boundary of just one domain of U. Also, the argument used in the second paragraph of the proof of Theorem 6.10 shows that U is the union of a finite set of domains whose closures are disjoint, so that the study of such a set U can be reduced to the study of a single ulc^{n-2} domain.

Suppose, then, that D is a ulc^{n-2} domain of an orientable n-gcm S, and that C is a component of the boundary B of D. It is possible that C is a point—as in the case where S is a 2-sphere and B is a single point. However, suppose D' is a component of $S - \overline{D}$. If we assume that $p^1(S) = 0$, then $F(D')$ is in one component of B by Lemma 6.4. Hence we can augment each component C of B by those components D' of $S - \overline{D}$ that have boundary points in C, to

form a connected set C^*, and every point of $S - \overline{D}$ will lie in such a C^*. Let us select a fixed C, say C', and denote by D^* the domain D augmented by all sets C^* different from C'^*. The set D^* is connected, and we assert it is open. Obviously points of D and points of the sets of type D' in D^* are interior to D^*. Consider an $x \in C \subset D^*$. Let P and Q be open sets such that $x \in Q \subset P \subset S - C'$, and such that all points of Q lie in one component of P. Then no point of a D' such that $F(D') \subset C'$ can lie in Q, since if it did, then there would exist in P a boundary point of the corresponding D'—i.e., a point of C' in P. Thus D^* is a domain whose boundary is the continuum C'. We shall show that D^* is ulc^{n-2}, and hence by Theorem 6.11 that C' is an orientable $(n - 1)$-gcm [if C' is not n-dimensional].

Let r be an integer such that $0 \leqq r < n - 1$. To show D^* r-ulc, it will be sufficient by Corollary 2.6 to show that if $x \in C'$, then $q^r(D^*, x) = 0$. Let P, Q and R be open sets such that $x \in R \subset Q \subset P$, compact r-cycles in R bound in Q, and compact r-cycles of $D \cap Q$ bound in $D \cap P$. Let γ^r be a compact cycle carried by a compact subset K of $R \cap D^*$. Then $\gamma^r \sim 0$ on a compact subset M of Q, and by Lemma VII 1.4 there exists a cycle Z^{r+1} mod K on M such that $\partial Z^{r+1} \sim \gamma^r$ on K. Now by application of Theorem IV 1.3, $K \cup [(B \cup C'^*) \cap \overline{P}] = A_1 \cup A_2$ separate, where $A_1 \supset C'^* \cap \overline{P}$ and $A_2 \supset K$. Let U and V be disjoint open subsets of S containing A_1 and A_2 respectively (Lemma IV 1.9). The portion of Z^{r+1} on the set $M_1 = M - U$ is a cycle mod $K \cup F(U)$ whose boundary is $\partial Z^{r+1} + Z^r$, where Z^r is on $M \cap F(U)$.

Now Z^r is a compact cycle of D in Q, since M lies in Q and no point of C'^* lies on $F(U)$. Consequently, by the choice of Q, $Z^r \sim 0$ on a compact subset L of $D \cap P$. Then $\gamma^r \sim 0$ on $M_1 \cup L \subset D^* \cap P$, and we conclude that $q^r(D^*, x) = 0$, D^* is ulc^{n-2}, and C' [if not n-dimensional] is an orientable $(n - 1)$-gcm. In particular, then, it follows from Theorem 2.13 and Lemma 6.5 that each nondegenerate component C of B is the boundary of a domain of type D^* and of a ulc^{n-1} domain in D' such that $S = C \cup D^* \cup D'$—the set $C^* - C$ being the domain D'.

Now if $\mathfrak{E}$ is an arbitrary covering of S, then at most a finite number of sets of type C^* fail to be of diameter $< \mathfrak{E}$. Suppose the contrary. Then there exists a covering $\mathfrak{E}$ and an infinite sequence of sets C_i^* such that no C_i^* is of diameter $< \mathfrak{E}$. Let $p \in \limsup C_i^*$ and E an element of $\mathfrak{E}$ that contains p. Let E', P and Q be open sets such that $p \in Q \subset P \subset E' \subset E$ and such that (1) every $(n - 1)$-cycle on $\overline{P}$ bounds in E', (2) every $(n - 2)$-cycle of $\overline{D} \cap F(P)$ bounds in $E' - \overline{Q}$ (Lemma 5.18), and (3) $S - E$ lies in one component of $S - E'$. Since infinitely many of the sets C_i^* meet both Q and $S - E$, it follows from Theorem IV 1.14 that $\limsup C_i^*$ contains a continuum M' that meets both Q and $S - E$ and contains P. Then $M' \subset B$, and M, the component of B containing M' [if not n-dimensional] is an orientable $(n - 1)$-gcm which is the common boundary of two domains D_1 and D_2 such that $S = M \cup D_1 \cup D_2$. Without loss of generality we may assume all C_i^* in D_2. For the remainder of the argument we need the following lemma and corollary.

8.1 LEMMA. *Let S be an orientable n-gcm and M an $(n - 1)$-gcm in S which bounds two disjoint domains D_1 and D_2 such that $S = M \cup D_1 \cup D_2$. Then M is orientable and its fundamental $(n - 1)$-cycle bounds on each of the sets $\overline{D}_1$, $\overline{D}_2$.*

PROOF. If Γ^n is the fundamental n-cycle of S, then Γ^n is a cycle mod $S - D_1$ and accordingly, by Lemma VII 1.16, there exists on $\overline{D}_1$ a cycle Z^n mod M such that $\Gamma^n \sim Z^n$ mod $S - D_1$. Let $\gamma^{n-1} = \partial Z^n$. Then $\gamma^{n-1} \nsim 0$ on M, since if it were there would exist by Lemma VII 1.6 a cycle γ^n on $\overline{D}_1$ such that $\gamma^n \sim Z^n$ mod M, and from this would follow $\Gamma^n \sim 0$ mod $S - D_1$; this is impossible by Lemma VII 3.6. It follows easily that the fundamental $(n - 1)$-cycle of M bounds on $\overline{D}_1$ and $\overline{D}_2$.

8.2 COROLLARY. *Under the hypothesis of Lemma 8.1, there exists on $\overline{D}_1$ a cycle Z^n mod M such that ∂Z^n is homologous on M to the fundamental $(n - 1)$-cycle Γ^{n-1} of M.*

Continuing with the argument preceding the lemma, and using the notation of the corollary, the portion of Z^n in P has as boundary a cycle Z^{n-1} such that $Z^{n-1} \sim \Gamma^{n-1}$ mod $S - P$, and the portion of Z^{n-1} in P has as boundary a cycle Z^{n-2} on $M \cap F(P)$ such that $Z^{n-2} \sim 0$ on a closed subset L_1 of $\overline{D}_1 \cap F(Q)$. By the choice of P and Q, $Z^{n-2} \sim 0$ also in $\overline{D} \cap (E' - \overline{Q})$ on a compact set L_2. It may now be shown that $L_1 \cup L_2$ separates p from $S - E$ (by methods similar to those used in Chapter VII in like situations).

Contradiction now results from the fact that $L_1 \cup L_2$ must meet (infinitely many of) the domains complementary to the sets C_i distinct from M, although this is not possible since $L_1 \cup L_2 \subset M \cup \overline{D} \cup \overline{D}_1$.

8.3 THEOREM. *Let U be a ulc^{n-2} open subset of an orientable n-gcm S such that $p^1(S) = 0$. Then* (1) *U consists of a finite number of domains $D_1, \cdots, D_m$ such that $\overline{D}_i \cap \overline{D}_j = 0$ if $i \neq j$;* (2) *each component of B_i, the boundary of D_i, $i = 1, \cdots, m$, is either a point or a continuum which [if not n-dimensional] is an orientable $(n - 1)$-gcm whose complement consists of two disjoint domains having the component as common boundary; moreover, $B_i = B_{i0} \cup B_{i1} \cup \cdots \cup B_{ik} \cup \{B_{i\alpha}\}$, where B_{i0} is the set of point components of B_i and B_{ij}, $j = 1, \cdots, k$ or α, is a component of B which, in case S is spherelike, for $j = \alpha$ is spherelike; and* (3) *if $\mathfrak{E}$ is an arbitrary covering of S, then at most a finite number of the sets B_{ij} fail to be of diameter $< \mathfrak{E}$.* (From this follows that the sets B_{ij} form a countable collection, S being perfectly normal. Cf. Theorem XI 2.15.)

8.4 COROLLARY. *Under the hypothesis of Theorem 8.3, if S is metric, then the nondegenerate components B_{ij}, $j = 1, \cdots, k$ or α, form a countable collection such that, for arbitrary $\epsilon > 0$, at most a finite number of the sets B_{ij} are of diameter $> \epsilon$.*

9. Decomposition of the spherelike n-gcm into two generalized closed n-cells. One of the disappointing aspects of the topology of euclidean spaces is the

failure of the analogue of the Schoenflies extension theorem in the 3-sphere. According to this theorem, if M is an S^1 in S^2, and A is one of the domains complementary to M, then $\bar{A}$ is a closed 2-cell and A is a 2-cell. However, as was shown by a well-known example of Alexander [c] in 1924, if M is an S^2 in S^3, and A is one of the domains complementary to M, then $\bar{A}$ is not necessarily a closed 3-cell nor A a 3-cell. However, in terms of our generalized concepts we can prove:

9.1 THEOREM. *If S is a spherelike n-gcm and M is a spherelike $(n - 1)$-gcm in S, then $S - M$ is the union of two disjoint generalized n-cells A_i, $i = 1, 2$; moreover, $\bar{A}_i = A_i \cup M$ is a generalized closed n-cell.*

PROOF. By Theorem 3.1, $S - M$ is the union of two disjoint domains A_1, A_2 that have M as common boundary. That the compact homology groups of these domains of dimension $< n$ reduce to the identity follows from Theorem VIII 6.4. By Lemma 8.1, if Γ^{n-1} is the fundamental cycle of M, then $\Gamma^{n-1} \sim 0$ on $\bar{A}_i$, $i = 1, 2$, and it follows from Theorem VII 2.22 and Corollary VII 2.21 that $\Gamma^{n-1} \not\sim 0$ on a proper closed subset of $\bar{A}_i$; thus A_i is an orientable n-gm. Consequently A_i is a generalized n-cell.

Henceforth denote either of the sets A_i by A. We shall show that $\bar{A}$ is a generalized closed n-cell. It is necessary only to show that $p_r(\bar{A}, x) = 0$ for $r \leqq n$ and $x \in M$, since the other properties of Definition IX 7.10 have already been established above. By Theorem 3.2, A is ulc^{n-1}, and by Theorem 5.8, $\bar{A}$ is lc^{n-1}. Hence by Lemma 5.18, $\bar{A}$ is completely r-avoidable at all points for $r < n - 1$. It follows immediately from Lemma IX 3.3 that $p_r(\bar{A}, x) = 0$ for $r \leqq n - 1$.

Suppose $x \in M$ such that $p_n(\bar{A}, x) > 0$. Then there exists an open set P containing x such that $p^n(x; P \cap \bar{A}, R \cap \bar{A}) > 0$ for all open sets R such that $x \in R \subset P$. Let Q, R be open sets such that $x \in R \Subset Q \Subset P$ and such that all $(n - 1)$-cycles of $F(Q)$ bound in $S - R$ (Corollary IX 2.2). Using only n-dimensional coverings, let γ^n be a cycle mod $S - P$ on $\bar{A}$. The portion of γ^n in Q has boundary $\partial\gamma^n$ on $F(Q)$ and $\partial\gamma^n \sim 0$ in $S - R$, so that by Lemma VII 1.6 there exists a cycle Z^n on S such that $\gamma^n \sim Z^n$ mod $S - R$. But on n-dimensional coverings "$\sim$" becomes identity, so that $\gamma^n = Z^n$ mod $S - R$. However, Z^n is on a closed proper subset of S, since there are points of $S - \bar{A}$ in R, and consequently $Z^n \sim 0$ on S. It follows that $\gamma^n = 0$ mod $S - R$, and a fortiori $\gamma^n \sim 0$ on $\bar{A}$ mod $S - R$. Thus $p^n(x; P \cap \bar{A}, R \cap \bar{A}) = 0$ in contradiction of the manner in which P was chosen.

As a sort of converse theorem, we can state the following:

9.2 THEOREM. *Let S be a space which is the union of two generalized closed n-cells $C_i = K \cup A_i$, $i = 1, 2$, where K and A_i satisfy the conditions stated relative to K and A in Definition* IX 7.10, *and such that $A_1 \cap A_2 = 0$. Then S is a spherelike n-gcm.*

PROOF. Condition A of the definition of n-gm is satisfied, since dim $S = n$.

(See, for instance, E. Čech [c].) So we first prove condition B of the definition of n-gm. Evidently this needs to be done only for $x \in K$. Let U be an open set containing x, and let r be an integer such that $0 < r < n$ (the case $r = 0$ is handled by use of Corollary VI 6.12). Since $p_r(C_i, x) = 0$ and $p_{r-1}(K, x) = 0$ there exist open sets P, Q, R and W such that $x \in W \subset R \subset Q \subset P \subset U$, such that $p_{r-1}(x; K \cap P, K \cap Q) = 0$, $p_r(x; C_1 \cap Q, C_1 \cap R) = 0$, and $p_r(x; C_2 \cap R, C_2 \cap W) = 0$. Consider any cycle Z^r mod $S - P$ on S. If Z^r is on either set C_i, then it is homologous to zero mod $S - W$. Otherwise, let Z_1^r denote the portion of Z^r in $A_1 \cap P$; then Z_1^r is a cycle mod $(S - P) \cup C_2$ whose boundary is a cycle γ^{r-1} on $K \cup F(P)$. And since γ^{r-1} is a cycle mod $S - P$, there exists, by the choice of Q, a cycle γ^r mod $S - Q$ on K such that $\partial\gamma^r \sim \gamma^{r-1}$ mod $S - Q$ on K. Since $\partial Z_1^r = \gamma^{r-1}$ mod $S - Q$, we therefore have

$$\text{(9.2a)} \qquad \partial(Z_1^r - \gamma^r) \sim 0 \qquad \text{mod } S - Q \qquad \text{on } K.$$

Hence there exists a cycle Γ_1^r mod $S - Q$ on $C_1 \cap Q$ such that $\Gamma_1^r \sim Z_1^r - \gamma^r$ on C_1 mod $(S - Q) \cup L$, where L is the carrier of the homology (9.2a) on $K \cap Q$. Now $\Gamma_1^r \sim 0$ mod $S - R$ on C_1, because of the way in which R was selected, and therefore we have

$$\text{(9.2b)} \qquad Z_1^r - \gamma^r \sim 0 \qquad \text{mod } (S - R) \cup L \qquad \text{on } C_1 .$$

Since γ^r is on K and $L \subset K$, relation (9.2b) implies that $Z_1^r \sim 0$ mod $(S - R) \cup C_2$, and it follows that $Z^r \sim 0$ mod $(S - R) \cup C_2$. Hence by Lemma VII 1.9, there exists a cycle Γ^r mod $(S - R)$ on $(S - R) \cup C_2$ such that $Z^r \sim \Gamma^r$ mod $S - R$. By the choice of W, $\Gamma^r \sim 0$ mod $S - W$ on C_2, and therefore $Z^r \sim 0$ mod $S - W$. We can therefore conclude that $p_r(S, x) = 0$ for $r < n$.

Turning to the case $r = n$, let x again be a point of K and U an arbitrary open set containing x. On C_i there exists by Lemma VII 1.4 a cycle Z_i^n mod K such that $\partial Z_i^n \sim \Gamma^{n-1}$ on K, $i = 1, 2$. Hence by Lemma VII 1.6, there exists a cycle Γ^n on S such that $\Gamma^n \sim Z_1^n + Z_2^n$ mod K. If for some open set V such that $x \in V \subset U$, $\Gamma^n \sim 0$ mod $S - V$, then, since dim $S = n$, we may assume $\Gamma^n = 0$ mod $S - V$. But this implies that $Z_1^n = 0$ mod $(S - V) \cup C_2$, which in turn implies that $\Gamma^{n-1} \sim 0$ on $C_1 - V$, contradicting the fact that C_1 is an irreducible membrane relative to Γ^{n-1}. Thus we conclude that $p_n(S, x) \geqq 1$.

Now since $p_n(C_1, x) = 0$ and $p_{n-1}(K, x) = 1$, there exist in U open sets P, Q, R and W such that $x \in W \subset R \subset Q \subset P \subset U$ and such that $p_{n-1}(x; K \cap P, K \cap Q) = 1$, $p_n(x; C_1 \cap Q, C_1 \cap R) = 0$, $p_n(x; C_2 \cap R, C_2 \cap W) = 0$. Suppose that Γ_1^n and Γ_2^n are cycles of S mod $S - P$. The portions of these cycles in $A_1 \cap P$, which we shall denote by γ_1^n and γ_2^n, respectively, have cycles γ_1^{n-1} and γ_2^{n-1} mod $S - P$ on K, respectively, as boundaries mod $S - P$. Because of the choice of Q, there exists a homology $a\gamma_1^{n-1} + b\gamma_2^{n-1} \sim 0$ mod $S - Q$ on K. But then there exists by Lemma VII 1.7 a cycle Z^n mod $S - Q$ on $C_1 \cap \bar{Q}$ such that $Z^n \sim a\Gamma_1^n + b\Gamma_2^n$ mod $(S - Q) \cup C_2$ on $C_1 \cap \bar{Q}$. And because of the choice of R, $Z^n \sim 0$ mod $S - R$ on C_1. This

implies that $a\Gamma_1^n + b\Gamma_2^n = 0 \bmod (S - R) \cup C_2$ on C_1. However, this in turn implies that $a\Gamma_1^n + b\Gamma_2^n$ is a cycle mod $S - R$ on C_2, and hence $= 0$ on C_2 mod $S - W$. A fortiori, $a\Gamma_1^n + b\Gamma_2^n \sim 0 \bmod S - W$. We conclude that $p_n(S, x) = 1$.

Now suppose that S has a proper closed subset F that carries an n-cycle $\gamma^n \not\sim 0$ on S. By Lemma VII 2.6 we may assume F to be the unique minimal carrier of γ^n. Evidently $F \not\subset K$, since $\dim K = n - 1$. Suppose $F \cap A_1 \neq 0$, but $F \not\supset A_1$. Then the portion of γ^n in A_1 is a cycle γ_1^n mod K. If $\partial\gamma_1^n \not\sim 0$ on K, then $\Gamma^{n-1} \sim c\partial\gamma_1^n$ on K, $c \neq 0$, where Γ^{n-1} is the fundamental cycle of K, and consequently $\Gamma^{n-1} \sim 0$ on the closed proper subset $F \cap C_1$ of C_1. This is impossible since C_1 is an irreducible membrane relative to Γ^{n-1}. Hence $\partial\gamma_1^n \sim 0$ on K. Then by Lemma VII 1.6, there exists a cycle Z^n on C_1 such that $\gamma_1^n \sim Z^n$ mod K. As S is n-dimensional, we may assume $\gamma_1^n = Z^n$ mod K, so that Z^n is a cycle on $(F \cap A_1) \cup K$. Let $x \in F \cap A_1$, and P and Q open sets such that $x \in Q \subset P \subset A_1$, and $p_n(x; P, Q) = 1$. Then with Z_1^n such that $\partial Z_1^n = \Gamma^{n-1}$, as above, there must exist a homology $aZ_1^n + bZ^n \sim 0$ mod $S - Q$. Neither a nor b can be zero, since $b = 0$ would imply C_1 not an irreducible membrane relative to Γ^{n-1}, and $a = 0$ would imply F not an irreducible carrier of γ^n. Then $Z_1^n = cZ^n$ mod $S - Q$, and the boundary of the portion of Z_1^n in Q is a cycle on $F(Q)$ equal to $c\partial Z_3^n$, where Z_3^n is the portion of Z^n in Q. But as $c\partial Z_3^n \sim 0$ on $F \cap C_1$, this implies that $\Gamma^{n-1} \sim 0$ on $C_1 - Q$, contradicting the fact that C_1 is an irreducible membrane relative to Γ^{n-1}.

If $F \cap A_2 \neq 0$, and $F \not\supset A_2$, a contradiction again results as above. This leaves the case where $F \supset A_1$ and $F \cap A_2 = 0$, say. But if $x \in K$, there exist open sets P and Q such that $p_n(x; C_1 \cap P, C_1 \cap Q) = 0$ since $p_n(C_1, x) = 0$ by definition, and $\gamma^n \sim 0$ on C_1 mod $C_1 - Q$, implying that F is not an irreducible carrier of γ^n. In any case, then, the assumption that a closed proper subset of S carries a nonbounding n-cycle leads to contradiction, and condition D of the definition of n-gcm in Chapter VIII is satisfied. That $p^n(S) > 0$ was proved above, and that $p^n(S) = 1$ follows from Theorem VIII 3.1. Hence S is an orientable n-gcm.

To conclude the proof that S is a spherelike n-gcm, it remains to show that $p^r(S) = 0$ if $r < n$. From the fact that K is spherelike, it follows that any cycle γ^r of S is homologous to the sum of two cycles γ_1^r and γ_2^r carried by C_1 and C_2, respectively. The sets A_1 and A_2 are ulc^{n-1} (see the Remark preceding Theorem 3.2). Hence by Theorem 5.12, $h_a^r(A_i) = H_a^r(\overline{A}_i)$ for $r < n - 1$, $i = 1, 2$. Since each A_i is a generalized n-cell, $h_a^r(A_i) = 0$ for $r \leqq n - 1$. Hence $H_a^r(\overline{A}_i) = 0$ and each cycle γ_i^r bounds on C_i, $i = 1, 2$. This concludes the proof.

9.3 Corollary. *If $K \cup A$ is a generalized closed n-cell, where K and A are as in Definition* 7.10, *then the generalized n-cell A is a ulc^{n-1} subset of $K \cup A$.*

Proof. It is only necessary to notice that $K \cup A$ can be imbedded in a compact space S which is the union of two generalized closed n-cells $C_i = K \cup A_i$, $i = 1, 2$, satisfying the conditions of Theorem 9.2 and where each set A_i is homeomorphic with the A of the hypothesis of the theorem.

9.4 COROLLARY. *With $K \cup A$ as above, the augmented homology groups of $K \cup A$ all reduce to the identity.*

REMARK. The reader will note here that since, by Theorem IX 2.3, the separable case of the orientable 2-gcm reduces to the classical 2-manifold, it follows from Theorem 9.2 that the separable case of the generalized closed 2-cell reduces to the ordinary closed 2-cell—a fact already proved independently in Chapter IX, §7.

9.5 We cited above the example of Alexander [c] of an S^2, M, in S^3 such that one of the domains, A, complementary to M is not a 3-cell. (The domain A has an infinite fundamental group.) Evidently $\bar{A}$ is a generalized 3-cell. Suppose S is a space formed of two sets $\bar{A}_1$, $\bar{A}_2$ homeomorphic with $\bar{A}$, having in common the sets corresponding to M. Moreover, suppose that there is a homeomorphism h between $\bar{A}_1$ and $\bar{A}_2$ such that if $M_1 \subset \bar{A}_1$, $M_2 \subset \bar{A}_2$ are the sets corresponding to M and $x \in M_1$, then $h(x) = x$. By Theorem 9.2, S is a spherelike 3-gcm. It would be interesting to know whether S is an S^3.

BIBLIOGRAPHICAL COMMENT

§1. Theorem 1.5 was given by Alexandroff [f], Čech [f] and Begle [b]. Theorems 1.4 and 1.7 were given by Begle [b; Theorem 6.2, Corollary 6.6].

§§2, 5. The "co-properties" defined here and their applications were discussed in an abstract, Wilder [A₆].

§3. The proof of Theorem 3.1 for S^n given by Brouwer was cited in I 6. For a manifold in the classical sense, it is a corollary of a duality theorem of Pontrjagin [d]. Theorem 3.2 was proved for S^n in Wilder [c] and [n]. Theorem 3.3 for S^n appeared in Wilder [n].

§4. Theorem 4.5 was abstracted for S^n in Wilder [A_7].

§5. Theorem 5.13 was stated for S^n and $k = n - 2$ in Wilder [n; Theorem 18]. Theorem 5.14 was proved for the euclidean case in Wilder [k] and [n; Theorem 5]. Theorem 5.15 was given for S^n in Wilder [n; Theorem 19].

§6. For S^n, Theorem 6.8 appeared in Wilder [n; Principal Theorem C]. The Moore Theorem 6.12 for the plane appeared in Moore [f].

§7. For S^n, see Wilder [t].

§8. The general ulc^{n-2} open subset of an S^n was discussed in Wilder [n].

§9. See Wilder [A_8].

CHAPTER XI

LC^k SUBSETS OF AN n-GM

In the last chapter we studied the positional properties of a k-gcm imbedded in an n-gcm, $k < n$. The k-gcm is a very special type of lc^k set. In the present chapter we consider the general lc^k subsets of a generalized manifold. We make connection here with the work of Schoenflies on locally connected subsets of the plane, as well as with later investigations of the set-theoretic properties of the plane (R. L. Moore, G. T. Whyburn, C. Kuratowski, etc.). We recall that Schoenflies characterized the Peano continua in E^2 by means of the diameters and accessibility properties of the complementary domains; similar work was done by Whyburn (see Theorems IV 7.7, IV 7.8). Moore used the S property to accomplish similar results (see Corollary IV 6.4, Theorem IV 6.12). It is our intention to fit these and like results into their proper places in the topology of generalized manifolds.

As in the last chapter, we shall generally use the orientable n-gcm as the scene of operations. However, as before, it is evident that in many cases the orientable n-gm could be used if suitable hypotheses are made regarding compactness or local compactness of the set under consideration. Here again the situation is like that regarding S^n and E^n—the former has the advantage of symmetry, whereas use of the latter frequently necessitates boundedness assumptions. This is not to gainsay the fact that we leave certain problems open regarding the type of imbedding space which may be employed. But as we have stated in the Preface, our purpose is only to lay the groundwork as to materials and methods, rather than to write a complete theory.

Frequently we wish to impose simple connectedness in certain dimensions, as we did in the theorems of the last chapter. The symbol defined below is very convenient for this purpose:

Definition. By the symbol $M^n_{r,s}$ we shall denote an orientable n-gcm S such that $p^i(S) = 0$ for all i such that $r \leqq i \leqq s$. (Thus $M^n_{1,n-1}$ means sphere-like n-gcm.)

1. Duality of the S properties. We shall commence with an important duality relation between the S properties of a set and those of its complement.

1.1 Theorem. *If M is a closed subset of an $M^n_{r,r+2}$, S, where $r < n - 1$, then a necessary and sufficient condition that M have property* S_r *rel. bounding cycles is that $S - M$ have property* S_{n-r-2} *rel. bounding cycles.*

(The sufficiency holds in an $M^n_{r,r+1}$ and the necessity in an $M^n_{r+1,r+2}$; how-

ever, if $r = n - 2$, the necessity requires only $M^n_{n-1,n-1}$. These requirements are due solely to the use of Theorem VIII 6.4 in the proof.)

PROOF OF SUFFICIENCY. By Theorems VII 8.2 and VII 8.6 it is sufficient to prove the closed subsets of M almost locally r-avoidable rel. bounding cycles. Suppose that M has a closed subset, K, and open subset, U, containing K, such that no matter what the open subsets V and W of M, $K \subset W \subset V \subset U$, there exist infinitely many r-cycles of M on $F(V)$ that bound in M and are lirh on $M - W$. Let P_1, P_2 and P_3 be open subsets of S such that $U \supset M \cap \overline{P}_1$, and $P_1 \supseteq P_2 \supseteq P_3 \supset K$. Let $M \cap P_1 = V$, $M \cap P_2 = V'$, and $M \cap P_3 = W$. Then on $F(V)$ there exist infinitely many r-cycles of M that bound in M and are lirh on $M - W$. By Corollary VI 3.8, these cycles can be grouped in finite linear combinations that bound in $S - \overline{P}_2$, and such linear combinations form an infinite set of r-cycles that bound in M and in $S - \overline{P}_2$, and are lirh on $M - W$. Let G denote the group of all r-cycles of M that lie on $F(V)$, bound in M, and bound in $S - \overline{P}_2$. Then, as just shown, there exist infinitely many cycles of G that are lirh on $M - W$.

By Theorem VIII 8.8, there exists a fundamental system of cycles γ^r_i, $i = 1, 2, \cdots$, of G rel. homologies on $M - W$, and a system of compact cycles Γ^{n-r-1}_i of $S - (M - W)$ such that every finite linear combination of cycles of $\{\Gamma^{n-r-1}_i\}$ is linked with a cycle of the system $\{\gamma^r_i\}$, etc. The portion of each cycle Γ^{n-r-1}_i in $S - \overline{P}_3$ is a cycle Γ^{n-r-1}_{i1} mod $F(P_3)$. Denote $\partial\Gamma^{n-r-1}_{i1}$ by Z^{n-r-2}_i; then Z^{n-r-2}_i is a cycle on $F(P_3)$. By Theorem VII 7.10, there exists an integer m such that every m of the $(n - r - 2)$-cycles of $S - M$ that lie on compact subsets of $F(P_3)$ and bound in $S - M$ are lirh in $P_2 \cap (S - M)$. Hence there exists a homology

$$\sum_{i=1}^{m} c^i Z^{n-r-2}_i \sim 0 \qquad \text{in } P_2 - M. \tag{1.1a}$$

Let E be a compact subset of $P_2 - M$ carrying the homology (1.1a), and let F be a closed subset of $S - (M - W)$ carrying the cycle $\sum_{i=1}^m c^i\Gamma^{n-r-1}_i$. Let A denote the union of E and the closure of the set $F \cap (S - \overline{P}_3)$, and B denote the union of E and the set $F \cap \overline{P}_3$. Then by Lemma VII 1.14, there exist cycles Z^{n-r-1}_1 and Z^{n-r-1}_2 on A and B, respectively, such that $Z^{n-r-1}_1 + Z^{n-r-1}_2 \sim \sum_{i=1}^m c^i\Gamma^{n-r-1}_i$ on $A \cup B$. It is to be noted that $A \subset S - M$ and $B \subset S - (M - W)$.

Without loss of generality we may assume $c^1 \neq 0$. Then $Z^{n-r-1}_1 + Z^{n-r-1}_2$ is linked with γ^r_1. However, let Z^j_{r+1}, $j = 1, 2$, denote a cocycle derived from a corealization $\tau^*Z^{n-r-1}_j$ such that Z^1_{r+1} is a compact cocycle of $S - M$ and Z^2_{r+1} a compact cocycle of $S - M - W$ in P_2. Let C^j_r be a chain (on some fcos of S) whose coboundary is Z^j_{r+1}. Since $\sum_{j=1}^2 Z^j_{r+1}$ is in the same cohomology class of $S - (M - W)$ as the cocycle associated with $\sum_{i=1}^m c^i\Gamma^{n-r-1}_i$ in the establishing of the dual systems γ^r_i, Γ^{n-r-1}_i (see VIII 7.6), the number $\sum_{j=1}^2 C^j_r\cdot\gamma^r_1 = 1$. However, $C^1_r\cdot\gamma^r_1 = 0$ since C^1_r is a cocycle mod $S - M$ and $\gamma^r_1 \sim 0$ on M, and $C^2_r\cdot\gamma^r_1 = 0$ since C^2_r is a cocycle mod P_2 and $\gamma^r_1 \sim 0$ on $S - P_2$.

Thus the assumption that the closed subsets of M are not almost locally r-avoidable rel. bounding cycles leads to contradiction.

PROOF OF NECESSITY. Let P and Q be open sets such that $P \supset Q$. We shall show that at most a finite number of the bounding $(n - r - 2)$-cycles of $S - M$ in Q are lirh in $P \cap (S - M)$, from which it will follow, by Theorem VII 7.9, that $S - M$ has property S_{n-r-2} rel. bounding cycles. Suppose this not to be the case. Let P_1 and P_2 be open sets such that $P \supset P_1 \supset P_2 \supset Q$. Since S is lc^{n-r-2}, only finitely many of the $(n - r - 2)$-cycles of S that lie in Q are lirh in P_2. Consequently, if H denotes the group of all compact $(n - r - 2)$-cycles of $Q \cap (S - M)$ each of which bounds in $S - M$ as well as in P_2, then infinitely many cycles of H are lirh in $P \cap (S - M)$.

By Theorem VIII 8.9, there exists a set $\{Z_i^{n-r-2}\}$ of cycles of H lirh rel. homologies in $P \cap (S - M)$ and a system $\{\gamma_i^{r+1}\}$ of cycles of $S - P \cap (S - M)$ such that the linking properties asserted by that theorem hold. The portion of each cycle γ_i^{r+1} in P_1 is a cycle γ_{i1}^{r+1} mod $F(P_1)$ on M, whose boundary is a cycle of $M \cap F(P_1)$. Since the closed subsets of M are almost locally r-avoidable rel. bounding cycles of M, there exists by Theorem VII 7.10 a homology

$$\sum_{i=1}^{m} c^i \partial \gamma_{i1}^{r+1} \sim 0 \qquad \text{on } M - \overline{P}_2. \tag{1.1b}$$

Let C be a closed subset of $M - \overline{P}_2$ carrying the homology (1.1b), and F a closed subset of $S - P \cap (S - M)$ carrying the cycle $\sum_{i=1}^{m} c^i \gamma_i^{r+1}$. Let A denote the union of C and the closure of the set $F \cap P_1$, and B the union of C and the set $F \cap (S - P_1)$. Then by Lemma VII 1.14, there exist cycles Z_1^{r+1} and Z_2^{r+1} on A and B, respectively, such that $Z_1^{r+1} + Z_2^{r+1} \sim \sum_{i=1}^{m} c^i \gamma_i^{r+1}$ on $A \cup B$. Note that $A \subset M$ and $B \subset S - P_2$.

Suppose c^1, for instance, $\neq 0$. Then $Z_1^{r+1} + Z_2^{r+1}$ is linked with Z_1^{n-r-2}. But as a cycle of the group H, $Z_1^{n-r-2} \sim 0$ in $S - M \subset S - A$, and $Z_1^{n-r-2} \sim 0$ in $P_2 \subset S - B$, and it follows by an argument similar to that used above that a contradiction results. This completes the proof.

1.2 COROLLARY. *In order that a subcontinuum M of an $M_{1,2}^n$* (if $n = 2$, then an $M_{1,1}^n$ is sufficient), *should be 0-lc, it is necessary and sufficient that its complement have property S_{n-2} rel. bounding cycles.*

PROOF OF SUFFICIENCY. Since M is a continuum, all its 0-cycles bound. Consequently by Theorem 1.1, M has property S_0, and by Theorem VII 7.13 is 0-lc.

For the necessity, we may use Theorem VII 7.17 (with $n = 0$) and Theorem 1.1.

For our present purposes, we need the following relation between property S_r rel. bounding cycles, and property S_r :

1.3 THEOREM. *In order that a set M in a compact space should have property*

S_r, *it is necessary and sufficient that* $p^r(M)$ *be finite*[1] *and* M *have property* S_r *rel. bounding cycles.*

PROOF. The necessity is obvious.

For the sufficiency, let F be any closed set and P any open set containing F. Let $p^r(M) = m - 1$, where m is a positive integer. Since M has property S_r rel. bounding cycles, there exists by Theorem VII 7.10 a positive integer k such that every k compact r-cycles of $M \cap F$ that bound in M satisfy a homology in $M \cap P$.

We assert that every mk r-cycles of $M \cap F$ satisfy a homology in $M \cap P$. For let Z_i^r, $i = 1, \cdots, mk$, be cycles of $M \cap F$. Since $p^r(M) = m - 1$, there exist homologies

$$\gamma_t^r = \sum_{i=m(t-1)+1}^{mt} c^i Z_i^r \sim 0 \qquad \text{on } M; \qquad t = 1, \cdots, k. \tag{1.3a}$$

Since each of the cycles γ_t^r is a bounding cycle of M on F, there exists a homology

$$\sum_{t=1}^{k} a^t \gamma_t^r \sim 0 \qquad \text{in } M \cap P. \tag{1.3b}$$

Combination of the homologies (1.3a) and (1.3b) gives the desired homology relating the cycles Z_i^r in $M \cap P$. Hence M has property S_r by Theorem VII 7.10.

Now with regard to a closed subset of a manifold and its complement, we can state the following duality:

1.4 THEOREM. *If* M *is a closed subset of an* $M^n_{r,r+2}$, S, *where* $r < n - 1$, *then a necessary and sufficient condition that* M *have property* S_r *is that* $S - M$ *have property* S_{n-r-2} *rel. bounding cycles and finite* $p^{n-r-1}(S - M)$.

PROOF. As for the necessity, if M has property S_r, then by Theorem 1.3, M has property S_r rel. bounding cycles and finite $p^r(M)$. Hence by Theorem VIII 6.4, $p^{n-r-1}(S - M)$ is finite, and by Theorem 1.1, $S - M$ has property S_{n-r-2} rel. bounding cycles. The proof of the sufficiency is an obvious converse argument.

Another form of the duality between the S-properties is embodied in the following theorem:

1.5 THEOREM. *In order that a closed subset* M *of an* $M^n_{r,r+2}$, S, *should have property* S_r *and finite* $p^{r+1}(M)$, *it is necessary and sufficient that* $S - M$ *have property* S_{n-r-2} *and finite* $p^{n-r-1}(S - M)$.

PROOF. By Theorem 1.4, for M to have property S_r is equivalent to $S - M$ having property S_{n-r-2} rel. bounding cycles and finite $p^{n-r-1}(S - M)$. If in addition M has finite $p^{r+1}(M)$, then by Theorem VIII 6.4, the number $p^{n-r-2}(S - M)$ is finite and by Theorem 1.3, $S - M$ has property S_{n-r-2}.

[1] Unless otherwise stated, $p^r(M)$ hereafter denotes the dimension of $h^r(M)$; or what amounts to the same thing, the maximum number of cycles on compact subsets of M that are lirh on compact subsets of M.

One may also state:

1.6 THEOREM. *With M and S as in* 1.5, *but such that $p^r(M)$ and $p^{r+1}(M)$ are finite, then a necessary and sufficient condition that M have property* S_r *is that $S - M$ have property* S_{n-r-2}.

In connection with the number $g(K; G^r)$ defined in VII 8.7, we may also state, in view of Theorem VII 8.9 and the theorems above:

1.7 THEOREM. *If M is a closed subset of an $M^n_{r,r+2}$, S, then for $g(K; B^r)$ to be $\leqq \omega$ for all closed subsets K of M (where B^r denotes the group of all bounding r-cycles of M), it is necessary and sufficient that $S - M$ have property* S_{n-r-2} *rel. bounding cycles. And if $p^r(M)$ is finite, then for $S - M$ to have property* S_{n-r-2} *rel. bounding cycles is equivalent to $g(K; Z^r) \leqq \omega$ (where Z^r is the group of r-cycles of M).*

2. **Duality between lc and S properties.** In Corollary 1.2 above, we have already established a duality between the 0-lc property for a subcontinuum of an n-gcm and property S_{n-2} rel. bounding cycles of its complement. We shall now give further dualities between local connectedness properties of a closed subset of a manifold and the S properties of its complement.

2.1 THEOREM. *In order that a closed subset M of an $M^n_{1,r+2}$, S, should be lc^r, where r is a fixed integer such that $0 \leqq r \leqq n - 2$, it is necessary and sufficient that $S - M$ should have property S^{n-2}_{n-r-2} rel. bounding cycles and that $p^s(S - M)$ be finite for $s = n - r - 1, \cdots, n - 1$.*

PROOF. By Theorem VII 7.17, the lc^r property of M is equivalent to the S^r_0 property of M. And by Theorem 1.4, property S^r_0 of M is equivalent to property S^{n-2}_{n-r-2} of $S - M$ rel. bounding cycles and finite $p^s(S - M)$ for $s = n - r - 1, \cdots, n - 1$.

Another form of this duality may be obtained by means of the following lemma:

2.2 LEMMA. *If U is an open subset of an $M^n_{1,1}$, S, than a necessary and sufficient condition that U have property S_{n-1} is that $p^{n-1}(U)$ be finite; or, what is equivalent, that $p^0(S - U)$ be finite.*

PROOF. The necessity is obvious. To prove the sufficiency, it suffices, by Theorem 1.3, to show that U has property S_{n-1} rel. bounding cycles. Let $P \supseteqq Q$ be open subsets of S and let G denote the group of all compact $(n - 1)$-cycles of S that lie in $U \cap Q$ and bound in U. As S is lc^{n-1}, there exists an integer m such that every m cycles of G satisfy a homology in P. But consider a cycle Z^{n-1} on a compact subset M of $U \cap Q$ which is homologous to zero on a compact subset A of U and on a compact subset B of P. We may assume A and B to be minimal relative to containing M and carrying such a homology (Lemma VII 2.8). If $A = B$, then $Z^{n-1} \sim 0$ in $U \cap P$. If $A \neq B$, then it follows from Theorem VII 2.19 that $A \cup B = S$ and hence $A \supset S - P$. But then

$U \supset S - P$ and hence $S - U \subset P$. If this case occurs, then, let V be an open set such that $P \supseteq V \supset (S - U) \cup \overline{Q}$. Then every cycle Z^{n-1} bounding in U that fails to bound in $U \cap P$ is homologous on $\overline{V}$ to a cycle γ^{n-1} on $F(V)$, and as $F(V)$ is a closed subset of the open set $P - (S - U)$, only a finite number of such cycles γ^{n-1} are lirh in $P - (S - U) = U \cap P$ (Corollary VI 3.8). Then only a finite number of the corresponding cycles Z^{n-1} are lirh in $U \cap P$.

2.3 Theorem. *In order that a closed subset M of an $M^n_{1,r+2}$, S, should be lc^r, $0 \leqq r \leqq n - 2$, it is necessary and sufficient that its complement have property S^{n-1}_{n-r-1}, as well as property S_{n-r-2} rel. bounding cycles.*

Remark. The restriction $r \leqq n - 2$ may be removed by the following considerations: Suppose that a closed subset M of an n-gcm has finite $p^{n-1}(M)$. Then by Corollary V 19.5, M is semi-$(n - 1)$-connected. If $x \in M$, let P be an open subset of S containing x such that $(S - M) \cap (S - P) \neq 0$ and such that every cycle Z^{n-1} of $M \cap P$ bounds on M. Let Q be an open set such that $x \in Q \subset P$ and $(n - 1)$-cycles of Q bound in P. Then using Theorem VII 2.19 it may be shown that every such Z^{n-1} of $M \cap Q$ bounds in $M \cap P$. Thus we have:

2.4 Lemma. *If M is a closed subset of an n-gcm such that $p^{n-1}(M)$ is finite, then M is $(n - 1)$-lc.*

Now with M as above, suppose $S - M$ has property S^{n-1}_0. Then we can state that $S - M$ has property S^{n-1}_1 and property S_0 rel. bounding cycles, which, by Theorem 2.3, is equivalent to the lc^{n-2} property of M. But in addition $p^0(S - M)$ is finite, so that $p^{n-1}(M)$ is finite, and hence by the above lemma, M is $(n - 1)$-lc. Conversely, we need only recall that if M is lc^{n-1}, then $p^{n-1}(M)$ is finite and hence $p^0(S - M)$ is finite. Consequently we can state:

2.5 Theorem. *In order that a closed subset M of a spherelike n-gcm S should be lc^{n-1}, it is necessary and sufficient that its complement have property S^{n-1}_0.*

In case M is a continuum, one can state:

2.6 Theorem. *If M is a subcontinuum of an $M^n_{1,r+2}$, S, then a necessary and sufficient condition that M be lc^r, $0 \leqq r \leqq n - 2$, is that $S - M$ have property S^{n-2}_{n-r-1} and property S_{n-r-2} rel. bounding cycles.*

(We make the convention that property S^j_i for $i > j$ imposes no condition on the set in question; thus for $r = 0$, property S^{n-2}_{n-r-1} may be ignored.)

And of importance are the special cases embodied in the following corollaries:

2.7 Corollary. *If M is a subcontinuum of a spherelike n-gcm S, then a necessary and sufficient condition that M be lc^{n-2} is that $S - M$ have property S^{n-2}_1, and property S_0 rel. bounding cycles.*

2.8 Corollary. *In order that a subcontinuum of an* $M^n_{1,2}$, S, *should be 0-lc, it is necessary and sufficient that its complement have property* S_{n-2} *rel. bounding cycles.*

The case $n = 2$ of Corollary 2.7 implies:

2.9 Corollary. *In order that a subcontinuum* M *of the 2-sphere* S^2 *should be peanian, it is necessary and sufficient that* $S - M$ *have property* S_0 *rel. bounding cycles.*

Let us investigate the nature of an open set that has property S_0 rel. bounding cycles. First we show:

2.10 Lemma. *If* S *is an lc space* (hence, in the case of a locally compact space, 0-lc), U *an open subset of* S, *and* C *a union of components of* U, *and* γ^r *a compact cycle of* C *which bounds in* U, *then* γ^r *bounds in* C.

Proof. Let F be a compact subset of U carrying the homology $\gamma^r \sim 0$. Then $F = F_1 \cup F_2$, where $F_1 = F \cap C$, $F_2 = F \cap (U - C)$. By Corollary II 3.2, C and $U - C$ are open sets. Hence neither F_1 nor F_2 contains a limit point of the other, and as F is closed, F_1 and F_2 are disjoint closed sets.

Let $\mathfrak{E}$ be a covering of S, of which no element meets both F_1 and F_2, and let Σ' denote the complete family consisting of refinements of $\mathfrak{E}$. If $\mathfrak{U} \in \Sigma'$, then there exists $C^{r+1}(\mathfrak{U})$ on F such that $\partial C^{r+1}(\mathfrak{U}) = \gamma^r(\mathfrak{U})$. Now $C^{r+1}(\mathfrak{U}) = C_1^{r+1}(\mathfrak{U}) + C_2^{r+1}(\mathfrak{U})$, where $C_i^{r+1}(\mathfrak{U})$ is on F_i, $i = 1, 2$. And $\partial C^{r+1}(\mathfrak{U}) = \partial C_1^{r+1}(\mathfrak{U}) + \partial C_2^{r+1}(\mathfrak{U}) = \gamma^r(\mathfrak{U})$. Since $\gamma^r(\mathfrak{U})$ is on F_1, and $\partial C_2^{r+1}(\mathfrak{U})$ is on F_2, $\partial C_2^{r+1}(\mathfrak{U}) = 0$. It follows that $\partial C_1^{r+1}(\mathfrak{U}) = \gamma^r(\mathfrak{U})$ where $C_1^{r+1}(\mathfrak{U})$ is on F_1. Hence $\gamma^r \sim 0$ in C.

2.11 Lemma. *If* S *is an lc space,* U *an open subset of* S, *and* C *a component of* U, *and* U *has property* S_r *rel. a group* G^r, *then* C *has property* S_r *rel.* G^r.

Proof. Let $P \supset Q$ be open subsets of S. Then by Theorem VII 7.9, at most a finite number of cycles of G^r in $U \cap Q$ are lirh in $U \cap P$. A fortiori, at most a finite number of cycles of G^r in $C \cap Q$ are lirh in $U \cap P$. Let $Z_1^r, \cdots, Z_k^r$ be a base of cycles of G^r in $C \cap Q$ rel. homologies in $U \cap P$, and consider any $Z^r \in G^r$ in $C \cap Q$. Then there exists a homology

$$Z^r \sim \sum_{i=1}^{k} c^i Z_i^r \qquad \text{in } U \cap P. \tag{2.11a}$$

Now $C \cap P$ is a union of components of $U \cap P$; for if $x \in C \cap P$, the component of $U \cap P$ determined by x, as a connected subset of U, is a subset of a single component of U, hence a subset of C. It follows from Lemma 2.10 that the homology (2.11a) also holds in $C \cap P$. Hence some subset of the set of cycles Z_i^r forms a base for cycles of $C \cap Q$ rel. homologies in $C \cap P$. It follows that C has property S_r rel. G^r.

2.12 Lemma. *In order that an open subset* U *of an lc compact space* S *should*

have property S_0 *rel. bounding cycles, it is necessary and sufficient that the components of U have property* S_0, *and that for arbitrary covering* $\mathfrak{C}$ *of S, at most a finite number of components of U are of diameter* $> \mathfrak{C}$.

PROOF OF NECESSITY. The components of U have property S_0 rel. bounding cycles by Lemma 2.11, and as every 0-cycle of a component bounds in that component, each component must have property S_0. Consider a finite covering $\mathfrak{C}$ of S, and suppose $C_1, C_2, \cdots, C_n, \cdots$ an infinite sequence of components of U all of diameter $> \mathfrak{C}$. Let $\mathfrak{U}$ and $\mathfrak{D}$ be fcos of S such that $\mathfrak{U} \gg \mathfrak{D} >^* \mathfrak{C}$. Then each C_n meets two elements of $\mathfrak{U}$ that have disjoint closures, and since $\mathfrak{U}$ is finite, we may assume the situation where every C_n meets both $P_1, Q_1 \in \mathfrak{U}$ such that $\bar{P}_1 \cap \bar{Q}_1 = 0$. Let P, Q be open subsets of S such that $P_1 \Subset P$, $Q_1 \Subset Q$, $P \cap Q = 0$. Since $P_1 \cup Q_1 \Subset P \cup Q$, and U has property S_0 rel. bounding cycles, at most a finite number of bounding 0-cycles of U in $P_1 \cup Q_1$ are lirh in $U \cap (P \cup Q)$. But if $x_n \in C_n \cap P_1$, and $y_n \in C_n \cap Q_1$, and Z_n^0 is a nontrivial 0-cycle on $x_n \cup y_n$, then Z_n^0 is a bounding cycle of U. However, the cycles Z_n^0 are lirh in $U \cap (P \cup Q)$.

PROOF OF SUFFICIENCY. Let $P \Supset Q$ be open subsets of S. Since S is compact, there exists an open set R such that $S - \bar{Q} \Supset R \supset S - P$. Let $\mathfrak{C}$ denote the covering of S consisting of the sets P and R. By hypothesis, at most a finite number of components of U are of diameter $> \mathfrak{C}$. Let U' be the union of those components of U that meet Q. Then $U' = (\bigcup_{i=1}^{k} U_i) \cup (\bigcup U_\nu)$, where the k sets U_i are those components of U that fail to lie in P—finite in number because of the condition placed on the diameters of the components of U.

Consider a cycle γ^0 in $U \cap Q$ with compact carrier F, such that $\gamma^0 \sim 0$ in U. Then $F = (\bigcup_{j=1}^{h} F_{i(j)}) \cup (\bigcup_{j=1}^{m} F_{\nu(j)})$, where $F_\mu = F \cap U_\mu$, these sets being finite in number by Corollary IV 3.4. Since $\gamma^0 \sim 0$ in U, $\gamma^0 \sim 0$ in $(\bigcup_{j=1}^{h} U_{i(j)}) \cup (\bigcup_{j=1}^{m} F_{\nu(j)})$ by Lemma 2.10. Moreover, if the portion of γ^0 in U_μ is denoted by γ_μ^0, then $\gamma_{i(j)}^0 \sim 0$ in $U_{i(j)}$ and $\gamma_{\nu(j)}^0 \sim 0$ in $U_{\nu(j)}$. Thus, every compact 0-cycle in $U \cap Q$ that bounds in U is the sum of a finite set of compact cycles that bound in the components of U'. Of these, the cycles in $\bigcup U_\nu$ bound in $U \cap P$. Now each U_i has property S_0, hence at most a finite number, m_i, of 0-cycles of $U_i \cap Q$ are lirh in $U_i \cap P$. Hence at most $\sum_{i=1}^{k} m_i$ of the 0-cycles of $U \cap Q$ that bound in U are lirh in $U \cap P$, and we conclude that U has property S_0 rel. bounding cycles.

2.13 REMARK. It is interesting to note that the combination of Theorem 2.6 (for $r = 0$) and Lemma 2.12 gives the exact form of the Schoenflies-Moore theorem (Theorem IV 7.7) without assuming any metric: *In order that a subcontinuum M of a spherelike 2-gcm S should be 0-lc, it is necessary and sufficient that each domain of $S - M$ have property* S_0 *and that for arbitrary fcos* $\mathfrak{C}$ *of S, not more than a finite number of such domains are of diameter* $> \mathfrak{C}$.

Now in a compact metric space every collection of disjoint open sets is

countable. The question arises, does this hold in the general compact spaces with which we are dealing?[2] This question is answered by the next theorem:

2.14 THEOREM. *Let S be a perfectly normal, compact space, and $\mathfrak{P}$ a collection of disjoint open subsets P_ν of S. Then $\mathfrak{P}$ is countable.*

PROOF. The set $P = \bigcup P_\nu$ is open, hence $S - P$ is closed. If $S - P$ is empty, then, since S is compact, a finite number of the sets P_ν covers S and $\mathfrak{P}$ is a finite collection. Hence we assume that $S - P \neq 0$. Since S is perfectly normal, there exists a sequence of open sets $U_1, \cdots, U_k, \cdots$ such that for each k, $U_k \supset U_{k+1}$ and $\bigcap U_k = S - P$.

The collection $\mathfrak{P}_k$ consisting of U_k and the elements of $\mathfrak{P}$ covers S, hence there exists a unique finite subcollection $\mathfrak{P}_k'$ of $\mathfrak{P}_k$ consisting of U_k and a finite number of the sets P_ν which we denote by $P_{\nu(1)}, \cdots, P_{\nu(k)}$, covering S, where each $P_{\nu(i)}$, $i = 1, \cdots, k$, contains a point not in U_k. Now for each P_ν there exists a k such that $P_\nu \in \mathfrak{P}_k'$. Hence the sequence $P_{\nu(i)}$, $i = 1, 2, 3, \cdots$, contains all elements of $\mathfrak{P}$.

For the noncompact spaces the following theorem holds:

2.15 THEOREM. *If S is a perfectly normal space, and $\mathfrak{P}$ is a collection of disjoint open sets P_ν of S such that if $\mathfrak{C}$ is an arbitrary covering of S, then at most a finite number of elements of $\mathfrak{P}$ are of diameter $> \mathfrak{C}$, then the collection $\mathfrak{P}$ is countable.*

PROOF. Let $U = \bigcup_\nu P_\nu$ and $F = S - U$. Then U is open and F is closed, and since S is perfectly normal, there exists a sequence of open sets $U_1, \cdots, U_n, \cdots$ such that $F = \bigcap_n U_n$.

For each ν and n, let $F_{\nu n} = P_\nu - U_n$. Then $F_{\nu n}$ is a closed set, because $\overline{P}_\nu - P_\nu \subset F \subset U_n$ and hence $P_\nu - U_n = \overline{P}_\nu - U_n$. Since S is normal, there exists an open set $Q_{\nu n}$ such that $P_\nu \supseteq Q_{\nu n} \supset F_{\nu n}$.

For each n, the set U_n, together with all sets $Q_{\nu n}$, constitutes a covering $\mathfrak{C}_n$ of S by open sets. By hypothesis, at most a finite number of the sets P_ν are of diameter $> \mathfrak{C}_n$—denote these by $P_{\nu(1)}, \cdots, P_{\nu(n)}$. But no P_ν is covered by a $Q_{\nu n}$; hence $P_{\nu(1)}, \cdots, P_{\nu(n)}$ are the only elements of $\mathfrak{P}$ not contained in U_n.

However, for each P_ν, there exists an integer n such that U_n fails to contain P_ν. Thus enumeration of the collections $P_{\nu(1)}, \cdots, P_{\nu(n)}$ will also enumerate all elements P_ν, and $\mathfrak{P}$ is therefore countable.

As a corollary of the above theorems we now have:

2.16 COROLLARY. *If an open subset U of a perfectly normal, lc, compact space S has property S_0 rel. bounding cycles, then the components of U have property S_0 and are countable in number.*

REMARK. Since the components of an open subset of an lc space are them-

[2]That it fails to hold in a space that is not perfectly normal is shown by the collection of all second class ordinal numbers, together with the first ordinal of the third class, which is a compact space when topologized by means of its open intervals, and contains an uncountable set of disjoint open sets. It fails to be perfectly normal only at the third class ordinal.

selves open, and the n-gms are lc spaces, Theorems 2.14 and 2.15 apply thereto. In particular, every open subset of an n-gcm consists of a countable set of components (a fact already known for the $M^n_{n-1,n-1}$ from the Alexander duality). More generally, however, it follows from Theorem 2.14 that

2.17 THEOREM. *If U is an open subset of a perfectly normal, lc, compact space S, then the components of U are countable in number.*

As a consequence of Corollary 2.7, Lemma 2.12, and Theorem 2.15 and Corollary VI 3.2, we have:

2.18 THEOREM. *If a subcontinuum M of a spherelike n-gcm S is lc^{n-2}, then* (1) *the domains complementary to M are countable in number, and* (2) *for every fcos $\mathfrak{E}$ of S at most a finite number of these domains are of diameter $> \mathfrak{E}$, and* (3) *all except a finite number of these domains are simply r-connected* (see V 19.4) *for all dimensions r.*

Corollary IV 6.5 (the "Torhorst theorem") is a special case of the following theorem for manifolds:

2.19 THEOREM. *If M is an lc^{n-2} closed subset of a spherelike n-gcm S, and D is a component of $S - M$, then the boundary of D is 0-lc.*

PROOF. As M is 0-lc, $S - M$ has property S_{n-2} rel. bounding cycles by Theorem 2.3; and hence D has property S_{n-2} rel. bounding cycles by Lemma 2.11. Consequently, if $S - \bar{D}$ also had property S_{n-2} rel. bounding cycles, then $S - F(D)$ would also. Since M is 0-lc and compact, it has only a finite number of components, and by Theorem II 4.13, $F(D)$ has only a finite number of components; thus $p^{n-1}(S - F(D))$ is finite by Theorem VIII 6.4. It would then follow from Theorem 2.1 that $F(D)$ is 0-lc. By the same theorem, for $S - \bar{D}$ to have property S_{n-2} rel. bounding cycles, it is sufficient that $\bar{D}$ be 0-lc. Hence all we need do is show that $\bar{D}$ is 0-lc.

Suppose $\bar{D}$ is not 0-lc. Then, since $\bar{D}$ is a continuum, there exist by Theorem IV 2.1 open sets P and R such that $P \supset\!\!\!\supset R$, and infinitely many components M_ν of $\bar{D} \cap (\bar{P} - R)$ that contain points of both $F(P)$ and $F(R)$. Let P_1, P_2, Q, R_1 and R_2 be open sets such that $P \supset\!\!\!\supset P_1 \supset\!\!\!\supset P_2 \supset\!\!\!\supset Q \supset\!\!\!\supset R_2 \supset\!\!\!\supset R_1 \supset\!\!\!\supset R$. Since D is dense in $\bar{D}$, there must exist infinitely many 0-cycles in $D \cap (P_2 - \bar{R}_2)$ that are lirh in $D \cap (P - \bar{R})$. As S is 0-lc, at most a finite number of such 0-cycles are lirh in $P_1 - \bar{R}_1$. Let G^0 denote the group of all compact 0-cycles of $D \cap (P_2 - \bar{R}_2)$ that bound in $P_1 - \bar{R}_1$. Denote $D \cap (P - \bar{R})$ by K. Then by Theorem VIII 8.9, there exists a base $\{\gamma^0_i\}$ for G^0 rel. to homologies in K, and a fundamental system $\{\gamma^{n-1}_i\}$ of cycles of $S - K$ such that if $\gamma^{n-1} \sim \sum_{i=1}^{k} c^i\gamma^{n-1}_{j(i)}$ in $S - K$, $c^i \neq 0$, then γ^{n-1} is linked with each $\gamma^0_{j(i)}$, etc.

The portion of each γ^{n-1}_i in $P - \bar{R}$ is a cycle γ^{n-1}_{i1} mod $F(P - \bar{R})$, whose boundary is in $S - D$. Now $S - M$ has property S^{n-2}_0 rel. bounding cycles, and $p^s(S - M)$ is finite for $s = 1, \cdots, n - 2$, by Theorem 2.1. The same

holds for D (Lemma 2.11). Hence $S - D$ is lc^{n-2}, and therefore at most a finite number of $(n - 2)$-cycles of $S - D$ on $F(P - \bar{R})$ are lirh on $S - D - (P_1 - \bar{R}_1)$. Hence there exists a homology

$$\sum_{i=1}^{m} a^i \partial \gamma_i^{n-1} \sim 0 \qquad \text{in } S - D - (P_1 - \bar{R}_1),^3 \tag{2.19a}$$

where not all a^i are zero. Without loss of generality we may assume $c^1 \neq 0$. Then if $\gamma^{n-1} = \sum_{i=1}^{m} a^i \gamma_i^{n-1}$, the cycles γ^{n-1}, γ_1^0 are linked.

But let F be a carrier of γ^{n-1} in $S - K$, and C a carrier of the homology (2.19a) in $S - D - (P_1 - \bar{R}_1)$. Let A denote $C \cup [F \cap (\bar{P} - R)]$ and B denote $C \cup (F \cap [S - (\bar{P} - R)])$. Then by Lemma VII 1.14 there exist cycles Z_1^{n-1} and Z_2^{n-1} on A and B respectively such that $Z_1^{n-1} + Z_2^{n-1} \sim \gamma^{n-1}$ on $A \cup B \subseteq S - K$. Thus $Z_1^{n-1} + Z_2^{n-1}$ is linked with γ_1^0. However, $\gamma_1^0 \sim 0$ in $P_1 - \bar{R}_1 \subset S - B$. Also $\gamma_1^0 \sim 0$ in $D \subset S - A$. [For $F \subset S - K = S - D \cap (P - \bar{R})$ implies $F \cap (P - \bar{R}) \subset S - D$; and $C \subset S - D$.] Thus, by an argument similar to that used above in the proof of Theorem 1.1, a contradiction may be obtained, and we conclude that $\bar{D}$ is 0-lc, completing the proof of the theorem.

As interesting corollaries we may state:

2.20 Corollary. *If M is an lc^{n-2} continuum in a spherelike n-gcm, and D is a domain complementary to M, then the boundary of D is a 0-lc continuum.*

[We recall that in a spherelike n-gcm the Phragmen-Brouwer properties hold.]

2.21 Corollary. *If M is an lc^{n-2} continuum in the n-sphere S^n, then the domains complementary to M are bounded by Peano continua.*

2.22 Theorem. *Let U be an open subset of a spherelike n-gcm S, such that $p^{n-1}(U)$ is finite. Then if U has property S_0^{n-2}, the boundary of every component of U is 0-lc.*

Proof. Let C be a component of U. By Lemma 2.11, C has property S_0^{n-2} and it follows from Lemma 2.10 that $p^{n-1}(C)$ is finite. Then by Theorems 1.3 and 2.1, $S - C$ is lc^{n-2}. Consequently, by virtue of Theorem 2.19, the boundary of C is 0-lc.

2.23 Corollary. *If a simply $(n - 1)$-connected domain D of a spherelike n-gcm S has property S_0^{n-2}, then the boundary of D is 0-lc.*

2.24 Corollary. *If D is a simply $(n - 1)$-connected domain in the n-sphere S^n, and D has property S_0^{n-2}, then the boundary of D is peanian.*

In particular, then,

2.25 Corollary. *If D is a bounded simply 1-connected domain in the euclidean plane, and D has property S, then the boundary of D is peanian.*

[It will be noted that Corollary 2.25 and Theorem IV 6.3 are identical.]

[3] Note that this set lies in $S - K$.

3. **Duality with S properties in terms of cohomology.** Up to now, in the present chapter, no use has been made, except for purposes of proofs, of the cocycles and their dual relationship to the cycles in the study of "S properties." In the present section we shall show how this may be done, and derive certain new results.

We recall that in Chapter VI a property which we called "$(P, Q)_n$" was introduced (VI 7.1). Since we shall be dealing with locally compact spaces, we may replace the $H_n(S: Q, 0; P, 0)$ of Definition VI 7.1 by $h_n(S: Q; P)$, the vector space of compact cocycles in Q mod cohomologies in P. We have seen (Theorem VII 7.9) that Property S_r is equivalent to what we called "Property $(P, Q)^r$" (Definition VII 7.12) in analogy with property $(P, Q)_n$. Definitions of the same concepts *relative* to some group of cycles should be obvious. The notations defined below will be found useful, however.

3.1 DEFINITION. A subset M of a space S will be said to have *property* $(P, Q, \sim)^r$ if for arbitrary open subsets P and Q of S such that $P \supset Q$, at most a finite number of the bounding compact r-cycles of M in Q are lirh in $M \cap P$.

3.2 DEFINITION. A subset M of a space S will be said to have *property* $(P, Q, \backsim)_r$ if for arbitrary open subsets P and Q of S such that $P \supset Q$, at most a finite number of the cobounding cocycles of M in Q are lircoh on M in P.

REMARK. In the new terminology, Theorem 1.1 may be restated as follows: *If M is a closed subset of an $M^n_{r,r+2}$, S, then a necessary and sufficient condition that M have property $(P, Q, \sim)^r$ is that $S - M$ have property $(P, Q, \sim)^{n-r-2}$.* Our next step will be to establish more dualities between "P, Q-properties."

3.3 THEOREM. *In order that a compact space S have property $(P, Q, \sim)^{r-1}$, it is necessary and sufficient that S have property $(P, Q, \backsim)_r$.*

PROOF. Let $P \supset Q$ be arbitrary open sets. Let us select open sets P_1, Q_1 such that $P \supset P_1 \supset Q_1 \supset Q$, and denote $P - \bar{Q}$, $P_1 - \bar{Q}_1$ by U, V respectively.

To prove the necessity, suppose Z^i_r, $i = 1, 2, 3, \cdots$, an infinite sequence of cocycles in Q that are lircoh in P, but which cobound in S. Since S has property $(P, Q, \sim)^{r-1}$, there exists an integer m such that every m bounding $(r-1)$-cycles of S that are on $F(P_1)$ satisfy a homology relation in U. Now by Theorem V 18.31 there exist cycles Z^r_j mod $S - P$, $j = 1, \cdots, m$, such that $Z^i_r \cdot Z^r_j = \delta^i_j$, $i, j \leqq m$. The portion of each Z^r_i in P_1 is a cycle γ^r_i mod $F(P_1)$ whose boundary is a cycle on $F(P_1)$ that obviously bounds in S. Hence there exists a homology relation

$$\sum_{i=1}^{m} c^i \partial \gamma^r_i \sim 0 \qquad \text{in } U, \tag{3.3a}$$

where not all c^i are zero. Because of relation (3.3a), there exists, by Lemma VII 1.6, a cycle Γ^r of S such that

$$\Gamma^r \sim \sum_{i=1}^{m} c^i \gamma^r_i \sim \sum_{i=1}^{m} c^i Z^r_i \qquad \text{mod } S - Q. \tag{3.3b}$$

But suppose $c^1 \neq 0$, for instance. Then $Z_r^1 \cdot \Gamma^r = c^1 \neq 0$. On the other hand, Z_r^1 is a cobounding cocycle of S, and by Theorem V 18.24, $Z_r^1 \cdot \Gamma^r = 0$.

The proof of the sufficiency is similar to the above: The Z_r^i are replaced by cycles Z_i^{r-1} that bound in S but are lirh *on* P, and the Z_j^r replaced by cocycles Z_{r-1}^i mod $S - \bar{P}$. And we use the portion of each Z_{r-1}^i *on* $\bar{Q}_1$, whose coboundary is in V. Relation (3.3a) is replaced by a cohomology in U.

3.4 THEOREM. *Let S be a compact space which has properties $(P, Q)_r$ and $(P, Q, \backsim)_{r+1}$, and let M be a closed subset of S. Then a necessary and sufficient condition for M to have property $(P, Q, \backsim)_r$ is that $S - M$ have property $(P, Q, \backsim)_{r+1}$.*

(It is interesting to compare the statement of this theorem with that of Theorem X 1.4. It is to be noted that by virtue of Theorem VIII 1.1, Theorem 3.4 applies to an n-gcm.)

PROOF OF SUFFICIENCY. Let $R \supseteq P \supseteq Q$ be open subsets of S. Now if Z_r is a cocycle on M which cobounds on M, then δZ_r cobounds in $S - M$. For $Z_r \backsim 0$ on M implies the existence of a chain C_{r-1} on some covering such that

$$\delta C_{r-1} = Z_r - U_r, \tag{3.4a}$$

where U_r is a chain in $S - M$, and applying the operator δ to relation (3.4a), we get $\delta Z_r = \delta U_r$.

Since $S - M$ has property $(P, Q, \backsim)_{r+1}$, there exists an integer m such that every m $(r + 1)$-cocycles of $S - M$ that lie in $Q \cap (S - M)$ and cobound in $S - M$ satisfy a cohomology in $P \cap (S - M)$. Suppose there exist $Z_r^1, \cdots, Z_r^i, \cdots$, an infinite sequence of cocycles of M in Q, that cobound on M. Then the cocycles δZ_r^i cobound in $S - M$, and consequently there exist relations $\delta C_r^j = \sum_{m(j-1)+1}^{mj} c_i \delta Z_r^i$ in $P \cap (S - M)$. The chain $\sum_{m(j-1)+1}^{mj} c_i Z_r^i - C_r^j$ is a cocycle of S in P. Now since S itself has property $(P, Q, \backsim)_r$, there exists an integer k such that every k r-cocycles in P cobound in R. Hence there exists a relation

$$\delta L_{r-1} = \sum_{j=1}^{k} a^j \sum_{i=m(j-1)+1}^{mj} [c_i Z_r^i - C_r^j] \qquad \text{in } R. \tag{3.4b}$$

Since C_r^j is in $S - M$, relation (3.4b) implies that the cocycles Z_r^i, $i = 1, \cdots, mk$, satisfy a cohomology on M in R, and we conclude that M has property $(P, Q, \backsim)_r$.

PROOF OF NECESSITY. With R, P and Q as before, suppose Z_{r+1}^i, $i = 1, 2, 3, \cdots$, a sequence of cocycles in $Q \cap (S - M)$ that cobound in $S - M$. Since S has property $(P, Q, \backsim)_{r+1}$, there exist relations

$$\delta C_r^j = \sum_{m(j-1)+1}^{mj} c_i Z_{r+1}^i \qquad \text{in } P. \tag{3.4c}$$

By hypothesis there exist chains L_r^i in $S - M$ such that $\delta L_r^i = Z_{r+1}^i$, and hence cocycles $C_r^j - \sum_{m(j-1)+1}^{mj} c_i L_r^i = \gamma_r^j$.

Now the fact that S has property $(P, Q, \smile)_r$ implies that $p_r(S)$ is a finite number $h - 1$, where h is a positive integer. Hence there exist chains N^k_{r-1} on S such that

$$\delta N^k_{r-1} = \sum_{h(k-1)+1}^{hk} \gamma^i_r . \tag{3.4d}$$

The chains C^i_r are cocycles mod $S - M$, and relations (3.4d) imply that the chains

$$D^k_r = \sum_{h(k-1)+1}^{hk} C^i_r \tag{3.4e}$$

constitute cobounding cocycles of M. As these cocycles all lie in P, and M has property $(P, Q, \smile)_r$, there exist relations

$$\delta E^t_{r-1} = \sum_{k=s(t-1)+1}^{st} D^k_r - H^t_r \qquad \text{in } R, \tag{3.4f}$$

where H^t_r is in $R \cap (S - M)$.

Relations (3.4c), (3.4e) and (3.4f) imply that $\delta H^t_r = \sum_{s(t-1)+1}^{st} \delta D^k_r = \sum_{s(t-1)+1}^{st} \sum_{h(k-1)+1}^{hk} \delta C^i_r = \sum_{s(t-1)+1}^{st} \sum_{h(k-1)+1}^{hk} \sum_{m(j-1)+1}^{mj} c_i Z^i_{r+1}$.

3.5 Theorem. *If U is an open subset of an $M^n_{r-2,r-1}$, S, and $1 < r \leqq n$, then a necessary and sufficient condition that U have property $(P, Q, \smile)_r$ is that U have property $(P, Q, \sim)^{n-r}$.*

Proof. Let $M = S - U$. By Theorem 1.1, for U to have property $(P, Q, \sim)^{n-r}$ implies that M has property $(P, Q, \sim)^{r-2}$. This in turn implies, by Theorem 3.3, that M has property $(P, Q, \smile)_{r-1}$; which in turn implies, by Theorem 3.4, that U has property $(P, Q, \smile)_r$. To summarize, for U to have property $(P, Q, \sim)^{n-r}$ implies that U has property $(P, Q, \smile)_r$. To complete the proof we need only show the converse.

Let $P \supseteq Q$ be open subsets of S, and suppose that U has property $(P, Q, \smile)_r$. Then there exists an integer m such that every m cobounding r-cocycles of U that lie in Q satisfy a cohomology in $U \cap P$. Consider cycles Z^{n-r}_i, $i = 1, \cdots, m$, in $U \cap Q$ that bound in U. By Lemma VIII 5.4, there exist cocycles Z^i_r in $U \cap Q$ such that $Z^i_r \frown \Gamma^n \sim Z^{n-r}_i$ in $U \cap Q$ and $Z^i_r \smile 0$ in U, Γ^n being the fundamental cycle of S. There exists a relation $\delta C_{r-1} = \sum_{i=1}^m c^i Z^i_r$ in $U \cap P$, which in turn implies $\partial(C_{r-1} \frown \Gamma^n) = (-1)^{n-r+1} \delta C_{r-1} \frown \Gamma^n$. Thus $\sum_{i=1}^m c^i Z^{n-r}_i \sim \sum_{i=1}^m c^i Z^i_r \frown \Gamma^n \sim 0$ in $U \cap P$.

Now by an argument parallel to that used in the case of Theorem 1.3, we may prove:

3.6 Theorem. *A necessary and sufficient condition that a subset M of a compact space S have property $(P, Q)_r$ is that $p^r(M)$ be finite and M have property $(P, Q, \smile)_r$.*

3.7 Theorem. *A necessary and sufficient condition that a compact space S have property $(P, Q)_r$, $r \geqq 1$, is that S have property $(P, Q, \sim)^{r-1}$ and finite $p^r(S)$.*

Proof. For S to have property $(P, Q)_r$ is equivalent, by Theorem 3.6, to S having property $(P, Q, \backsim)_r$ and finite $p^r(S)$. By Theorem 3.3, this is equivalent to S having property $(P, Q, \sim)^{r-1}$ and finite $p^r(S)$.

3.8 Corollary. *If a compact space S is lc^r, then S has property $(P, Q)_r$.*

Proof. By Corollary VI 3.2, $p^r(S)$ is finite, and by Corollary VI 3.7, S has property $(P, Q)^{r-1}$. Hence by Theorem 3.7, S has property $(P, Q)_r$.

3.9 Corollary. *If a compact space S is lc^r and $p^{r+1}(S)$ is finite, then S has property $(P, Q)_{r+1}$.*

From Theorems 3.7, 1.3 and V 18.18 we have:

3.10 Theorem. *In order that a compact space S should have property $(P, Q)_r$ and finite $p_{r-1}(S)$, $r \geqq 1$, it is necessary and sufficient that S have property $(P, Q)^{r-1}$ and finite $p^r(S)$.*

It seems advisable at this point to summarize some of the incidental intrinsic equivalences that have been established in Chapter VII and in this chapter:

3.11 Theorem. *For a compact space S, the following sets of properties are equivalent:*

I. *S is lc^n, and $p^{n+1}(S)$ is finite.*
II. *S has property S_0^n, and $p^{n+1}(S)$ is finite.*
III. *S has property $(P, Q)^r$ for $r = 0, 1, \cdots, n$ and $p^{n+1}(S)$ is finite.*
IV. *S has property $(P, Q)_r$ for $r = 1, 2, \cdots, n+1$ and $p_0(S)$ is finite.*
V. *For all closed subsets K of S, $g(K; Z^r) \leqq \omega$ (where Z^r is the group of r-cycles of S), $r = 0, 1, \cdots, n$, and $p^{n+1}(S)$ is finite.*

Proof. The equivalence of I and II was shown in Theorem VII 7.17; that of II and III in Theorem VII 7.9; that of II and V in Theorem VII 8.9; and that of III and IV follows from the results just established above.

3.12 Theorem. *In order that a closed subset M of an $M^n_{r-1,r+1}$, S, should have property $(P, Q)_r$, $1 \leqq r \leqq n-1$, it is necessary and sufficient that $S - M$ have property $(P, Q)^{n-r-1}$.*

Proof. By Theorem 3.7, for M to have property $(P, Q)_r$ is equivalent to M having property $(P, Q, \sim)^{r-1}$ and finite $p^r(M)$. By Theorems 1.1 and VIII 6.4, this is equivalent to $S - M$ having property $(P, Q, \sim)^{n-r-1}$ and finite $p^{n-r-1}(S)$, which in turn is equivalent, by Theorem 1.3, to $S - M$ having property $(P, Q)^{n-r-1}$.

3.13 Theorem. *If the open subset U of an $M^n_{n-r-3,n-r}$, S, has property $(P, Q)^r$, $0 \leqq r \leqq n-3$, and $q^{r+1}(U, x) \leqq \omega$ for all $x \in F(U)$, then U has property $(P, Q)^{r+1}$.* (Compare Theorem VI 7.2.)

Proof. By Theorem 3.12, for U to have property $(P, Q)^r$ is equivalent to $S - U$ having property $(P, Q)_{n-r-1}$. By Theorem X 1.5, $q^{r+1}(U, x) \leqq \omega$ for

all $x \in F(U)$ is equivalent to $p_{n-r-2}(S - U, x) \leqq \omega$ for all $x \in S - U$. Applying Theorem VI 7.2, we see that $S - U$ has property $(P, Q)_{n-r-2}$, and hence, by Theorem 3.12, that U has property $(P, Q)^{r+1}$.

3.14 Corollary. *If M is a subcontinuum of an $M^n_{1,k+2}$, S, such that $S - M$ has property S_{n-k-2} and $q^r(S - M, x) \leqq \omega$ for all $x \in F(S - M)$ and $r = n - k - 1, \cdots, n - 2$, then M is lc^k.*

[Cf. Theorem 2.6.]

3.15 Corollary. *If the open subset U of a spherelike n-gcm S has property S_0, finite $p^{n-1}(U)$, and $q^r(U, x) \leqq \omega$ for $r = 1, \cdots, n - 2$ and all $x \in F(U)$, then U has property S_0^{n-2} and, consequently, the boundary of every component of U is 0-lc.*

[Cf. Theorem 2.22.]

4. **Relation of avoidability properties at a point to S-properties of the complement of a closed set.**

4.1 Definition (see X 4.3). If M is a closed subset of a space S, $x \in M$, and U an open subset of $S - M$, then by $q^r(U, x; G^r)$ we denote a number defined exactly like the number $q^r(U, x)$ except that only the cycles of a certain subgroup G^r of the group of all r-cycles of U are used. In particular, $q^r(U, x, \sim)$ will denote the number obtained in the case where G^r is the group of all compact r-cycles of U that bound in U.

By an argument analogous to that used in the "necessity" part of the proof of Theorem 1.1, we can prove:

4.2 Theorem. *If M is a closed subset of an $M^n_{r+1,r+2}$, S, U a union of components of $S - M$, and $x \in M$ such that M is almost locally r-avoidable rel. bounding cycles at x, $r \leqq n - 2$, then $q^{n-r-2}(U, x, \sim) \leqq \omega$.*

4.3 Corollary. *Under the same hypothesis, except that there exists an open set P containing x such that only finitely many $(n - r - 2)$-cycles of $U \cap P$ are lirh in U, then $q^{n-r-2}(U, x) \leqq \omega$.*

4.4 Corollary. *With $r \leqq n - 2$, if M is a closed subset of an $M^n_{r+1,r+2}$, S, such that $p^{r+1}(M)$ is finite, and x a point of M at which M is almost locally r-avoidable rel. bounding cycles, then $q^{n-r-2}(S - M, x) \leqq \omega$.*

Remark. Corollary 4.4 is also a corollary of Theorem X 1.5 and the following lemma:

4.5 Lemma. *If M is a locally compact space which is almost locally r-avoidable rel. bounding cycles at $x \in M$, and $p^{r+1}(M)$ is finite, then $p^{r+1}(M, x) \leqq \omega$.*

Proof. If $U \supset V \supset W$ are open sets containing x such that only finitely many r-cycles of $F(V)$, which bound on M, are lirh in $M - W$ then

$p^{r+1}(M\colon M, M - U; M, M - W)$ is finite. Otherwise, there exist infinitely many cycles Z_i^{r+1} mod $M - U$ that are lirh mod $M - W$. The portion of each Z_i^{r+1} in V is a cycle γ_i^{r+1} mod $F(V)$ whose boundary, γ_i^r, is a cycle of $F(V)$ that obviously bounds on M. In finite linear combinations the γ_i^r bound on $M - W$, and consequently the corresponding linear combinations of the γ_i^{r+1} are portions of absolute cycles of M that cannot be lirh on M since $p^{r+1}(M)$ is finite. It follows that the γ_i^{r+1} and hence the Z_i^{r+1} are not lirh mod $M - W$.

4.6 Corollary. *For an n-dimensional continuum M to be* lc^n, *it is necessary and sufficient that for* $r = 1, 2, \cdots, n$, $p^r(M)$ *be finite and M be almost locally* $(r - 1)$*-avoidable rel. bounding cycles of M at all points.*

[The necessity follows from Corollaries VI 3.2, VI 3.8; the sufficiency from Corollary VI 6.12, Theorem VI 7.9 and the above lemma.]

4.7 Remark. It will be noted that from Theorems VII 7.17, VII 8.2, VII 8.3, and the above, it follows that *if M is an n-dimensional continuum such that for* $r = 1, \cdots, n$, $p^r(M)$ *is finite and M is almost locally* $(r - 1)$*-avoidable at all points rel. bounding cycles of M, then the closed subsets of M are almost completely r-avoidable as well as almost locally r-avoidable.*

Now resuming with the line of thought in Theorem 4.2: If in addition we utilize a proof analogous to the sufficiency proof of Theorem 1.1, we may prove:

4.8 Theorem. *If M is a closed subset of an* $M^n_{r,r+2}$, *S, and* $x \in M$, *then a necessary and sufficient condition that M be almost locally r-avoidable rel. bounding cycles at x,* $r \leqq n - 2$, *is that* $q^{n-r-2}(S - M, x, \sim) \leqq \omega$.

As a consequence of the above theorems we may derive certain relations between the avoidability properties at a point and the S-properties of the complement of a closed set:

4.9 Theorem. *Let M be a closed subset of an* $M^n_{n-1,n-1}$, *S, such that M is almost locally* $(n - 2)$*-avoidable rel. bounding cycles. Then each domain complementary to M has property* S_0.

Proof. Consider any domain D complementary to M and any covering $\mathfrak{U}$ of S. Let $x \in F(D)$. By Theorem 4.2, if U is an element of $\mathfrak{U}$ that contains x, there exists an open set V containing x and lying in U, such that $V \cap D$ is contained in a finite number of connected subsets of $U \cap D$. It follows that D has property S_0.

4.10 Theorem. *Let M be a closed subset of a spherelike n-gcm S such that* $p^r(M)$ *is finite and M is almost locally r-avoidable rel. bounding cycles for* $r = 0, 1, \cdots, n - 2$. *Then every complementary domain of M has property* S_0^{n-2} *and, consequently, the boundary of every such domain is 0-lc.*

Proof. Let D be a domain complementary to M. By Theorem 4.9, D has property $(P, Q)^0$.

Since M is almost locally $(n - 3)$-avoidable rel. bounding cycles and $p^{n-2}(M)$ is finite, it follows from Corollary 4.4 that $q^1(S - M, x) \leqq \omega$ for all $x \in M$. Hence by Theorem 3.13, D has property $(P, Q)^1$.

Continuing in this manner, we finally arrive at the case where D has property $(P, Q)^{n-3}$. Then since M is almost locally 0-avoidable rel. bounding cycles and $p^1(M)$ is finite, $q^{n-2}(D, x) \leqq \omega$ and hence by Theorem 3.13, D has property $(P, Q)^{n-2}$.

Thus D has property S_0^{n-2}, and since $p^0(M)$ is finite, it follows (Theorem 2.6) that $S - D$ is lc^{n-2}. That $F(D)$ is 0-lc now follows from Theorem 2.19.

If instead of Corollary 4.4, as employed in the above proof, use is made of Corollary 4.3, we may prove:

4.11 Theorem. *If M is a subcontinuum of a spherelike n-gcm S which is almost locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n - 2$, and D is a domain complementary to M such that for some fcos $\mathfrak{E}$ of S at most a finite number of r-cycles of D of diameter $< \mathfrak{E}$ are lirh in D, then D has property S_0^{n-2} and its boundary is 0-lc.*

4.12 Remark. In connection with Theorems 4.9 and 4.10 (Theorem 4.10, incidentally, is a generalization of Theorem 2.19, inasmuch as every compact lc^{n-2} satisfies the conditions of its hypothesis), it is interesting to note the relations to what Whyburn terms "semi-local-connectedness." (See Whyburn [Wh; p. 19]; also VII 6.27 above.) According to definition (Whyburn's definition is slightly rephrased here so as to apply to the nonmetric, nonconnected case), a space M is *semi-locally-connected* at $x \in M$ if for arbitrary open set U containing x there exists a neighborhood V of x in U such that $M - V$ has only a finite number of components. In the case of continua, this property is identical with the property of being almost 0-avoidable at x. Whyburn develops (loc. cit. Chap. IV) the cyclic element theory of semi-locally-connected metric continua, of which the cyclic element theory of Peano spaces is a special case. However, of special interest from the standpoint of positional properties is the following theorem proved by Whyburn [a; Theorem 14]: In the euclidean 2-sphere, every complementary domain of a semi-locally connected continuum has property S, and consequently the boundaries of such domains are Peano continua (cf. Corollary 2.25 above). This is obviously a generalization of the Torhorst theorem (Corollary IV 6.5).

Now in the case of a continuum, semi-local-connectedness, almost 0-avoidability rel. bounding cycles and almost local 0-avoidability rel. bounding cycles are all equivalent (compare Corollary VII 4.12[a]). (In nonconnected spaces, semi-local-connectedness is generally a stronger property. For example, in the space of real numbers, let $M = \{x \mid (x = 1/n) \vee (x = 1 - 1/n)\}$, together with 0 and 1.) However, for $r > 0$, the almost local r-avoidability property is weaker than the nonlocal type. (A condition intermediate between the two types of avoidability mentioned here may be obtained by requiring that for "small enough" neighborhood U of the point x under consideration, there exist a

neighborhood V of x contained in U such that only a finite number of the r-cycles of $F(U)$ are lirh on $S - V$. For example, in the 3-dimensional coordinate space (ρ, θ, φ), let $K_n = \{(\rho, \theta, \varphi) \mid \rho = 1/n\}$, and on K_n let M_n be a set consisting of the points on an infinite number of disjoint circles. Between the spheres of radius $1/n$ and $1/(n + 1)$ let R_n be the set of all points such that ρ, θ and φ are rational. Let $M = p \cup \bigcup M_n \cup \bigcup R_n$, where $p = (0, 0, 0)$. Then M is almost locally 1-avoidable at p, but is not avoidable in the sense just defined, nor is it almost 1-avoidable at p. This example is easily modified so as to make of M a compact space.) In a simply r-connected space, almost r-avoidability and almost local r-avoidability are equivalent, but in a semi-r-connected space the latter is again weaker. (For instance, in the example just given above, let M_1 be a disjoint, denumerable set of circles all of diameter $> 1/4$ on K_1. Inside the sphere of radius 1 let R be the set of all "rational points", and $S = p \cup M_1 \cup R$. With $U = \{(\rho, \theta, \varphi) \mid \rho < 1\}$, a V does not exist satisfying the almost 1-avoidability condition.)

In view of the above facts, it appears to be desirable to utilize the almost local r-avoidability property wherever possible in extensions to the case $r > 0$. And in Theorem 4.9 we have an interesting obvious generalization of the theorem of Whyburn cited above.

4.13 The following example is instructive: In the (x, y)-plane, for each positive integer n, let $A_n = \{(x, y) \mid x = 1/n, 0 < y < 1\}$, $A_0 = \{(0, y) \mid 0 \leqq y \leqq 1\}$, $A = \bigcup A_n$, $B = \{(x, 0) \mid 0 \leqq x \leqq 1\}$, $C = \{(x, 1) \mid 0 \leqq x \leqq 1\}$; and finally $M = A \cup B \cup C$. The set M is both locally 0- and 1-avoidable rel. bounding cycles, yet considered as a subset of 3-space, it constitutes the boundary of its single complementary domain and is not peanian. The hypothesis of Theorem 4.10 does not apply, inasmuch as $p^1(M)$ is not finite.

If in the hypothesis of Theorem 4.10 we had also assumed $p^{n-1}(M)$ finite, then since each complementary domain has property S_0 and $p^0(S - M)$ is finite, $S - M$ has property S_0 by Theorem 1.3. We then have:

4.14 THEOREM. *If M is a closed subset of a spherelike n-gcm S such that $p^r(M)$ is finite for $r = 0, 1, \cdots, n - 1$, and M is almost locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n - 2$, then M is lc^{n-1}.*

[Cf. Lemma 2.4.]

4.15 THEOREM. *A necessary and sufficient condition that a closed subset M of a spherelike n-gcm S should be lc^{n-2} is that it satisfy, in addition to the hypothesis of Theorem* 4.10, *the condition that for arbitrary covering $\mathfrak{C}$ of S, there exist at most a finite number of domains complementary to M of diameter $> \mathfrak{C}$.*

PROOF. To prove the sufficiency, by Theorem 4.9 and Lemma 2.12, $S - M$ has property S_0 rel. bounding cycles; the proof then proceeds as in the case of Theorem 4.10. The necessity follows from Theorem 2.3, Lemma 2.12 and Corollaries VI 3.2 and VI 3.8.

(For the case where S is the 2-dimensional euclidean sphere, see Whyburn [a, Corollaries 2, 3].)

4.16 The following example is of interest in connection with Theorems 4.10, 4.14 and 4.15: In cartesian 3-space, let A denote the surface of the unit cube $\{(x, y, z) \mid 0 \leqq x \leqq 1, 0 \leqq y \leqq 1, 0 \leqq z \leqq 1\}$, and for each positive integer n, let $A_n = \{(1/n, y, z) \mid 0 \leqq y \leqq 1, 0 \leqq z \leqq 1\}$. Let $M = \bigcup A_n \cup A$. The point set M is almost locally 0- and 1-avoidable, but considered as a configuration in the 3-sphere, it is readily seen that $S - M$ does not have property S_1, although its individual complementary domains do, in conformity with Theorem 4.10. The set M is not 0-lc; it fails to satisfy the hypothesis of Theorem 4.14, in that $p^2(M)$ is not finite, and also fails to satisfy the condition concerning the size of the domains in the hypothesis of Theorem 4.15.

4.17 Another interesting example in S^n is afforded by a sequence of successively tangent 2-spheres, whose diameters form a null sequence and which converge to a point p. Both Theorems 4.10 and 4.15 apply, the set so formed being lc^1 and the boundaries of the complementary domains being 0-lc. However, the hypothesis of Theorem 4.14 does not apply.

By use of Corollary 4.3 we may prove:

4.18 THEOREM. *In order that a subcontinuum M of a spherelike n-gcm S should be lc^{n-2}, it is necessary and sufficient that* (1) *M be almost locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n - 2$;* (2) *there exists a fcos $\mathfrak{C}$ of S such that at most a finite number of compact r-cycles of $S - M$ of diameter $< \mathfrak{C}$ are lirh in $S - M$, $r = 1, \cdots, n - 2$; and* (3) *if $\mathfrak{U}$ is an arbitrary covering of S, at most a finite number of the domains complementary to M are of diameter $> \mathfrak{U}$.*

The following three theorems may be proved by use of the same methods employed above:

4.19 THEOREM. *If M is a closed subset of a spherelike n-gcm S such that* (1) *for arbitrary covering $\mathfrak{C}$ of S, at most a finite number of the complementary domains of M are of diameter $> \mathfrak{C}$,* (2) *M is almost locally r-avoidable rel. bounding cycles for $r = n - k - 2, \cdots, n - 2$,* (3) *there exists a covering $\mathfrak{U}$ of S such that at most a finite number of the cycles of $S - M$ of diameter $< \mathfrak{U}$ are lirh in $S - M$; then $S - M$ has property S_1^k, and property S_0 rel. bounding cycles.*

4.20 THEOREM. *If M is a closed subset of a spherelike n-gcm S, then a necessary and sufficient condition that $S - M$ have property S_0^k is that $p^r(M)$ be finite for $r = n - k - 1, \cdots, n - 1$, and that M be almost locally r-avoidable rel. bounding cycles for $r = n - k - 2, \cdots, n - 2$.*

4.21 THEOREM. *If the closed subset M of a spherelike n-gcm S is almost locally r-avoidable rel. bounding cycles for $r = n - k - 2, \cdots, n - 2$, and D is a domain complementary to M such that for some covering $\mathfrak{C}$ of S, at most a finite number of the r-cycles of D of diameter $< \mathfrak{C}$ are lirh in D, $r = 0, \cdots, k$, then D has property S_0^k.*

Although the numbers $q^r(S - M, x, \sim)$ play a leading role above, due to their relation to avoidability in M, the numbers $q^r(S - M, x)$ may be utilized to advantage in certain cases:

4.22 Lemma. *If M is an lc^k, $k \leqq n - 1$, closed subset of an n-gm, then $q^r(S - M, x) \leqq \omega$ for all $x \in M$ and $r = n - k - 1, n - k, \cdots, n - 1$.*

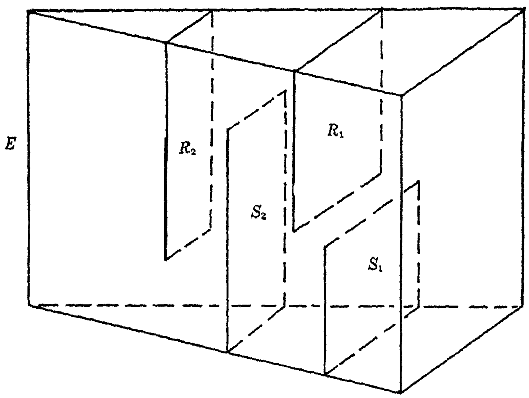
This lemma is a consequence of Theorem VII 2.26 and Theorem X 1.5.

4.23 Theorem. *In order that a k-dimensional closed subset of an n-gm should be lc^k, $k \leqq n - 1$, it is necessary and sufficient that $q^r(S - M, x) \leqq \omega$ for $r = n - k - 1, n - k, \cdots, n - 1$ and all $x \in M$.*

The necessity follows from the above lemma, and the sufficiency follows from Theorem X 1.5 and Theorem VII 2.25.

4.24 Corollary. *If M is a closed subset of an n-gm S such that $q^r(S - M, x) \leqq \omega$ for all $x \in M$ and $r = 0, 1, \cdots, n - 1$, then M is lc^{n-1}.*

(It is easy to show that if M is a closed subset of an n-gm and $x \in M$, then $p^n(M, x) \leqq 1$. Hence if M is n-dimensional, it is lc^n!)



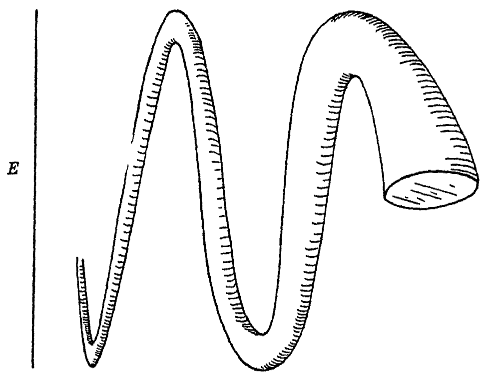
5. **Weak S-properties; recognition of lc^k boundaries from properties of the domain.** We have shown above how the dualities between the S-properties of a closed set and its complement in a gcm lead to relations between the local connectedness properties of the closed set and the S-properties of its complement. (As in Theorem 2.1, for instance.) We have found properties of the complement of a closed point set which are equivalent to local connectedness properties of the set. We have not yet determined, however, what properties of a *single domain* are equivalent to the local connectedness of its boundary. The analogous problem, for the case of a domain bounded by an $(n - 1)$-gcm, was solved in Chapter X (Theorem X 6.8, for instance). In the present section we shall find conditions which characterize those domains whose boundaries are 0-lc, those whose boundaries are lc^k, etc.

For the case of the euclidean 2-sphere, the problem was solved by R. L. Moore [d] in 1922; he showed that a necessary and sufficient condition that a simply 1-connected domain in the 2-sphere should have a Peano continuum as

boundary is that it have property S. In Corollary 2.23 above, we found that if a simply $(n - 1)$-connected domain in a spherelike n-gcm S has property S_0^{n-2}, then its boundary is 0-lc; but this condition is not, in case $n > 2$, a necessary condition in order that the boundary of a domain should be 0-lc. Even in the case of ordinary euclidean 3-space we have the following example:

5.1 The figure on p. 336 is composed of the surface, A, of a wedge, with plane rectangles R_1, R_2, $\cdots$ and their interiors inserted so as to extend downward from the top into the interior of the wedge, in such a manner as to converge to the sharp edge E of the wedge. Similar rectangles S_i are inserted extending from the bottom up into the interior of the wedge, occurring alternately between the rectangles R_1, R_2, $\cdots$, and not meeting the latter. Let $M = A \cup \bigcup R_n \cup \bigcup S_i$, and let D denote the domain interior to M in 3-space. Then M is 0-lc, but D does not have property S_0.

Now what we should like to find is a set of properties which, for $n = 2$, give Moore's result exactly, as well as give an analogous result for the case of the general n. Suppose we contrast example 5.1 with the following example:



5.2 See figure above; this consists of the surface, M', of a hollow tube with one closed end, converging on a line segment E. It may be considered as obtained from a portion of the sin $1/x$ curve by the device of replacing the curve by a tube whose thickness approaches zero as x approaches zero. The set M' is not 0-lc, and if D' is the domain interior to M', D' does not have property S_0.

What is the essential difference between D' and the domain D of example 5.1? In each case, the domain oscillates as it converges to E along with the boundary. But in the second figure, it is noteworthy that the successive curved portions of the domain D' are well separated, whereas in the case of D they are separated only by pieces of 2-dimensional surfaces (the rectangular pieces R_n and S_i).

In order to get at the heart of the matter, we recall an old definition of connectedness due to Cantor:

Definition. A point set M in a metric space is called *connected in the sense of Cantor* if for every pair $x, y \in M$ and $\epsilon > 0$, there exists an ϵ-chain of points

of M from x to y; i.e., a finite sequence $x_1, \cdots, x_k \in M$ such that $x = x_1$, $y = x_k$, and $\rho(x_i, x_{i+1}) < \epsilon$ for $i = 1, \cdots, k - 1$.

In case M is imbedded in a compact metric space S, then for M to be connected in the sense of Cantor is equivalent to $p^0(\overline{M}) = 0$, as we show below:

5.3 LEMMA. *In order that a subset M of a compact metric space S should be connected in the sense of Cantor, it is necessary and sufficient that $\overline{M}$ be connected.*

To prove the necessity, we note that if $\overline{M} = A \cup B$ separate, and $a \in M \cap A$, $b \in M \cap B$, $\epsilon = \rho(A, B)$, then there exists no ϵ-chain of points of M from a to b. And to prove the sufficiency, given $a, b \in M$ and $\epsilon > 0$, let $\mathfrak{U}$ be a covering of S consisting of open sets of diameter $< \epsilon/2$ and (Corollary I 12.5) $U_1, \cdots, U_k \in \mathfrak{U} \cap \overline{M}$ constitute a simple chain from a to b. If $x_i \in U_i \cap M$, then the points $x_1, \cdots, x_k$ form an ϵ-chain from a to b.

5.4 COROLLARY. *In order that a subset M of a compact metric space should be connected in the sense of Cantor, it is necessary and sufficient that $p^0(\overline{M}) = 0$.*

[Theorem V 11.2.]

5.5 THEOREM. *In order that a subset M of a compact metric space S should be connected in the sense of Cantor, it is necessary and sufficient that every Čech 0-cycle on M bound on M.*

PROOF. If Z^0 is a Čech 0-cycle on M, and M is connected in the sense of Cantor, then for every fcos $\mathfrak{U}$ of S, there exists a chain $C^1(\mathfrak{U})$ on $\mathfrak{U} \wedge \overline{M}$ such that $\partial C^1(\mathfrak{U}) = Z^0(\mathfrak{U})$ by Corollary 5.4. But if $C^1(\mathfrak{U})$ is on $\overline{M}$, it must also be on M. And conversely, if every 0-cycle on M bounds on M, then every 0-cycle on $\overline{M}$ bounds on $M \subset \overline{M}$, so that $p^0(\overline{M}) = 0$, whence M is connected in the sense of Cantor by Corollary 5.4.

EXAMPLE. Those elements of the interval $0 \leqq x \leqq 1$ which are rational form a totally disconnected subset of the space of real numbers that is connected in the sense of Cantor.

The above considerations regarding metric spaces lead to the following definitions:

5.6 DEFINITION. A subset M of a compact space S will be called *Cantor-connected*, or simply *C-connected*, if every Čech 0-cycle of S on M bounds on M.

And we have:

5.7 LEMMA. *A necessary and sufficient condition that a subset M of a compact space S be C-connected is that $\overline{M}$ be a continuum.*

PROOF OF NECESSITY. If $\overline{M} = A \cup B$ separate, let $\mathfrak{U}$ be a fcos of S consisting of $S - A \cup B$ and of two disjoint open sets U, V containing A, B respectively [III 1.27]. Let $a \in A \cap M$, $b \in B \cap M$, and Z^0 a nontrivial 0-cycle on $a \cup b$. Then $Z^0(\mathfrak{U}) \not\sim 0$ on $\mathfrak{U} \wedge M$ and M is not C-connected.

Proof of sufficiency. Let $\overline{M}$ be a continuum. Then every 0-cycle of S on M is also on $\overline{M}$ and must bound on $\overline{M}$. A fortiori, it bounds on M.

5.8 Definition. A subset M of a compact space S will be said to have *weak property* S if for every fcos $\mathfrak{C}$ of S, M is the union of a finite number of C-connected sets of diameter $< \mathfrak{C}$.

One easily proves:

5.9 Theorem. *In order that a subset M of a compact space S should have weak property* S, *it is necessary and sufficient that $\overline{M}$ have property* S.

Note, in connection with the sufficiency proof, that by Theorem IV 4.5, if $\mathfrak{C}$ is a fcos of S, then $\overline{M} = \bigcup_{i=1}^{k} M_i$, where each M_i is a connected open subset of $\overline{M}$, and that then $\overline{M}_i \cap M$ is C-connected.

5.10 Corollary. *In order that a subset M of a compact space S should have weak property* S, *it is necessary and sufficient that $\overline{M}$ be* 0-*lc*.

[Cf. Corollary IV 3.9.]

5.11 Corollary. *If M is a* 0-*lc closed subset of a compact space S, then every subset dense in M has weak property* S.

Before proceeding to the general r-dimensional case, we prove the following theorem:

5.12 Theorem. *In order that the boundary, M, of a simply $(n - 1)$-connected domain D in an $M_{1,2}^n$, S, should be* 0-*lc, it is necessary and sufficient that D have weak property* S *as well as property* S_{n-2} *rel. bounding cycles.*

Proof of necessity. If M is 0-lc, then $\overline{D} = D \cup M$ is 0-lc, and D has weak property S by Corollary 5.11. By Corollary 2.8, $S - M$ has property S_{n-2} rel. bounding cycles and hence by Lemma 2.11, D has the same property.

Proof of sufficiency. If D has weak property S, then $\overline{D} = D \cup M$ is 0-lc by Corollary 5.10. Then $S - \overline{D}$ has property S_{n-2} rel. bounding cycles by Corollary 2.8, and hence $S - M = (S - \overline{D}) \cup D$ has the same property. That M is 0-lc follows from Corollary 2.8.

Remark. Example 5.1 is a good illustration of a case where Theorem 5.12 holds, for $n = 3$. The domain D' of Example 5.2 does not have weak property S; it does have property S_1 rel. bounding cycles, since its complement is 0-lc. In the case $n = 2$, for D to have property S_{n-2} rel. bounding cycles obviously implies that D has weak property S. That the converse fails to hold is shown by the following example: In the (x, y)-plane, let R denote the interior of the rectangle bounded by the coordinate axes and the lines $x = 2$, $y = 2$. For each positive integer n, let $A_n = \{(x, y) \mid (x = 1/n) \& (0 < y \leqq 1)\}$. Let $D = R - \bigcup A_n$. Then D has weak property S but not property S_0. However, since property S_0 for a domain implies weak property S, we have:

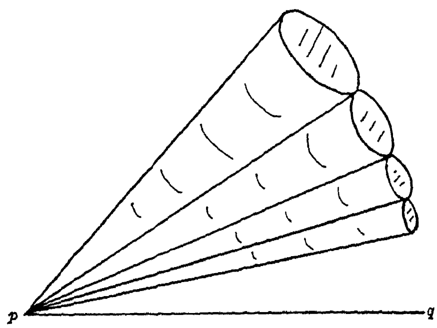
5.13 Corollary. *In order that the boundary, M, of a simply 1-connected domain D in a spherelike 2-gcm S should be 0-lc, it is necessary and sufficient that D have property* S_0 .

Also, one notes that the role of the simple $(n - 1)$-connectedness of D in Theorem 5.12 is only to make the boundary of D a continuum. However, one may assume, instead, that $p^{n-1}(D)$ is finite:

5.14 Lemma. *Let D be a domain in an $M^n_{0,1}$, S, and let M be the boundary of D. Then $p^{n-1}(S - M) = p^{n-1}(D)$.*

Proof. Since D is connected, $D \cup M = \overline{D}$ is connected. Hence by Theorem VIII 6.4, all $(n - 1)$-cycles in $S - \overline{D}$ bound in $S - \overline{D}$. Hence $p^{n-1}(S - M) = p^{n-1}(S - \overline{D}) + p^{n-1}(D) = p^{n-1}(D)$.

As a consequence we may prove, using Theorem 2.1 instead of Corollary 2.8 as in the proof of Theorem 5.12:



5.15 Theorem. *In order that the boundary of a domain D in an $M^n_{1,2}$, S, should be 0-lc, it is necessary and sufficient that $p^{n-1}(D)$ be finite and that D have weak property* S *as well as property* S_{n-2} *rel. bounding cycles.*

A nice application of Theorem 5.15 may be made to give the following theorem:

5.16 Theorem. *If M is a 0-lc subcontinuum of an $M^n_{1,2}$, S, and M is almost locally $(n - 2)$-avoidable rel. bounding cycles, then the boundaries of the domains complementary to M are all 0-lc.*

Proof. If D is a domain complementary to M, then D has property S_0 by Theorem 4.9, and a fortiori D has property S (Theorem VII 7.7) and hence weak property S. Also, since M is 0-lc, $S - M$ has property S_{n-2} rel. bounding cycles by Theorem 2.1, and hence D has the same property by Lemma 2.11. Since S has the Phragmen-Brouwer property, $F(D)$ is a continuum and therefore $p^{n-1}(D) = 0$. The theorem now follows from Theorem 5.15.

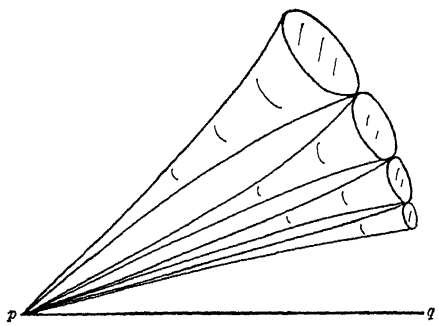
Remark. When S is the ordinary 2-sphere, Theorem 5.16 gives the Torhorst theorem (Corollary IV 6.5), since every Peano continuum is almost locally 0-

avoidable. Thus Theorem 5.16 may be considered as another generalization of this theorem.

Of greater importance, however, is the fact that Theorem 5.16 is another generalization of the theorem (Theorem 2.19) that the domains complementary to an lc^{n-2} continuum in a spherelike n-gcm S have boundaries that are 0-lc.

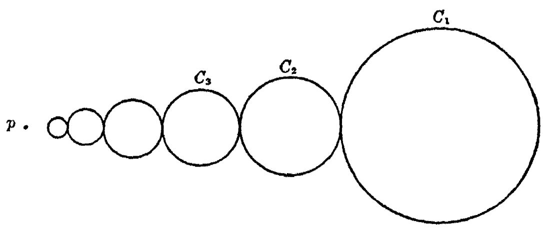
5.17 EXAMPLE. The figure on p. 340 may be called "the infinite Greek pipes." The conical "pipes", with closed ends, converge to the line segment pq, each two successive cones having in common a line element of each. The surfaces of the cones, together with pq, form a continuum M which is 0-lc and almost locally 1-avoidable. Hence, with M considered as a subset of S^3, Theorem 5.16 applies. However, M is not almost completely 1-avoidable and is therefore not 1-lc, and consequently Theorem 2.19 fails to apply.

Consider also the following example:



5.18 EXAMPLE. This is a modification of Example 5.17 (see figure above), where the cones have been separated along their original lines of contact, except at the extreme ends of these. Here Theorem 5.16 does not apply, since the continuum is not 0-lc, but inasmuch as the continuum is almost locally 0- and 1-avoidable, Theorem 4.11 applies to show that the domains interior to the cones must have Peano boundaries. (Note why the hypothesis of Theorem 4.11 fails to be fulfilled for the exterior domain!)

5.19 EXAMPLE. This consists of a denumerable set of circles C_n in the



3-sphere such that for each n, C_n is tangent to C_{n+1}, and the circles C_n converge to a point p. Here Theorem 5.16 applies, but Theorem 4.11 does not.

Now in order to define a weak property S_r for $r > 0$, we may go back to the original form of the definition of property S_r in terms of "pairs" (U, V), or

use the equivalent "$(P, Q)^{r}$" formulation. It will be more direct to use the latter form of the definition.

5.20 Definition. A subset M of a compact space S will be said to have *weak property* S_r—in symbols, WS_r —if for arbitrary open sets P and Q such that $P \supset Q$, at most a finite number of r-cycles (not necessarily compact) *on* $M \cap Q$ are lirh *on* $M \cap P$. The definition of "WS_r rel. G^r", where G^r is a special group of cycles, should be obvious. If a set has property WS_r for $r = 0, 1, \cdots, k$, we say it has property WS_0^k.

5.21 Theorem. *In order that a subset M of a compact space S should have property* WS_r *rel.* G^r, *it is necessary and sufficient that* $\overline{M}$ *have property* S_r *rel.* G^r.

Proof of necessity. Given open sets $P \supset Q$, let R be an open set such that $P \supset R \supset Q$. Since M has property WS_r rel. G^r, there exists an integer m such that every m cycles of G^r on $M \cap Q$ satisfy a homology on $M \cap R$. Then a set of m compact r-cycles of G^r on $\overline{M}$ in $\underline{Q}$, being also on $M \cap Q$, satisfy a homology on $M \cap R$ which is a fortiori on $\overline{M}$ in P.

Proof of sufficiency. With P, Q and R as before, if $\overline{M}$ has property S_r rel. G^r, there exists an integer m such that every m cycles of G^r on $\overline{M}$ in R satisfy a homology on $\overline{M}$ in P. Then any set of m cycles of G^r on $M \cap Q$ lie on $\overline{M}$ in R, hence satisfy a homology on $\overline{M}$ in P which is also a homology on $M \cap P$.

5.22 Corollary. *If a closed subset M of a compact space S has property* S_r, *then every set dense in M has property* WS_r.

5.23 Corollary. *A necessary and sufficient condition that a closed subset M of a compact space S be* lc^k *is that every set dense in M have property* WS_0^k.

[We recall that by Theorem VII 7.17, for M to be lc^k is equivalent to M having property S_0^k.]

5.24 Corollary. *For subsets of a compact space, weak property* S *and property* WS_0 *are equivalent.*

Proof. By Theorem 5.9, for a set M to have weak property S is equivalent to $\overline{M}$ having property S, which by Theorems VII 7.7 and VII 7.8 is equivalent to $\overline{M}$ having property S_0; the latter is by Theorem 5.21 equivalent to M having property WS_0.

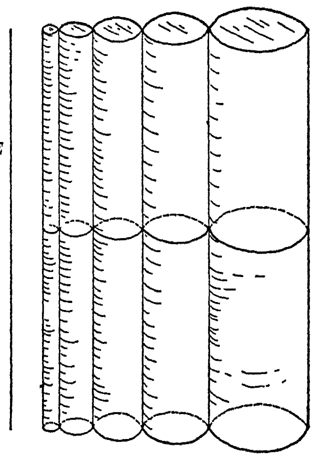
5.25 Theorem. *Property* S_0 *is stronger than property* WS_0. *For $r > 0$, properties* S_r *and* WS_r *are generally independent; for example, a euclidean domain may have either one of these properties and not the other. Indeed a domain in S^3 may have property* S_0^1 *(and consequently* WS_0*) and yet not have property* WS_0^1.

Proof. By Theorem VII 7.7, if a point set M has property S_0, then it has property S; hence it has weak property S and, by Corollary 5.24, has property WS_0. That the converse fails is shown by the following example: In the (x, y)-plane, for each positive integer n let $A_n = \{(x, y) \mid (x = 1/n) \& (0 < y \leqq 1)\}$;

$A = \{(x, y) \mid (x = 0)\&(0 < y \leqq 1)\}$; $B = \{(x, y) \mid (0 \leqq x \leqq 1)\&(y = 0)\}$. Let $M = A \cup B \cup \bigcup A_n$. As a configuration in S^2, the complement of M has property WS_0 but not property S_0.

Turning to the case $r > 0$, consider the point set M just defined above as a configuration in S^3. Then $S^3 - M$ does not have property S_1 (since M is not 0-lc), but does have WS_1 by Theorem 5.21 (the closure of $S^3 - M$ is S^3). Thus a euclidean domain may have property WS_1 and not S_1. On the other hand, consider the following example (see figure below): In S^3 let M be a continuum

E



consisting of a denumerable set of finite circular cylinders, closed at both ends and successively tangent along a common line element, converging to a line segment E. Here M is 0-lc and $p^1(S^3 - M) = 0$, so that $S^3 - M$ has property S_1. But the domain whose boundary is the complete set M does not have property WS_1; this follows from a theorem presently to be proved, but can also be seen from the presence of 1-cycles on circles which approximate the circles indicated in the figure half-way up the cylinders and which lie in the domain in question, except that they intersect M where the circles cut the tangent lines of the cylinders. This domain also has property S_0.

5.26 THEOREM. *In order that the boundary of a domain D in an $M^n_{1,k+2}$, S, should be lc^k, it is necessary and sufficient that* (1) *D have property* WS^k_0 *and* (2) *D have property* S^{n-1}_{n-k-1} *as well as property* S_{n-k-2} *rel. bounding cycles.*

PROOF OF SUFFICIENCY. By Corollary 5.23, $\overline{D}$ is lc^k. Hence by Theorem 2.3, $S - \overline{D}$ has property S^{n-1}_{n-k-1} as well as property S_{n-k-2} rel. bounding cycles. Since by (2) D has the same properties, it follows that $S - F(D)$ has these properties, and hence by Theorem 2.3 that $F(D)$ is lc^k.

REMARK. In view of Lemma 2.2, we may replace the condition that D have property S^{n-1}_{n-k-2} in Theorem 5.26 by the condition that $p^{n-1}(D)$ be finite and D have property S^{n-2}_{n-k-1}.

5.27 Corollary. *In order that a common boundary of (at least) two domains in an $M^n_{1,k+2}$, S, should be lc^k, it is necessary and sufficient that these domains have property* WS$^k_0$.

Proof of sufficiency. If D is one of the domains mentioned, then $\overline{D}$ is lc^k by Corollary 5.23 and hence $S - \overline{D}$ has property S^{n-1}_{n-k-1} as well as property S_{n-k-2} rel. bounding cycles by Theorem 2.3. By Lemma 2.11, if E is the other domain mentioned in the theorem, then E also has these S-properties, and since by hypothesis E also has property WS$^k_0$, its boundary is lc^k by Theorem 5.26.

From Corollaries 5.23 and 5.27 we also have:

5.28 Corollary. *In order that a common boundary of (at least) two domains A and B in an $M^n_{1,k+2}$, S, should be lc^k, it is necessary and sufficient that $\overline{A}$ and $\overline{B}$ should both be lc^k.*

5.29 Corollary. *If M is a closed, lc^k subset of an $M^n_{1,k+2}$, S, and D a domain complementary to M, then a necessary and sufficient condition that $F(D)$ be lc^k is that D have property* WS$^k_0$.

[The necessity is obvious and the sufficiency follows from Theorems 2.3 and 5.26.]

Analogous to Theorem 1.3, and proved similarly, we have:

5.30 Theorem. *In order that a set M should have property* WS$_r$, *it is necessary and sufficient that $p^r(\overline{M})$ be finite and that M have property* WS$_r$ *rel. bounding cycles.*

5.31 Corollary (of Theorems 5.26 and 5.30). *In order that the boundary of a domain D in an $M^n_{1,k+2}$, S, should be lc^k it is necessary and sufficient that* (1) *D have property* WS$^k_0$ *rel. bounding cycles and property* S^{n-2}_{n-k-2} *rel. bounding cycles, and that* (2) *the numbers $p^r(\overline{D})$, $r = 0, 1, \cdots, k$ and $p^s(D)$, $s = n - k - 1, \cdots, n - 1$ be finite.*

6. **Weak uniform local connectedness.** In a manner similar to that in which we introduced in §5 a "weak" type of property S_r, we may introduce a "weak" type of uniform local connectedness:

6.1 Definition. A subset M of a space S will be called *weakly r-ulc,* in symbols *r-wulc*, if for arbitrary fcos $\mathfrak{E}$ of S there exists a fcos $\mathfrak{D} > \mathfrak{E}$ such that every r-cycle (not necessarily compact) on M of diameter $< \mathfrak{D}$ bounds *on* a subset of M of diameter $< \mathfrak{E}$. By wulc$^k_0$ we denote property r-wulc for $r = 0, 1, \cdots, k$.

6.2 Theorem. *In order that a subset M of a compact space S should be r-wulc, it is necessary and sufficient that $\overline{M}$ be r-lc.*

6.3 Corollary. *In order that a closed subset M of a compact space S should be r-lc, it is necessary and sufficient that every set dense in M be r-wulc.*

6.4 Corollary. *For subsets of a compact space S, properties* WS_0^k *and* $wulc_0^k$ *are equivalent.*

In view of Corollary 6.4, wherever WS_0^k occurs in theorems concerning compact spaces above, we may replace it by wulc_0^k. For example, from Corollary 5.27 we have:

6.5 Theorem. *In order that a common boundary of (at least) two domains in* $M^n_{1,k+2}$, *S, should be* lc^k, *it is necessary and sufficient that these domains have property* $wulc_0^k$.

Remark. It was shown by R. L. Moore [g; Theorem 1] that in S^3 a common boundary of (exactly) two 0-ulc domains must be 0-lc, and later the present author showed [n; Theorem 2] that a common boundary of (at least) two ulc^k domains in S^n must be lc^k. These theorems are corollaries of the sufficiency part of Theorem 6.5 which, even in the euclidean case, is a stronger theorem. For consider the following theorem:

6.6 Theorem. *If U is a* ulc^k *open subset of an orientable n-gcm S, then U is* $wulc_0^k$. *The converse does not in general hold.*

Proof. If U is a ulc^k open subset of an orientable n-gcm, then $\overline{U}$ is lc^k by Theorem X 5.8. Hence by Theorem 6.2, U is wulc_0^k.

To see that the converse does not generally hold, we may easily construct in S^2 a domain that is not 0-ulc but that is 0-wulc; for instance, the example given in the proof of Theorem 5.25.

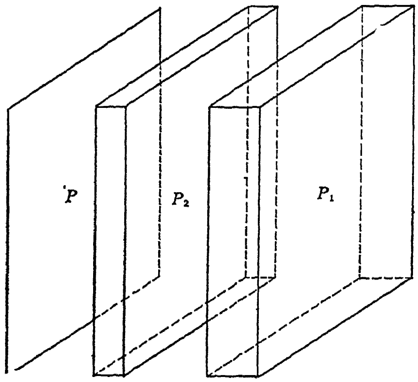
However, it is not difficult to give, in S^3, an example of *a continuum M whose complement is exactly two 0-wulc domains, neither of which is 0-ulc.* This may be done by a modification of the continuum in the figure accompanying the proof of Theorem 5.25; all that is necessary is to introduce a cone-shaped funnel half-way up the cyclinders with apex half-way up E, connecting the domains inside the cylinders; small pieces of the cylinders inside the cone have to be deleted from M in order to accomplish this.

Although, in view of Theorem 6.2 and its corollaries, it seems as though little is gained from introducing the "wulc" notion, it is of interest to note that for isolated values of $r > 0$ the r-wulc property is definitely weaker than property WS_r:

6.7 Theorem. *For subsets of a compact space S, property r-wulc is weaker than property* WS_r, *even in the case of the euclidean domains, with the exception that for subsets of a compact space, 0-wulc and* WS_0 *are equivalent.*

Proof. The exception has already been taken care of in Corollary 6.4. For $r > 0$, we first prove that property WS_r implies property r-wulc. By Theorem 5.21, if M has property WS_r, then $\overline{M}$ has property S_r; by Theorem VII 7.13, if $\overline{M}$ has property S_r, then $\overline{M}$ is r-lc; and by Theorem 6.2 above, if $\overline{M}$ is r-lc, then M is r-wulc. Hence if M has property WS_r, then M is r-wulc.

To see that, even in the case of a euclidean domain, property r-wulc does



not imply property WS_r, consider the following example (see figure above): In S^3, let M consist of a sequence of square plates P_n of thickness t_n, such that $\lim t_n = 0$, and which converge to a set P consisting of a square plus its interior. The domain D complementary to M is not WS_1, but is 1-wulc.

THEOREM 6.8. *If U is a ulc^k open subset of an orientable n-gcm S, then U has property S_0^k; the converse does not generally hold.*

PROOF. By Theorem X 5.8, $\overline{U}$ is lc^k. Let $P \supseteq Q$ be open subsets of S, and r an integer $\leqq k$. Then by Corollary VI 3.8, at most a finite number of r-cycles of $U \cap Q$ are lirh on $\overline{U} \cap P$. Let $Z_1^r, \cdots, Z_m^r$ be a finite set of cycles of $U \cap Q$ such that if Z^r is an arbitrary cycle of $U \cap Q$, there exists a homology

$$Z^r \sim \sum_{i=1}^{m} a^i Z_i^r \qquad \text{on } \overline{U} \cap P. \tag{6.8a}$$

Suppose Z^r fixed and relation (6.8a) determined. Let K be a compact subset of $U \cap Q$ carrying the cycle $\gamma^r = Z^r - \sum_{i=1}^m a^i Z_i^r$, and M a closed subset of $\overline{U} \cap P$ containing K and carrying the homology (6.8a). By Lemma VII 1.4, there exists a cycle γ^{r+1} mod K on M such that $\partial\gamma^{r+1} \sim \gamma^r$ on K, and by Lemma X 5.10, there exists in $U \cap P$ a compact set M' carrying a cycle Z^{r+1} mod K such that $Z^{r+1} \sim \gamma^{r+1}$ mod K in P. The latter homology implies (Lemma VII 1.2) that $\partial Z^{r+1} \sim \partial\gamma^{r+1}$ on K and hence $\partial Z^{r+1} \sim \gamma^r$ on K. Hence $\gamma^r \sim 0$ on $K \cup M' \subset U \cap P$. That is, relation (6.8a) holds in $U \cap P$.

7. Lc sets whose complementary domains are bounded by manifolds. Since a generalized manifold is merely a space having especially strong local connectedness properties, it is logical to inquire next into the conditions under which the boundaries of the complementary domains of a continuum are not merely locally connected, but are actually manifolds. In the study of plane point sets, this was a question of under what conditions a continuum, usually

peanian, had complementary domains whose boundaries were all simple closed curves.

We first prove another duality theorem of a type similar to those which were stated above for "in the large" situations (for example, 1.1, 1.7, 3.12), but more strictly analogous to Theorem 4.8.

7.1 THEOREM. *In order that a closed subset M of an $M^n_{r,r+2}$, S, should be locally r-avoidable rel. bounding cycles at $x \in M$, $0 \leqq r \leqq n-2$, it is necessary and sufficient that $q^{n-r-2}(S - M, x, \sim) = 0$.*

PROOF OF NECESSITY. Given $x \in M$ and an open set P containing x, there exist open sets Q and R such that $x \in R \Subset Q \Subset P$ and such that every r-cycle of $M \cap F(Q)$ bounding on M bounds on $M - R$. Let U be an open set such that $x \in U \Subset R$ and cycles of S in U bound in R. Suppose Z^{n-r-2} a cycle of $U - M$ that bounds in $S - M$ but fails to bound in $P - M$. Then by Theorem VIII 6.4, there exists a cycle γ^{r+1} on $M \cup (S - P)$ that is linked with Z^{n-r-2}. The portion of γ^{r+1} in Q is a cycle Z^{r+1} mod $F(Q)$ on $M \cap \bar{Q}$, whose boundary, ∂Z^{r+1}, bounds on a closed subset F of $M - R$. Let A be a closed subset of $S - R$ carrying $\gamma^{r+1} - Z^{r+1}$ and containing F, and B a closed subset of M carrying Z^{r+1} and containing F. By Lemma VII 1.14, there exist cycles Z_1^{r+1} and Z_2^{r+1} on A and B respectively, such that $Z_1^{r+1} + Z_2^{r+1} \sim \gamma^{r+1}$ on $A \cup B$. But Z^{n-r-2} bounds in $S - M$, hence in $S - B$; and also Z^{n-r-2} bounds in R, hence in $S - A$.

PROOF OF SUFFICIENCY. Let P be an open set containing x, and let Q and R be open sets such that $x \in R \Subset Q \Subset P$ and such that (1) r-cycles of S on $F(Q)$ bound on $S - R$ and (2) $(n - r - 2)$-cycles of $(S - M) \cap \bar{R}$ that bound in $S - M$ also bound in $Q - M$. Then if γ^r is a bounding cycle of M that lies on $F(Q)$, and fails to bound on $M - R$, there exists a cycle Z^{n-r-1} in $S - (M - R)$ that is linked with γ^r. The portion of Z^{n-r-1} in R is a cycle Z_1^{n-r-1} mod $F(R)$ whose boundary bounds in $S - M$. We leave the remainder of the proof to the reader.

7.2 THEOREM. *In order that a continuum M in a spherelike n-gcm S should have only complementary domains* (1) *whose boundaries are orientable $(n-1)$-gcms all but a finite number of which are simply r-connected for $r = 1, \cdots, n-2$, and* (2) *such that if $\mathfrak{E}$ is an arbitrary fcos of S, then at most a finite number of these domains are of diameter $> \mathfrak{E}$, it is necessary and sufficient that M be lc^{n-2} and locally $(n-2)$-avoidable rel. bounding cycles, and that $p_r(M, x) = 0$ for $r = 1, \cdots, n-2$, and no boundary of a domain complementary to M be n-dimensional.*

PROOF OF NECESSITY. By Theorem 6.8 and Theorem X 3.2, each domain complementary to M has property S_0^{n-2}. If $P \supset Q$ are open sets, then P and $S - \bar{Q}$ form a fcos of S and by (2) only a finite number of domains complementary to M that meet Q fail to lie in P. It follows easily that $S - M$ has property S_1^{n-2} rel. bounding cycles. By condition (2) of the hypothesis and

Lemma 2.12, $S - M$ has property S_0 rel. bounding cycles. By condition (1) of the hypothesis, $p^r(S - M)$ is finite for $r = 1, \cdots, n-2$, and hence by Theorem 1.3, $S - M$ has property S_1^{n-2}. It now follows from Corollary 2.7 that M is lc^{n-2}.

To show that $p_r(M, x) = 0$ for $r = 1, \cdots, n-2$, $x \in M$, let P be an open set containing x. Let Q and R be open sets such that $x \in R \Subset Q \Subset P$, and let $\mathfrak{C}$ denote the covering of S consisting of the two open sets P, $S - \bar{Q}$. By condition (2) of the hypothesis, only a finite number of domains complementary to M meet both R and $S - P$; denote these by D_i, $i = 1, \cdots, m$. Since each D_i is $(n - r - 1)$-ulc by Theorem X 3.2, there exists an open set R_1 such that $x \in R_1 \subset R$ and such that every $(n - r - 1)$-cycle of $R_1 \cap D_i$, $i = 1, \cdots, m$, bounds in $P \cap D_i$. As a consequence of condition (1), there also exists an open set R_2 such that $x \in R_2 \subset R_1$ and such that R_2 contains no nonbounding $(n - r - 1)$-cycles of $S - M$. Then any $(n - r - 1)$-cycle of $R_2 - M$ bounds in $P - M$. Hence we conclude that $q^{n-r-1}(S - M, x) = 0$ and therefore by Theorem X 1.5, $p_r(M, x) = 0$.

To show that M is locally $(n - 2)$-avoidable rel. bounding cycles, we show that $q^0(S - M, x, \sim) = 0$ and apply Theorem 7.1. We select Q and R as before, and select R_1 so that every 0-cycle of $R_1 \cap D_i$, $i = 1, \cdots, m$, bounds in $P \cap D_i$. Then if Z^0 is a bounding cycle of $R_1 - M$, Z^0 is the sum of cycles each of which either lies in an open set of type $R_1 \cap D_i$, or in a domain D which lies wholly in P; in either case $Z^0 \sim 0$ in $P - M$.

Proof of sufficiency. Since $p_r(M, x) = 0$ for $r = 1, \cdots, n-2$, it follows that $q^{n-r-1}(S - M, x) = 0$ by Theorem X 1.5. Hence by Corollary X 2.6, each domain complementary to M is $(n - r - 1)$-ulc. By Theorem 7.1, $q^0(S - M, x, \sim) = 0$ for all $x \in M$, hence if D is a domain complementary to M, $q^0(D, x) = 0$ for every $x \in F(D)$ and by Corollary X 2.6, D is 0-ulc. Thus every domain complementary to M is ulc^{n-2} and hence the boundary of every such domain is an $(n - 1)$-gcm by Theorem X 6.8. Since M is lc^{n-2}, $p^r(M) = p^{n-r-1}(S - M)$ is finite for $r = 1, \cdots, n-2$, and condition (1) follows. Condition (2) is a consequence of the lc^{n-2} property of M and Theorem 4.15.

Since by Lemma IX 3.4, if S is lc^{n-2} then the properties of being completely r-avoidable for $r = 0, 1, \cdots, n-3$ and of having $p_r(M, x) = 0$ for all x and $r = 1, \cdots, n-2$, are equivalent, we can state:

7.2a Theorem. *Theorem 7.2 remains true if the condition on $p_r(M, x)$ is replaced by the requirement that M be completely r-avoidable for $r = 0, 1, \cdots, n - 3$.*

7.3 Corollary. *In order that an lc^{n-2} continuum in a spherelike n-gcm should have only orientable $(n - 1)$-gcms as boundaries of all its complementary domains, it is necessary and sufficient that it be completely r-avoidable for $r = 0, 1, \cdots, n - 3$ and locally $(n - 2)$-avoidable rel. bounding cycles, and that no boundary of a complementary domain be n-dimensional.*

7.4 THEOREM. *Let M be a subcontinuum of a spherelike n-gcm S and D a domain complementary to M such that* (1) *if $n > 2$, then for some fcos $\mathfrak{E}$ of S all r-cycles of D of diameter $< \mathfrak{E}$ bound in D, $r = 1, 2, \cdots, n - 2$;* (2) *M is locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n - 2$. Then if the boundary of D is not n-dimensional, it is an orientable $(n - 1)$-gcm.*

PROOF. By Theorem 7.1, $q^r(S - M, x, \sim) = 0$ for $r = 0, 1, \cdots, n - 2$. Since by (1) "small" r-cycles of D bound in D, it follows that $q^r(D, x) = 0$ for all $x \in F(D)$. Then by Corollary X 2.6, D is ulc^{n-2} and by Theorem X 6.8, $F(D)$ is an orientable $(n - 1)$-gcm.

The following corollaries—special cases—of Theorem 7.4 are of interest:

7.5 COROLLARY. *In S^2, if M is a continuum all of whose points are locally 0-avoidable, then the boundaries of the complementary domains of M are simple closed curves.*

7.6 COROLLARY. *In S^3, if M is a continuum all of whose points are locally 0- and 1-avoidable rel. bounding cycles, and D is a complementary domain of M whose "small" 1-cycles bound in D, then the boundary of D is a 2-dimensional closed manifold.*

The following theorem is of interest in comparison with Theorem 4.10 and the theorems immediately following it:

7.7 THEOREM. *In order that the boundary B of a simply $(n - 1)$-connected domain D in a spherelike n-gcm should be an orientable $(n - 1)$-gcm, it is necessary and sufficient that* (1) *B be locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n - 2$,* (2) *for some fcos $\mathfrak{E}$ of S, all r-cycles of D of diameter $< \mathfrak{E}$ bound in D for $r = 1, \cdots, n - 2$, and* (3) *B not be n-dimensional.*

PROOF. The necessity follows readily from the properties of an orientable $(n - 1)$-gcm stated in Corollary IX 2.2 and Lemma IX 3.1, and from Corollary X 1.8. As for the sufficiency, condition (1) implies by Theorem 7.1 that $q^{n-r-2}(D, x, \sim) = 0$ for all $x \in B$, which together with condition (2) implies $q^{n-r-2}(D, x) = 0$. Then B is an $(n - 1)$-gcm by Corollary X 2.6 and Theorem X 6.8.

As a corollary of Theorem 2.18 and 7.4 we have:

7.8 THEOREM. *If a subcontinuum M of a spherelike n-gcm S is lc^{n-2} and locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n - 2$, then all but a finite number of the domains complementary to M are bounded by orientable $(n - 1)$-gcm's that are simply-connected in all dimensions $< n - 1$, provided these boundaries are not n-dimensional.*

7.9 THEOREM. *In order that a simply r-connected, $r = 0, 1, \cdots, n - 2$, closed subset M of a spherelike n-gcm S should have only simply r-connected orientable $(n - 1)$-gcm's as boundaries of its complementary domains, it is sufficient that M be locally r-avoidable rel. bounding cycles and that these boundaries not be n-dimensional.*

Proof. That the domains complementary to M are bounded by orientable $(n-1)$-gcm's is a corollary of Theorem 7.4, and that these domains are simply connected in the dimensions stated is a consequence of Corollary VIII 8.6.

7.10 Corollary. *In S^3, if M is a simply 1-connected continuum which is locally r-avoidable rel. bounding cycles for $r = 0, 1$, then the domains complementary to M are all bounded by 2-spheres.*

7.11 Theorem. *In order that a simply r-connected, $r = 0, 1, \cdots, n-2$, lc^{n-2} closed subset M of a spherelike n-gcm S should have only simply r-connected orientable $(n-1)$-gcm's as boundaries of its complementary domains it is necessary and sufficient that all points of M be non-r-cut points of M and that these boundaries not be n-dimensional.*

Proof of necessity. Let $p \in M$ and Z^r a cycle on a compact subset of $M - p$, $0 \leqq r \leqq n-2$. By Corollary 7.3, M is locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n-2$, and hence by Theorem 7.1, $q^{n-r-2}(S - M, x, \sim) = 0$ for all $x \in M$. Let P, Q and R be open sets such that $x \in R \subset Q \subset P$ and such that (1) Z^r is on $S - \bar{P}$, (2) cycles of S on Q bound in P, (3) bounding cycles of $S - M$ on $\bar{R}$ bound in $Q - M$. If $Z^r \nsim 0$ on $M - R$, then there exists by Corollary VIII 8.6 a cycle $\underline{\gamma}^{n-r-1}$ in $S - (M - R)$ that is linked with Z^r. The portion of γ^{n-r-1} on $S - \bar{R}$ is a cycle γ_1^{n-r-1} mod $F(R)$ with boundary Z^{n-r-2} on $F(R)$. By the selection of R, $Z^{n-r-2} \sim 0$ on a closed subset H of $Q - M$. Let A be a closed subset of $S - M$ that carries γ_1^{n-r-1} and contains H; and let B be a closed subset of Q that carries $\gamma^{n-r-1} - \gamma_1^{n-r-1}$ and contains H. By Lemma VII 1.14, there exist cycles Z_1^{n-r-1} and Z_2^{n-r-1} on A and B respectively such that $\gamma^{n-r-1} \sim Z_1^{n-r-1} + Z_2^{n-r-1}$ on $A \cup B$. But Z_2^{n-r-1} lies in Q and hence bounds in P. It follows that Z^r and Z_1^{n-r-1} are linked—which is impossible since $Z^r \sim 0$ on M and Z_1^{n-r-1} is on $A \subset S - M$.

Proof of sufficiency. As M is lc^{n-2} and simply r-connected for $r = 0, \cdots, n-2$, it follows that a non-r-cut point is a locally r-avoidable point. Therefore Theorem 7.9 applies.

7.12 Corollary. *In order that a Peano continuum M in S^2 should have only simple closed curves as boundaries of its complementary domains, it is necessary and sufficient that M have no cut points.*

[We recall that by Corollary VII 6.6, non-cut points and non-0-cut points are identical in such a set M.]

7.13 Corollary. *In order that a simply 1-connected, lc^1 subcontinuum of S^3 should have only 2-spheres as boundaries of its complementary domains, it is necessary and sufficient that it have no cut points and no 1-cut points.*

An interesting consequence of Theorems 4.15 and 7.8 is as follows:

7.14 Theorem. *If a subcontinuum M of a spherelike n-gcm is locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n-2$, $p^r(M)$ is finite for $r = 1$,*

$\cdots$, $n-2$, and for arbitrary fcos $\mathfrak{E}$ of S at most a finite number of the domains complementary to M are of diameter $> \mathfrak{E}$, then M is lc^{n-2}, and all but a finite number of the complementary domains of M are bounded by orientable $(n-1)$-gcms that are simply connected in all dimensions $< n-1$ (provided these boundaries are not n-dimensional).

7.15 Corollary. *If the complementary domains of a locally 0-avoidable subcontinuum M of S^2 have diameters that form a null sequence, then M is peanian and almost all the complementary domains of M are bounded by simple closed curves.*

And from Theorem 4.15 and 7.9 we get,

7.16 Theorem. *In order that a simply r-connected $(r = 0, 1, \cdots, n-2)$, closed subset M of a spherelike n-gcm S should be lc^{n-2} and have only simply r-connected orientable $(n-1)$-gcm's as boundaries of its complementary domains, it is necessary and sufficient that for arbitrary fcos $\mathfrak{E}$ of S only a finite number of these domains be of diameter $> \mathfrak{E}$, their boundaries not be n-dimensional, and that M be locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n-2$.*

7.17 Corollary. *In order that a subcontinuum M of S^2 should be a Peano continuum all of whose complementary domains are bounded by simple closed curves, it is necessary and sufficient that M be locally 0-avoidable and that the diameters of the complementary domains of M form a null sequence.*

Remark. The reader may consider why, in S^2, a continuum M, that consists of two tangent circles, fails to satisfy the conclusion of Theorem 7.17.

From Theorems 4.18 and 7.8 we have:

7.18 Theorem. *If (1) a subcontinuum M of a spherelike n-gcm S is locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n-2$, (2) for arbitrary fcos $\mathfrak{E}$ of S at most a finite number of the domains complementary to M are of diameter $> \mathfrak{E}$, and (3) there exists a fcos $\mathfrak{U}$ of S such that for $r = 1, \cdots, n-2$, at most a finite number of the r-cycles of $S - M$ of diameter $< \mathfrak{U}$ are lirh in $S - M$; then M is lc^{n-2} and all but a finite number of its complementary domains are bounded by orientable $(n-1)$-gcm's that are simply connected in all dimensions $< n-1$ (provided these boundaries are not n-dimensional).*

7.19 Theorem. *In an $M^n_{1,n-k-1}$, S, let M be a continuum and D a domain complementary to M such that (1) D is ulc^k, $k \leqq n-3$; (2) there exists a fcos $\mathfrak{E}$ of S such that r-cycles of D of diameter $< \mathfrak{E}$ bound in D, $r = k+1, \cdots, n-2$; (3) M is locally r-avoidable rel. bounding cycles for $r = 0, 1, \cdots, n-k-3$. Then if $F(D)$ is not n-dimensional, it is an orientable $(n-1)$-gcm.*

Proof. By Theorem 7.1, $q^s(S - M, x, \sim) = 0$ for $s = k+1, \cdots, n-2$. Hence, by (2), $q^s(D, x) = 0$ for all $x \in F(D)$. It follows from Corollary X 2.6 that D is s-ulc. Consequently D is ulc^{n-2} and the theorem follows from Theorem X 6.8.

7.20 THEOREM. *In a spherelike n-gcm S, let M be a common boundary of (at least) two domains* D_1 *and* D_2 *such that* (1) D_i *is ulc*^{n_i}, $i = 1, 2$, *where* $n_1 + n_2 < n - 3$; (2) *there exists a fcos* $\mathfrak{C}$ *of S such that r-cycles of* D_1 *of diameter* $< \mathfrak{C}$ *bound in* D_1, $r = n_1 + 1, \cdots, n - n_2 - 2$; (3) *M is locally r-avoidable rel. bounding cycles for* $r = n_2 + 1, \cdots, n - n_1 - 3$. *Then M, if not n-dimensional, is an orientable* $(n - 1)$*-gcm.*

PROOF. Applying Theorem 7.1 as in the proof of Theorem 7.19, we may show that D_1 is $\mathrm{ulc}_{n_1+1}^{n-n_2-3}$ and hence ulc^{n-n_2-3}. The theorem then follows from Theorem X 7.2.

BIBLIOGRAPHICAL COMMENT

§1. Theorem 1.1 and complementary material for S^n were abstracted in Wilder [A_9].

§3. For the euclidean case, cf. Wilder [A_4].

§4. For the euclidean case, cf. Wilder [A_{10}].

§§5, 6. For the euclidean case, cf. Wilder [A_{11}].

§7. Some of the theorems will be found, for the euclidean case, in Wilder [p].

CHAPTER XII

ACCESSIBILITY AND ITS APPLICATIONS

An important segment of the literature on positional invariants, beginning with Schoenflies, is concerned with accessibility. The original form of the concept was given in II 5.36; in a separable space S, a point x was said to be accessible from a point set M if for each $y \in M$ there is an arc A which has x and y as end points, and such that $A - x \subset M$. Schoenflies noticed that in S^2 the simple closed curve is characterized by the fact that, in addition to satisfying the Jordan Curve Theorem, all its points are accessible from each of the two complementary domains. (See Theorem II 5.38.) He later modified the notion ("all-sided accessibility") in order to characterize the subcontinua of S^2 that are peanian (cf. IV 7).

1. **Regular r-accessibility.** Further modifications and new types of accessibility were introduced later by other authors. For example, in the above definition one may replace "arc" by "continuum", thus making the definition applicable to nonseparable spaces. (For most of the usual metric cases, the definition thus modified is equivalent to the old; thus if a point x of the boundary of a domain D in S^n is accessible from D in the new sense, it is accessible in the old sense.) Of special interest for our purposes is regular accessibility, due to G. T. Whyburn (cf. VII 7); in a metric space, a point x is *regularly accessible* from a point set M if for arbitrary positive number ϵ there exists a positive number δ such that if $y \in M \cap S(x, \delta)$, then there exists a continuum[1] K such that $x \cup y \subset K \subset [M \cap S(x, \epsilon)] \cup x$. Whyburn showed, in particular, that for a domain D in S^2, the Schoenflies all-sided accessibility of all points of $F(D)$ from D is equivalent to the regular accessibility of these points from D; moreover, these properties are in turn equivalent to D having property S, as well as to $F(D)$ being 0-lc [Wh; 112, Theorem (4.2)]. He also noticed [b; 510, Corollary] that in order that a boundary point x of a domain D in S^n should be regularly accessible from D, it is necessary and sufficient that $D \cup x$ be lc. Subsequently, P. Alexandroff [f; 14, Theorem 1] gave an equivalent property which, though stated by him for subsets of S^n, may be formulated as follows:

1.1 Theorem. *In order that a boundary point x, of countable character* [III 1.18], *of a connected open subset D of a locally compact 0-lc space should be regularly accessible from D, it is necessary and sufficient that for arbitrary open set P*

---

[1]By using "continuum" instead of "arc", the definition may be extended to nonmetric spaces in obvious manner.

containing x there exist an open set Q such that $x \in Q \subset P$ and such that every compact 0-cycle on $D \cap Q$ bounds on a compact subset of $(D \cap P) \cup x$.

Proof of necessity. Given P, let $Q \subset P$ be such that every point of $D \cap Q$ lies with x on some subcontinuum of $(D \cap P) \cup x$. Now suppose Z^0 is a 0-cycle on a compact subset K of $D \cap Q$. By Corollary IV 3.4, K may be assumed to have only a finite number of components K_i, $i = 1, \cdots, m$. For each i, let C_i be a subcontinuum of $(D \cap P) \cup x$ containing $x \cup x_i$ for some $x_i \in K_i$. Then $C = \bigcup C_i \cup \bigcup K_i$ is a subcontinuum of $(D \cap P) \cup x$ carrying Z^0 and therefore $Z^0 \sim 0$ in $(D \cap P) \cup x$.

Proof of sufficiency. Since x is of countable character, there exists a sequence of open sets P_i such that P_1 is any preassigned open set containing x, $P_i \supset P_{i+1}$ for all i, constituting a complete neighborhood system for x and such that if Z^0 is a compact cycle of $D \cap P_{i+1}$, then $Z^0 \sim 0$ on a compact subset of $(D \cap P_i) \cup x$. Let $x_i \in D \cap P_i$, $i > 1$. For each $i > 1$, there exists (by application of Theorems V 11.5, V 11.6 and IV 1.1) a continuum C_i that lies in $(D \cap P_{i-1}) \cup x$ and contains $x_i \cup x_{i+1}$. Then $C = \bigcup_{i=2}^{\infty} C_i$ is a subcontinuum of $(D \cap P_1) \cup x$ that contains $x_2 \cup x$.

The theorem of Whyburn mentioned above may be formulated here as follows:

1.2 Theorem. *In order that a boundary point x, of countable character, of a connected open subset D of a locally compact, 0-lc space should be regularly accessible from D, it is necessary and sufficient that $D \cup x$ be 0-lc in the sense of compact cycles.*

Proof. The sufficiency follows from the sufficiency of Theorem 1.1. To prove the necessity, let P_0 be a neighborhood of x and let P_i, $i = 1, 2, \cdots$, be a sequence of neighborhoods of x forming a complete system for x, such that $P_i \supset P_{i+1}$ and such that if $y \in P_{i+1}$ then there exists a subcontinuum of $(D \cap P_i) \cup x$ containing $x \cup y$. Let Z^0 be a cycle carried by a compact subset K of $(D \cap P_1) \cup x$. If $x \notin K$, the proof given for the necessity in Theorem 1.1 still holds. Otherwise, notice that for each $i > 0$, by Corollary IV 3.4, the closed set $K \cap (\overline{P}_i - P_{i+1})$ is contained in a finite collection C_i of subcontinua of $D \cap P_{i-1}$, and using the argument employed in the necessity proof of Theorem 1.1, the collections C_i all lie in one subcontinuum K_i of $(D \cap P_{i-1}) \cup x$. Then $F = \bigcup K_i$ is a subcontinuum of $(D \cap P_0) \cup x$ containing Z^0.

The equivalence proved in Theorem 1.1 led Alexandroff to a definition which may be phrased as follows:

1.3 Definition. A subset M of a space S is called *regularly r-accessible* at $x \in M$ if for arbitrary open set P containing x there exists an open set Q such that $x \in Q \subset P$ and such that every compact cycle Z^r of $Q - M$ bounds on a compact subset of $(P - M) \cup x$. If M is regularly r-accessible at all of its points, we say simply that M is regularly r-accessible. (Alexandroff used the term "r-accessible" instead of "regularly r-accessible" as we are doing.

We prefer to reserve the former term for a stronger type of accessibility to be considered later.)

If D is an open subset of S and $x \in F(D)$, then x is called *regularly r-accessible from D* if $F(D)$ is regularly r-accessible at x in the space $\bar{D}$. If all points of $F(D)$ are regularly r-accessible from D, we say simply that $F(D)$ is regularly r-accessible from D. Obviously if D is "r-ulc at x," i.e., if $q^r(D, x) = 0$, then x is regularly r-accessible from D.

By Theorem 1.1, evidently regular accessibility and regular 0-accessibility are equivalent for the boundaries of open subsets of the 0-lc, locally compact spaces—hence, in particular, in the generalized manifolds.

1.4 LEMMA.[2] *A necessary and sufficient condition that a closed subset M of a compact space S of countable character should be regularly r-accessible at $x \in M$ from a set $D \subset S - M$ is that for arbitrary open set P containing x there exist an open set Q such that $x \in Q \subset P$ and such that for every cycle Z^r in $D \cap Q$ and every open set R such that $x \in R \subset Q$, there exists a cycle γ^r in $D \cap R$ such that* $Z^r \sim \gamma^r$ in $D \cap P$.

PROOF OF NECESSITY. If x is regularly r-accessible from D, then for arbitrary P there exists Q such that if Z^r is in $D \cap Q$, then $Z^r \sim 0$ on a compact subset F of $(D \cap P) \cup x$. Consider any open sets R and R' such that $x \in R' \Subset R \subset Q$. By LemmaVII 1.13, there exists a cycle γ^r on $F \cap F(R)$ such that $Z^r \sim \gamma^r$ on $F - R'$.

PROOF OF SUFFICIENCY. Since S is of countable character, there exists a countable collection of open sets P_i, $i = 1, 2, 3, \cdots$, closing down on x such that P_1 is a given arbitrary open set and for every i, P_i and P_{i+1} may play the part of the P and Q of the hypothesis. Consider a compact cycle Z^r in $D \cap P_2$. There exists a compact cycle Z_1^r in $D \cap P_3$ and a compact set F_1 in $D \cap P_1$ such that $Z^r \sim Z_1^r$ on F_1. Again, there exists a compact cycle Z_2^r in $D \cap P_4$ and a closed set F_2 in $D \cap P_2$ such that $Z_1^r \sim Z_2^r$ on F_2. Evidently $Z^r \sim Z_2^r$ on $F_1 \cup F_2$. Continuing in this manner there is established the existence of a closed subset $F = \bigcup_{i=1}^{\infty} F_i$ of $(D \cap P_1) \cup x$ and a sequence Z_i^r, where Z_i^r is a compact cycle of $D \cap P_{i+2}$ such that for any $i = k$, $Z^r \sim Z_k^r$ on $\bigcup_{i=1}^{k} F_i \subset D \cap P_1$.

We assert that $Z^r \sim 0$ on F. For suppose not. Then there exists a covering $\mathfrak{U}$ of S such that $Z^r(\mathfrak{U}) \not\sim 0$ on F. Let U_j, $j = 1, \cdots, m$, denote the elements of $\mathfrak{U}$ that contain x, and let k be an integer such that $P_{k+2} \subset \bigcap_{j=1}^{m} U_j$. Now $Z^r \sim Z_k^r$ on $\bigcup_{i=1}^{k} F_i$. But since Z_k^r is in P_{k+2}, $Z_k^r(\mathfrak{U})$ is a cycle on the simplex E^{m-1} whose vertices are $U_1, \cdots, U_m$, and consequently by Corollary V 6.2, $Z_k^r(\mathfrak{U}) \sim 0$ on E^{m-1}—i.e., $Z_k^r(\mathfrak{U}) \sim 0$ on x. It follows that $Z^r(\mathfrak{U}) \sim 0$ on F, contrary to our original supposition.

In order to establish an internal condition for regular r-accessibility, Alex-

---

[2] Compare Alexandroff, loc. cit., p. 15.

androff introduced the following concept. (Loc. cit. Definition 5, p. 16; as usual, we have rephrased the definition in nonmetric terms.)

1.5 DEFINITION. A set M will be said to *have no r-dimensional condensation* at $x \in M$ if for arbitrary open set P containing x there exists an open set Q such that $x \in Q \subset P$ with the property that if Z^r is a cycle mod $M - P$ on M which is not ~ 0 mod $M - Q$ on M, then for every open set R such that $x \in R \subset Q$, $Z^r \nsim 0$ mod $M - R$ on M.

An equivalent definition, which throws further light (see also Theorem 1.11 below) on the inherent meaning of the concept just defined, is embodied in the following lemma:

1.6 LEMMA. *In order that a locally compact space S should have no r-dimensional condensation at $x \in S$, it is necessary and sufficient that if P is an arbitrary open set containing x, there exist an open set Q such that $x \in Q \subset P$ and such that every cycle γ^r mod $S - P$ which is carried by a compact subset of $S - x$ is ~ 0 mod $S - Q$.*

PROOF OF NECESSITY. With P and Q as in Definition 1.5, let γ^r be a cycle mod S − P carried on a compact subset F of $S - x$. There exists an open set R such that $x \in R \subset Q$ and such that $R \cap F = 0$, and hence $\gamma^r \sim 0$ mod $S - R$. But this must imply, because of the way in which Q was defined, that $\gamma^r \sim 0$ mod $S - Q$.

PROOF OF SUFFICIENCY. With P and Q as in the statement of the lemma, let γ^r be a cycle mod $S - P$. If γ^r has a compact carrier F such that $x \notin F$, then $\gamma^r \sim 0$ mod $S - Q$ and such cycles do not have to be considered. And if $\gamma^r \nsim 0$ mod $S - R$ for all open sets R such that $x \in R \subset Q$, the requirement for absence of r-dimensional condensation is satisfied. Suppose $\gamma^r \nsim 0$ mod $S - Q$ and that $\gamma^r \sim 0$ mod $S - R$ for some R. Then by Lemma VII 1.9, there exists a cycle γ_1^r mod $S - P$ on $S - R$ such that $\gamma_1^r \sim \gamma^r$ mod $S - P$. But by the choice of Q, $\gamma_1^r \sim 0$ mod $S - Q$, and hence $\gamma^r \sim 0$ mod $S - Q$, contradicting the properties of γ^r.

1.7 EXAMPLE. In the (x, y)-plane, for each natural number n let $M_n = \{(x, y) \mid (x - 1/2^n)^2 + y^2 = 1/4^n\}$, and $M = \bigcup M_n$. With $x = (0, 0)$, M has no 1-dimensional condensation at x (Alexandroff, loc. cit. p. 24, no. 7).

1.8 EXAMPLE. In the (x, y)-plane, let $M_n = \{(x, y) \mid (x = 1/n)\&(-1 \leqq y \leqq 1)\}$, $M_0 = \{(x, y) \mid (x = 0)\&(-1 \leqq y \leqq 1)\}$, and $M = \bigcup_{n=0}^{\infty} M_n$. Then M has 1-dimensional condensation at (0, 0).

1.9 THEOREM. *In order that a closed subset M of an $M^n_{r,r+1}$, S, should be regularly r-accessible, $r \leqq n - 1$, at $x \in M$, it is necessary and sufficient that M have no $(n - r - 1)$-dimensional condensation at x.*

(For the euclidean case, this was proved by Alexandroff, loc. cit. p. 16, Theorem II.)

The case $r = n - 1$ will be left to the reader; the case $r \leqq n - 2$ may be handled as follows:

PROOF OF NECESSITY. Suppose that M is regularly r-accessible at $x \in M$. Let us denote the number $n - r - 1$ by s. Let P_1 be an arbitrary neighborhood of x, and let P be an open set such that $x \in P \subset P_1$ and such that r-cycles of $S - P_1$ bound on $S - P$ (this is possible since S is simply r-connected, and by virtue of Lemma IX 3.1). Since, by Lemma IX 3.1, S is completely $(s - 1)$-avoidable, there exist open sets Q and R such that $x \in R \subset Q \subset P$ and such that $(s - 1)$-cycles on $F(Q)$ bound on $\overline{P} - R$. Since M is regularly r-accessible at x, there exists by Lemma 1.4 an open set R_1 such that $x \in R_1 \subset R$ and such that if Z^r is a cycle of $R_1 - M$, then for all open sets R' such that $x \in R' \subset R_1$, there exists a cycle γ^r of $R' - M$ such that $Z^r \sim \gamma^r$ on a compact subset of $R - M$.

Let Z^s be a cycle mod $S - P_1$ on M which does not bound mod $S - R_1$ on M. The portion of Z^s in Q is a cycle Z_1^s mod $F(Q)$ such that $\partial Z_1^s \sim 0$ on $\overline{P} - R$. Denote $(\overline{P} - R) \cup (M \cap R)$ by M' and let M'' denote the set $(\overline{P}_1 - R_1) \cup (M \cap R_1)$; note that $M'' \supset M'$. Then by Lemma VII 1.6 there exists a cycle γ^s on M' such that $\gamma^s \sim Z^s$ mod $S - R$ on M and on M'. Then $\gamma^s \nsim 0$ mod $S - R_1$ on M', and in particular, then, $\gamma^s \nsim 0$ on M''. By Corollary VIII 8.6, there exists in $S - M''$ a cycle Z^r linked with γ^s. We may write $Z^r = Z_1^r + Z_2^r$ where Z_1^r is in $R_1 - M$ and Z_2^r is in $S - \overline{P}_1$.

Suppose there exists an open set R' such that $x \in R' \subset R_1$ and such that $Z^s \sim 0$ mod $S - R'$ on M. Then $\gamma^s \sim 0$ mod $S - R'$ on M'. This implies by Lemma VII 1.9 that there exists a cycle γ_1^s on $M' - R'$ such that $\gamma_1^s \sim \gamma^s$ on M', and a fortiori on M''. Then γ_1^s and Z^r are linked. Now let R'' be an open set such that $x \in R'' \subset R'$ and such that r-cycles of S in R'' bound in R'. By the choice of R_1, there exists in $R'' - M$ a cycle γ^r such that $Z_1^r \sim \gamma^r$ in $R - M$, and since $\gamma^r \sim 0$ in R' it follows that $Z_1^r \sim 0$ in $S - F$, where F is a carrier of γ^s on M'. Also, by the choice of P, $Z_2^r \sim 0$ on $S - P$ and hence in $S - F$. It follows that $Z^r \sim 0$ in $S - F$, contradicting the fact that γ^s and Z^r are linked.

PROOF OF SUFFICIENCY. Let M have no s-dimensional condensation at x. Then for arbitrary open set P containing x, there exists an open set Q as in Definition 1.5 (with s replacing r). Let Q' be an open set such that $x \in Q' \subset Q$ and such that s-cycles on $S - Q$ bound on $S - Q'$. Suppose there exists an open set R such that $x \in R \subset Q'$, and a cycle Z^r on a compact subset K of $Q' - M$ such that Z^r is not homologous on a closed subset of $P - M$ to any compact cycle of $R - M$. We shall show this is impossible.

Let R' be any open set such that $x \in R' \Subset R$, and let $M' = M \cap \overline{P} - R'$. Then $Z^r \nsim 0$ in $P - M'$. For if F were a closed subset of $P - M'$ containing K such that $Z^r \sim 0$ on F, there would exist by Lemma VII 1.4 a cycle C^{r+1} mod K on F such that $\partial C^{r+1} \sim Z^r$ on K. The boundary of the portion of C^{r+1} in R' would be a compact cycle of $R - M$ to which Z^r would be homologous

on $F - R' \subset P - M$. It follows then that Z^r must be linked with a cycle γ^s of $M' \cup F(P)$.

Now the portion of γ^s on M is a cycle Z^s mod $F(P)$ on $M \cap \overline{P}$. If $Z^s \sim 0$ mod $M - Q$ on M, then there exists a cycle Z_1^s mod $F(P)$ on $M \cap \overline{P} - Q$ such that $Z^s \sim Z_1^s$ mod $F(P)$ on $M \cap \overline{P}$. But this would imply the existence of a cycle γ_1^s on $(M - Q) \cup F(P)$ such that $\gamma^s \sim \gamma_1^s$ on $(M \cap \overline{P}) \cup F(P)$, and since by the choice of Q, $\gamma_1^s \sim 0$ on $S - Q'$, it would follow that $\gamma^s \sim 0$ in $S - K$. This contradicts the fact that γ^s and Z^r are linked. We must conclude, then, that $Z^s \nsim 0$ mod $M - Q$ on M, and that therefore, by the choice of Q, $Z^s \nsim 0$ mod $M - R'$ on M. But Z^s is on $M - R'$ so that $Z^s \sim 0$ mod $M - R'$.

1.10 Corollary. *The property of being regularly r-accessible at a certain point is topologically invariant for closed subsets of an* $M^n_{r,r+1}$.

1.11 Theorem. *If a locally compact space S has r-dimensional condensation at $x \in S$, x of countable character, then there exist an open set P containing x and a sequence Z_i^r, $i = 1, 2, 3, \cdots$, of compact cycles mod $S - P$ on $S - x$ such that for each open set Q such that $x \in Q \subset P$, there exists an integer $k(Q)$ such that for $i > k(Q)$, the cycles Z_i^r are lirh mod $S - Q$ on S.*

Proof. By Lemma 1.6, there exists an open set P containing x such that for every open set Q such that $x \in Q \subset P$, there exists a compact cycle Z^r mod $S - P$ of $S - x$ such that $\gamma^r \nsim 0$ mod $S - Q$. Let Q_1 be such a set Q and Z_1^r such a cycle γ^r. Since Z_1^r is on a closed subset F_1 of $S - x$, there exists an open set Q_2 such that $x \in Q_2 \subset Q_1$ and $Q_2 \cap F_1 = 0$. Then there exists a compact cycle Z_2^r mod $S - P$ of $S - x$ such that $Z_2^r \nsim 0$ mod $S - Q_2$. We continue in this manner, using, however, a collection of open sets Q_i forming a complete neighborhood system for x. Then if Q is given as in the statement of the theorem, there exists an integer k such that $Q_k \subset Q$. Suppose there exists a relation

$$\sum_{j=1}^{m} a^{i(j)} Z^r_{i(j)} \sim 0 \text{ mod } S - Q,\ k < i(1) < \cdots < i(m),\ a_{i(j)} \neq 0. \tag{1.11a}$$

Then since the cycles Z_i for $i < i(m)$ all lie on $S - Q_{i(m)}$, relation (1.11a) implies $Z^r_{i(m)} \sim 0$ mod $S - Q_{i(m)}$, contradicting the choice of $Z^r_{i(m)}$.

1.12 Corollary. *If S is a locally compact space and x a point of S of countable character such that $p^r(S, x) \leqq \omega$, then S has no r-dimensional condensation at x.*

1.13 Theorem. *If M is a closed subset of an $M^n_{r,r+1}$, S, and $x \in M$ such that $p^r(M, x) \leqq \omega$, $r \leqq n - 1$, then M is regularly $(n - r - 1)$-accessible at x.*

[This is a consequence of Corollary 1.12 and Theorem 1.9.]

1.14 Corollary. *If M is a k-gm, $k \leqq n - 1$, in an $M^n_{r,r+1}$, S, then every point of M is regularly r-accessible from $S - M$ for all $r \leqq n - 1$.*

1.15 Definition. If G^r is a special group of cycles, then by regular accessibility *rel.* G^r will be meant the same notion as defined in Definition 1.3, except that consideration is restricted to the cycles of G^r.

It will be noted that the following analogue of Lemma 1.4 holds:

1.4a Lemma. *A necessary and sufficient condition that a closed subset M of a locally compact space S of countable character should be regularly r-accessible rel. bounding cycles of D at $x \in M$ from a set $D \subset S - M$ is that for arbitrary open set P containing x there exist an open set Q such that $x \in Q \subset P$ and such that for every cycle Z^r in $D \cap Q$ that bounds in D, and every open set R such that $x \in R \subset Q$, there exists a cycle γ^r in $D \cap R$ such that $Z^r \sim \gamma^r$ in $D \cap P$.*

If an open subset U of a spherelike n-gcm has property S_r, $r \leqq n - 1$, then $q^r(U, x) \leqq \omega$ for all $x \in F(U)$, hence (by Theorem X 1.5), $p^{n-r-1}(S - U, x) \leqq \omega$ and by Theorem 1.13, $S - U$ is regularly r-accessible at x.

1.16 Theorem. *If the open subset U of an $M^n_{r,r+1}$ has property S_r, $r \leqq n - 1$, then every point of the boundary of U is regularly r-accessible from U; if U has property S_r rel. bounding cycles, then the boundary of U is regularly r-accessible rel. bounding cycles from U.*

1.17 Corollary. *If a closed subset M of an $M^n_{1,r+2}$, S, is lc^r, $r \leqq n - 2$, then it is regularly $(n - r - 1)$-accessible, and regularly $(n - r - 2)$-accessible rel. bounding cycles from $S - M$.*

[This is a consequence of Theorem XI 2.3 and Theorem 1.16.]

1.18 Corollary. *If a closed subset M of a spherelike n-gcm S is lc^{n-2}, and D is a domain complementary to M, then every point of $F(D)$ is regularly* 0*-accessible from D.*

Remark. The applications of the above to the euclidean cases are obvious. Thus, the ordinary manifolds, when imbedded in S^n, come within the scope of Corollary 1.14. Or if M is an lc^r subset of S^n, then by Corollary 1.17, M is regularly $(n - r - 1)$-accessible at all its points.

It is interesting to note, however, that a result stronger than the first half of Corollary 1.17 is obtainable:

1.19 Theorem. *If $r \leqq n - 1$, and M is a closed, lc^{r-1}, semi-r-connected subset of an $M^n_{r,r+1}$, then M is regularly $(n - r - 1)$-accessible.*

Proof. By Theorem VII 2.26, $p^r(M, x) \leqq \omega$ for all $x \in M$. Hence by Theorem 1.13, M is regularly $(n - r - 1)$-accessible.

Returning to Theorem 1.2, we note that the sufficiency still holds in the following form:

1.2a Theorem. *If M is a point set and x is a point of countable character such that $M \cup x$ is r-lc in the sense of compact cycles at x, then x is regularly r-accessible from M.*

However, the converse does not hold, even in the case where M is an open subset of S^3 and $r = 1$:

1.20 EXAMPLE. In ordinary 3-space, using spherical coordinates (ρ, ϕ, θ), for each natural number n let $M_n = \{(\rho, \phi, \theta) \mid (0 \leqq \rho \leqq 1/n) \& (\theta = \pi/n)\}$. If x denotes the origin, then the set $M = \bigcup M_n$ is both regularly 0- and 1-accessible at x, but $(S - M) \cup x$ is not 1-lc at x.

However, we can state:

1.2b THEOREM. *If D is a point set and x is a point of countable character such that x is regularly r-accessible, $r \geqq 1$, from D, and $D \cup x$ is completely $(r-1)$-avoidable rel. bounding cycles of D at x, then $D \cup x$ is r-lc at x.*

[As usual when dealing with noncompact sets, we make the convention that unless otherwise specified, cycles and homologies are on compact sets. In particular, $D \cup x$ is completely $(r-1)$-avoidable relative to bounding cycles of D at x if for arbitrary open set P containing x, there exist open sets Q and R such that $x \in R \Subset Q \Subset P$ and such that every compact $(r-1)$-cycle of $F(Q) \cap D$ that bounds in D is homologous to zero on a compact subset of $(P - \bar{R}) \cap D$.]

PROOF. Let P be an arbitrary open set containing x, and let P_1 be an open set such that $x \in P_1 \subset P$ and such that compact r-cycles of $D \cap P_1$ bound on a compact subset of $(D \cap P) \cup x$. Let P_2 be an open set such that $x \in P_2 \subset P_1$ and such that compact r-cycles of $D \cap P_2$ bound on a compact subset of $(D \cap P_1) \cup x$. Let Q_2 and R_2 be open sets such that $x \in R_2 \Subset Q_2 \Subset P_2$ and such that $(r-1)$-cycles of $F(Q_2) \cap D$ that bound in D also bound on a compact subset of $(P_2 - \bar{R}_2) \cap D$.

In general, having defined P_i, Q_i and R_i, $i \geqq 2$, let P_{i+1}, Q_{i+1} and R_{i+1} be open sets such that (1) $x \in R_{i+1} \Subset Q_{i+1} \Subset P_{i+1} \subset R_i$, (2) compact r-cycles of $D \cap P_{i+1}$ bound on a compact subset of $D \cap P_i$, (3) $(r-1)$-cycles of $F(Q_{i+1}) \cap D$ that bound in D also bound on a compact subset of $(P_{i+1} - \bar{R}_{i+1}) \cap D$; (4) the open sets P_i form a complete neighborhood system of x.

Let Z^r be a compact cycle of $(D \cap P_1) \cup x$. We may assume that Z^r has no carrier that fails to contain x, else, by definition of r-accessibility, $Z^r \sim 0$ on a compact subset of $(D \cap P) \cup x$. Let K be a carrier of Z^r in $(D \cap P_1) \cup x$. The portion of Z^r on $S - Q_2$ is a cycle Z_1^r mod $F(Q_2)$ with boundary on $F(Q_2)$. Since $\partial Z_1^r \sim 0$ on a compact subset C_1 of $(P_2 - \bar{R}_2) \cap D$, there exist by Lemma VII 1.14 cycles γ_1^r and Γ_1^r on the compact sets $A_1 = C_1 \cup (K - Q_2)$ and $B_1 = C_1 \cup (K \cap \bar{Q}_2)$, respectively, such that $Z^r \sim \gamma_1^r + \Gamma_1^r$ on $A_1 \cup B_1 = K \cup C_1$.

In general, having obtained cycles $\gamma_2^r, \cdots, \gamma_{i-1}^r$, and a cycle Γ_{i-1}^r on a compact subset of $(D \cap P_i) \cup x$, such that $Z^r \sim \gamma_1^r + \cdots + \gamma_{i-1}^r + \Gamma_{i-1}^r$ on a compact subset of $(D \cap P) \cup x$, the portion of Γ_{i-1}^r on $S - \bar{Q}_{i+1}$ is a cycle Z_i^r mod $F(Q_{i+1})$ such that $\partial Z_i^r \sim 0$ on a compact subset C_i of $(P_{i+1} - \bar{R}_{i+1}) \cap D$, and there exist cycles γ_i^r and Γ_i^r on the compact sets $A_i = C_i \cup (B_{i-1} - Q_{i+1})$

and $B_i = C_i \cup (K \cap \overline{Q}_{i+1})$, respectively, such that $\Gamma_{i-1}^r \sim \gamma_i^r + \Gamma_i^r$ on $A_i \cup B_i$. Then $Z^r \sim \gamma_1^r + \cdots + \gamma_i^r + \Gamma_i^r$ on the set $K \cup \bigcup_{j=1}^i C_j$.

Now each cycle γ_i^r bounds on a compact subset F_i of $(D \cap P_{i-1}) \cup x$ (where $P_0 = P$). Let $C = K \cup \bigcup_{i=1}^\infty C_i \cup \bigcup_{i=1}^\infty F_i$; this is a compact subset of $(D \cap P) \cup x$. Then $Z^r \sim 0$ on C. For suppose not. Then there exists a covering $\mathfrak{U}$ of S such that $Z^r(\mathfrak{U}) \nsim 0$ on C. But let i be such that P_{i+1} lies in the nucleus of the simplex constituted by all the elements of $\mathfrak{U}$ that contain x. Then $\Gamma_i^r(\mathfrak{U}) \sim 0$ on x, and since $Z^r(\mathfrak{U}) \sim \sum_{j=1}^i \gamma_j^r(\mathfrak{U}) + \Gamma_i^r(\mathfrak{U})$ on $K \cup \bigcup_{j=1}^i C_j \subset C$, and $\gamma_j^r(\mathfrak{U}) \sim 0$ on F_j, it follows that $Z^r(\mathfrak{U}) \sim 0$ on C. We must conclude, then, that $Z^r \sim 0$ on C, and, as a consequence, that $D \cup x$ is r-lc at x.

Remark. That *almost* complete $(r - 1)$-avoidability would not have been sufficient for the hypothesis of Theorem 1.2b is shown by the following example: In the polar coordinate plane S, for each natural number n let $M_n = \{(\rho, \theta) \mid (\rho \leqq 1/n) \& (\theta = 1/n)\}$; let $M = \bigcup M_n$, $D = S - M$, and $x = (0, 0)$. Then M is regularly 0- and 1-accessible at all points. The set $D \cup x$ is almost completely 0-avoidable rel. bounding cycles of D at x, but $D \cup x$ is not 1-lc at x.

That the set $D \cup x$ of Theorem 1.2b may be r-lc at x without being completely $(r - 1)$-avoidable, or even almost completely $(r - 1)$-avoidable rel. bounding cycles of D, is shown by the following example: In coordinate 3-space, let $D' = \{(x, y, z) \mid |x| < 1, |y| < 1, 0 < z < 1\}$. Also, for each natural number n, let $M_n = \{(x, y, z) \mid 1/n^2 \leqq x^2 + y^2 \leqq 1, z = 1/n\}$, and $D = D' - \bigcup M_n$. If $x = (0, 0, 0)$, then $D \cup x$ is 0- and 1-lc at x, as well as regularly 0- and 1-accessible at x, but is not almost completely 0-avoidable rel. bounding cycles of D at x.

2. Stronger types of accessibility; their interrelations and topological invariance. We now introduce a type of accessibility that is much stronger than that of regular r-accessibility, and which is consequently of value in characterizing, by means of positional invariants, many of the more common topological configurations in higher dimensions. We found, as for instance in 1.13, 1.14 and 1.17, that sets having certain local connectedness properties also have regular accessibility properties, but we stated no converse theorems. This was because of the fact that, while the regular accessibility property in S^2 is strong enough to afford such converses, this is not the case in higher dimensions. For example, if T is a torus and C is a meridional circle on T, let C be deformed to a point p—the remainder of T remaining topologically unchanged. The resulting surface T', when imbedded in S^3, is the common boundary of two domains from each of which T' is regularly r-accessible for $r = 0, 1, 2$, yet possesses the "singular" point p. This is to be contrasted with the case in S^2, where a common boundary of two domains which is merely "arcwise accessible" from each of these domains is an S^1.

Let us reconsider the "arcwise accessibility" (cf. II 5.36); if in a space S, M is a point set and p a point of $S - M$, say, then p is arcwise accessible

from M if for $q \in M$ there exists an arc having end points p and q and lying, except for p, wholly in M. We can conceive of this as indicating that if Z^0 is a nontrivial 0-cycle on $p \cup q$, then $Z^0 \sim 0$ on a compact set A such that $A \subset M \cup p$. This idea forms the genesis of the following definition:

2.1 Definition. A point p of a point set K will be called *r-accessible* from a subset E of the complement of K if every compact r-cycle of $E \cup p$ bounds on a compact subset of $E \cup p$. In other words, p is r-accessible from E if the set $E \cup p$ is simply r-connected in the sense of compact cycles and homologies on compact sets. If for all $p \in K$, p is r-accessible from E, then we say K is r-accessible from E. When E is the complement of K, we may omit the words "from E," calling K simply an r-accessible set. Following the same order of ideas, the localization of this property, which we call *local r-accessibility*, is merely the r-lc property of $E \cup p$ at p in terms of compact cycles and homologies on compact sets. As we have seen in Theorem 1.2, the property which we now call local 0-accessibility is, for the boundary points of a domain D in a locally compact, 0-lc space of countable character, equivalent to regular accessibility. And in Theorem 1.2b we have established a connection between regular r-accessibility and local r-accessibility, although, as pointed out in Theorem 1.2a and the remarks following that theorem, for $r > 1$ local r-accessibility is the stronger property.

We shall first investigate some of the implications of the "in the large" type of r-accessibility.

2.2 Theorem. *Let K be an $(n - 1)$-accessible closed subset of an $M^n_{1,1}$, S, containing at least three points. Then K is a continuum which has no cut points.*

Proof. Suppose $K = A \cup B$ separate. Then by Theorem VIII 8.6, there exists a cycle γ^{n-1} on a compact subset of $S - K$ which is linked with a nontrivial zero cycle Z^0 carried by a pair of points p, q, where $p \in A$ and $q \in B$. Let x be a point of K distinct from both p and q. Since K is $(n - 1)$-accessible, $\gamma^{n-1} \sim 0$ on a compact subset L of $(S - K) \cup x$. But $L \subset S - (p \cup q)$ so that γ^{n-1} and Z^0 cannot be linked.

Let $p \in K$, and suppose that $K - p = K_1 \cup K_2$ separate. Since S has the Phragmen-Brouwer property (or cf. Lemma X 6.6) there exists, in $(S - K) \cup p$, a continuum C such that points of K lie in different components of $S - C$; in particular, there exist C, $p_1 \in K_1$ and $p_2 \in K_2$ such that p_1 and p_2 lie in different components of $S - C$. But by Theorem VIII 8.6, a nontrivial cycle Z^0 on $p_1 \cup p_2$ is linked with some γ^{n-1} of C, although since p is $(n - 1)$-accessible from $S - K$, γ^{n-1} would have to bound on $(S - K) \cup p \subset S - p_1 - p_2$.

2.3 Theorem. *If a closed subset K of an lc space S is 0-accessible from a set $D = \bigcup C_\nu$, C_ν a component of $S - K$, then K is the common boundary of all C_ν.*

Proof. By virtue of Corollary V 11.11, if $p \in K$ and x a point of some

component C_ν of D, then $D \cup p$ has a subcontinuum H that contains both p and x. Suppose that $H - p \not\subset C_\nu$. Then, since S is lc, C_ν is open (Theorem III 3.1) and $H - p = (H \cap C_\nu) \cup H_1$ separate. By Theorem I 9.8, $(H \cap C_\nu) \cup p$ is a continuum, and since it lies in $C_\nu \cup p$ we conclude that $p \in F(C_\nu)$.

Preliminary to investigating the relation between 0-accessibility and local 0-accessibility, we note the following lemmas:

2.4 LEMMA. *In order that a boundary point x (of countable character) of a connected open subset D of a locally compact, 0-lc space should be locally 0-accessible from D, it is necessary and sufficient that for arbitrary open set P containing x there exist an open set Q such that $x \in Q \subset P$ and such that $Q \cap (D \cup x)$ lies in one constituant of $P \cap (D \cup x)$.*

PROOF. The necessity is trivial. The sufficiency follows from the following considerations: As observed above, for x to be locally 0-accessible from D it is sufficient that x be regularly accessible from D. If, for arbitrary P, a Q exists as in the statement of the lemma, then x is by definition regularly accessible from D and consequently locally 0-accessible from D.

2.5 LEMMA. *In order that a boundary point x (of countable character) of a connected open subset D of a locally compact, 0-lc space should be locally 0-accessible from D, it is necessary and sufficient that each compact subset of $D \cup x$ lie in a subcontinuum of $D \cup x$.*

PROOF. The necessity is proved by methods analogous to those used in the proof of Theorem 1.2. (Using the sets P_i of Theorem 1.2, if K denotes the compact set in question, then $K - P_0$ is in a subcontinuum K_0 of D by Theorem IV 3.3. Each $K \cap (\overline{P}_i - P_{i+1})$ lies in a subcontinuum K_i of $(D \cap P_{i-1}) \cup x$ as in the proof of Theorem 1.2.)

To prove the sufficiency, suppose x not locally 0-accessible from D. Then there exists an open set P containing x such that no Q of the type described in Lemma 2.4 exists. As x is of countable character, there exists a sequence of points $x_1, \cdots, x_n, \cdots$ of $D \cap P$ having x as sequential limit point and such that no x_n lies with x in the same constituant of $(D \cup x) \cap P$. Since $x \cup \bigcup x_n$ is a compact point set, there exists a subcontinuum M of $D \cup x$ containing it. Let Q be an open set such that $x \in Q \Subset P$, and let $K = M \cap F(Q)$. Then by Theorem IV 1.8 each point x_n is in a subcontinuum M_n of $M \cap \overline{Q}$ that meets K. And by Theorem IV 1.14, lim sup M_n contains a subcontinuum M_0 of $M \cap \overline{Q}$ that contains x and meets K. But if $y \in M_0 \cap K$, then y is a point of $D \cap P$; and as D is 0-lc, there exists in $D \cap P$ a continuum C containing y and a point $y_n \in M_n \cap F(Q)$ for some n. Obviously x_n and x are in the same constituant of $(D \cup x) \cap P$, contrary to the definition of x_n.

2.6 LEMMA. *If D is a connected open subset of a locally compact, 0-lc space, x is a boundary point of D of countable character, and x is not locally 0-accessible, then there exist an open set P containing x and a sequence $x_1, \cdots, x_n, \cdots \in P \cap D$ such that no two points x_n lie in the same constituant of $P \cap (D \cup x)$.*

Proof. Suppose the conclusion of the theorem false. Since x is not locally 0-accessible, there exists by Lemma 2.4 an open set P_1 containing x such that no Q of the type described in that lemma exists. However, under our supposition, there exists an open set P_2 such that $x \in P_2 \subset P_1$, only a finite number of constituants of $P_1 \cap (D \cup x)$ meet P_2, and x is either an element of or a limit point of each of these constituants. Let C_1 be one of these constituants that fails to contain x—we may assume such a C_1 exists, else only one constituant of $P_1 \cap (D \cup x)$ meets P_2 and P_2 is a Q of the type described in Lemma 2.4.

As x is of countable character, there exists a sequence $p_1, \cdots, p_n, \cdots \in C_1$ having x as a sequential limit point. Now arguing as above, we show that there exists an open set P_3 such that $x \in P_3 \subset P_2$, only a finite number of constituants of $P_2 \cap (D \cup x)$ meet P_3, and x is an element of or a limit point of each of these constituants. Since "almost all" the points p_n lie in P_3, one of the constituants of $P_2 \cap (D \cup x)$ that meets P_3 must be a subset of C_1; we denote such a constituant by C_2.

In this manner we may prove the existence of a sequence of open sets P_n such that (1) the collection $\{P_n\}$ forms a complete neighborhood system of x, (2) there exists a constituant C_n of $P_n \cap (D \cup x)$ that has x as a limit point but does not contain x; (3) for each n, $C_n \supset C_{n+1}$. Now let $q_n \in C_n$. Then for each n, C_n contains a continuum K_n such that $q_n \cup q_{n+1} \subset K_n$. Evidently $x \cup \bigcup K_n$ is a subcontinuum of $P_1 \cap (D \cup x)$. But this implies that $x \in C_1$, contrary to definition.

2.7 Theorem. *For a boundary point of a connected open subset of a locally compact, 0-lc space of countable character, 0-accessibility and local 0-accessibility are equivalent.*

Proof. By Lemma 2.5, local 0-accessibility of a boundary point x of such an open set D is equivalent to the condition that every compact subset of $D \cup x$ lie in a subcontinuum of $D \cup x$. Hence if such a point x is locally 0-accessible, every compact 0-cycle Z^0 of $D \cup x$ has a carrier C which is a subcontinuum of $D \cup x$ and $Z^0 \sim 0$ on C.

Conversely, if x is 0-accessible from D, then x is locally 0-accessible from D. For suppose not. Then by Lemma 2.6 there exists an open set P containing x and a sequence of points $x_n \in P \cap D$ such that no two points x_n lie in the same constituant of $P \cap (D \cup x)$. If $\mathfrak{U}$ is a covering of S, let $Z^0(\mathfrak{U})$ be a 0-cycle $\sigma_1 - \sigma_2 + \cdots + \sigma_{k-1} - \sigma_k$ obtained as follows: σ_k is an element U_k of $\mathfrak{U}$ that contains x. If every $x_n \in U_k$, then $Z^0(\mathfrak{U}) = \sigma_k - \sigma_k$. If $x_1 \notin U_k$, then σ_1 is a $U_1 \in \mathfrak{U}$ that contains x_1. If $x_2 \in U_k$ and $x_3 \notin U_k$, for instance, then σ_2 is a $U_2 \in \mathfrak{U}$ that contains x_3. The collection $\{Z^0(\mathfrak{U})\}$ is a cycle carried by $x \cup \bigcup x_n$. As x is 0-accessible from D, there exists a compact subset M of $D \cup x$ such that $Z^0 \sim 0$ on M.

Let Q be an open set such that $x \in Q \subset P$. Now it may be shown (cf. the sufficiency proof of Lemma 2.5) that for each n, $M \cap \overline{Q}$ contains a continuum

M_n which contains x_n, as well as a point y_n of $F(Q)$. Proceeding as in the last part of the sufficiency proof of Lemma 2.5, we obtain the point y, from whose existence we readily see that the points x_n cannot all lie in different constituants of $P \cap (D \cup x)$.

2.8 Corollary. *For boundary points of connected open subsets of a locally compact, 0-lc space of countable character, regular accessibility, local 0-accessibility and 0-accessibility are all equivalent.*

2.9 Theorem. *Let M be a nondegenerate, 0- and 1-accessible closed subset of a spherelike 2-gcm S. Then M [if not 2-dimensional] is a spherelike 0- or 1-gcm.*

Proof. Suppose M is not an S^0. By Theorem 2.2, M is a continuum, and by Theorem 2.3, M is the common boundary of all its complementary domains.

Let $p \in M$, D be a domain complementary to M, and P an arbitrary open set containing p. Denote $S - M$ by E. Then by Theorem 2.7, there exists an open set Q such that $x \in Q \subset P$ and such that every compact 0-cycle of $Q \cap (E \cup x)$ bounds on a compact subset of $P \cap (E \cup x)$. Let Z^0 be a cycle on a compact set F in $Q \cap D$. Then there exists a compact subset A of $P \cap (E \cup x)$ such that $Z^0 \sim 0$ on A.

Since D is a domain, there exists a compact subset B of D such that $Z^0 \sim 0$ on B.

Let γ_1^1 and γ_2^1 be cycles mod F on A and B, respectively, such that $\partial\gamma_i^1 \sim Z^0$ on F, $i = 1, 2$ (Lemma VII 1.4). There exists on $A \cup B$ by Lemma VII 1.6 a cycle γ^1 such that $\gamma^1 \sim \gamma_1^1 + \gamma_2^1$ mod F on $A \cup B$. Since x is 1-accessible from E, it follows that $\gamma^1 \sim 0$ on a compact subset of $E \cup p$. From Theorem VII 9.1, with the A_1 and A_2 of that theorem denoting $F(P)$ and M respectively, it follows that $Z^0 \sim 0$ on a compact subset of $E - F(P)$ and hence of $D \cap P$.

Thus D is 0-ulc and M is a spherelike 1-gcm by Theorem X 6.11. In any case, then, M is a spherelike 0- or 1-gcm.

2.10 Corollary. *If M is a nondegenerate 0- and 1-accessible closed subset of S^2, then M is an S^0, S^1 or S^2.*

We shall now show that for $r > 0$, r-accessibility is stronger than local r-accessibility. First it should be noted that the latter does not imply the former, as is shown by simple examples: For instance, in the cartesian plane let $D' = \{(x, y) \mid x^2 + y^2 < 1\}$; $M = \{(x, y) \mid 0 \leqq x < 1, y = 0\}$ and $D = D' - M$. If $p = (1, 0)$, then p is locally 1-accessible from D but not 1-accessible from D.

2.11 Lemma. *In any space S, let $Z_1^r, \cdots, Z_n^r, \cdots$ be a countable collection of cycles such that if $\mathfrak{U}$ is any fcos of S, then at most a finite number of the coordinates $Z_i^r(\mathfrak{U})$ are not zero. Then the collection $Z^r = \{\sum_{n=1}^{\infty} Z_n^r(\mathfrak{U})\}$ is a Čech cycle of S.*

(Z^r may fail to have a compact carrier if S is not compact, however. If a compact carrier is desired, one must of course provide for this by carriers of the Z_n^r whose totality will have a compact closure.)

The proof is immediate.

2.12 Corollary. *If $x \in S$ and $\{Z_n^r\}$ is a countable collection of cycles such that if P is an arbitrary open set containing x, then all but a finite number of the cycles Z_n^r lie in P, then the collection $\{\sum_{n=1}^{\infty} Z_n^r(\mathfrak{U})\}$ is a Čech cycle of S.*

The following lemma, analogous to Lemma 1.4, is proved in the same way as the latter:

2.13 Lemma. *If D is an open subset of a locally compact space S and $x \in F(D)$ is of countable character, then a necessary and sufficient condition that x be locally r-accessible from D is that for arbitrary open set P containing x there exist an open set Q such that $x \in Q \subset P$ and such that for every compact cycle Z^r of $Q \cap (D \cup x)$ and every open set R such that $x \in R \subset Q$, there exist a compact cycle γ^r in $R \cap (D \cup x)$ such that $Z^r \sim \gamma^r$ in $P \cap (D \cup x)$.*

Now the following theorem includes the fact, proved in Theorem 2.7, that 0-accessibility implies local 0-accessibility, except that one has to assume that the space under consideration is perfectly normal; the reader may be interested in contrasting the proof for this case with the proof used in Theorem 2.7.

2.14 Theorem. *If x is an r-accessible boundary point of a connected open subset D of a locally compact, perfectly normal, lc^r space S, then x is locally r-accessible from D.*

Proof. Suppose x is not locally r-accessible from D. Then by Lemma 2.13, there exists an open set P containing x such that $\bar{P}$ is compact, and if Q is any open subset of P such that $x \in Q$, then there exists an open set R such that $x \in R \subset Q$ and a compact cycle Z^r of $Q \cap (D \cup x)$ which is not homologous to any compact cycle of $R \cap (D \cup x)$ on a compact subset of $P \cap (D \cup x)$. Since x is of countable character, there exists a sequence of open sets $\{P_n\}$ forming a complete neighborhood system for x, and such that (1) $P \supset P_n \supset P_{n+1}$ for all n, (2) there exists in $P_n \cap (D \cup x)$ a compact cycle Z_n^r which is not homologous to any compact cycle of $P_{n+1} \cap (D \cup x)$ on a compact subset of $P \cap (D \cup x)$.

Consider the collection $\{\Gamma_\nu^r\}$ of all forms of type $\Gamma_\nu^r = \sum_{n=1}^{\infty} b_\nu^n Z_n^r$, where $b_\nu^n = 0$ or 1. This is an uncountable collection, and by Corollary 2.12 each Γ_ν^r is a compact cycle of $P_1 \cap (D \cup x)$. By hypothesis, each $\Gamma_\nu^r \sim 0$ on a compact subset F_ν of $D \cup x$.

Let Q be an open set such that $P \supset Q \supset P_1$, and let $K = F(Q) - D$. Since S is perfectly normal, there exists a sequence of open sets $U_1, \cdots, U_n, \cdots$ such that $\bigcap_n U_n = K$. And since no F_ν meets K, there exists for each F_ν a set U_n such that $U_n \cap F_\nu = 0$. Also, since the F_ν form an uncountable collection, there is a value $n = k$ such that uncountably many of the sets F_ν fail to meet U_k; denote the collection of these by $\{F_\mu\}$.

Now $P \cap D \supset F(Q) - K \supset F(Q) - U_k$; hence $F(Q) - U_k$ is a compact subset of the open set $P \cap D$ in the lc^r space S and by Corollary VI 3.8 there

exists an integer m representing the maximum number of r-cycles on $F(Q) - U_k$ that are lirh in $P \cap D$. By Corollary VII 1.13, there exists for each μ a cycle γ_μ^r on $F(Q) - U_k$ such that $\Gamma_\mu^r \sim \gamma_\mu^r$ on $F_\mu \cap \bar{Q}$.

Suppose that $\Gamma_1^r, \cdots, \Gamma_{m+1}^r$ are any $m + 1$ elements of $\{\Gamma_\mu^r\}$. Between the corresponding cycles γ_i^r, $i = 1, \cdots, m + 1$, there exists a homology

$$\sum_{i=1}^{m+1} a^i \gamma_i^r \sim 0 \qquad \text{in } P \cap D, \qquad a^i \in \mathfrak{F}, \tag{2.14a}$$

where not all a^i are zero. We also have homologies

$$a^i \Gamma_i^r \sim a^i \gamma_i^r \qquad \text{on } F_i \cap \bar{Q} \subset P \cap (D \cup x). \tag{2.14b}$$

Combining (2.14a) and (2.14b) we get

$$\sum_{i=1}^{m+1} a^i \Gamma_i^r \sim 0 \qquad \text{on } M, \tag{2.14c}$$

where M is a compact subset of $P \cap (D \cup x)$.

From the form of the summations constituting the Γ_i^r we see that $\sum_{i=1}^{m+1} a^i \Gamma_i^r$ is a form $\sum_{n=1}^{\infty} c^n Z_n^r$. And relation (2.14c) becomes

$$\sum_{n=1}^{\infty} c^n Z_n^r \sim 0 \qquad \text{on } M. \tag{2.14d}$$

Suppose not all $c^n = 0$. Let c^t be the c^n of smallest index n such that $c^n \neq 0$. Then (2.14d) gives

$$Z_t^r \sim \sum_{n=t+1}^{\infty} e^n Z_n^r \qquad \text{on } M \tag{2.14e}$$

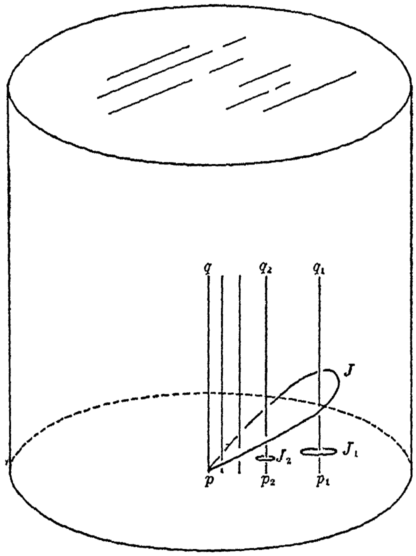
where $e^n = c^n/c^t$.

But $\sum_{n=t+1}^{\infty} e^n Z_n^r$ is a compact cycle of $P_{t+1} \cap (D \cup x)$ whereas Z_t^r is a compact cycle of $P_t \cap (D \cup x)$, and (2.14e) implies that Z_t^r is homologous to a compact cycle of $P_{t+1} \cap (D \cup x)$ on a compact subset M of $P \cap (D \cup x)$. This is impossible because of the stipulation (2) above.

It remains to ascertain, then, that there exists at least one form of type (2.14d) for which some c^n is not zero. This is the case for the following reasons: Each Γ_μ^r is determined by a set B_μ of elements b_μ^n of $\mathfrak{F}$ by the relation $\Gamma_\mu^r = \sum_{n=1}^{\infty} b_\mu^n Z_n^r$, where $b_\mu^n = 0$ or 1. Hence a form $\sum_{\mu=1}^{m+1} a^\mu \Gamma_\mu^r$ has for its nth term $\sum_{\mu=1}^{m+1} a^\mu b_\mu^n$, so that for all c^n in a form of type (2.14d) to be zero implies that elements a^μ of the field $\mathfrak{F}$ exist so as to render $\sum_{\mu=1}^{m+1} a^\mu b_\mu^n = 0$ for all n—i.e., the corresponding sets B_μ are linearly dependent rel $\mathfrak{F}$. We need only recognize, then, that there exist at least $m + 1$ linearly independent sets B_μ. Since the latter are infinite in number and consist only of 0's and 1's, this follows from elementary algebraic considerations.

2.15 Remarks. The following example is instructive in regard to Theorem 2.14 and its proof. In cartesian 3-space let $D' = \{(x, y, z) \mid (x^2 + y^2 < 4) \&$

$(0 < z < 2)\}$. For each n, let $M_n = \{(1/n, 0, z) \mid 0 < z \leqq 1\}$. Let $D = D' - \bigcup_{n=0}^{\infty} M_n$, $p = (0, 0, 0)$, $q = (0, 0, 1)$, $p_n = (1/n, 0, 0)$, and $q_n = (1/n, 0, 1)$ (see figure). The point p is neither 1-accessible nor locally 1-accessible. For



consider a small simple closed curve J_n in D near p_n encircling the line p_nq_n as indicated in the figure. If Z_n^1 is a nontrivial 1-cycle on J_n, then the cycles Z_n^r may form a set of cycles analogous to the cycles Z_n^r of the above proof. It is interesting to note that the Γ_ν^r of form $\sum_{n=1}^{\infty} Z_n^1$ does bound in $D \cup p$, since it is homologous in $D \cup p$ to a cycle Z^1 on the curve J of the figure, which in turn clearly bounds in $D \cup p$. However, a cycle of the form $\sum_{n=1}^{\infty} Z_{2n-1}^1$ does not bound in $D \cup p$.

It is also interesting to note that in proving Theorem 2.14 we used only the following property: If x is a boundary point of D, then there exists an open set P containing x such that if Z^r is a compact cycle of $P \cap (D \cup x)$, then $Z^r \sim 0$ on a compact subset of $D \cup x$. In other words, not all of the implications of r-accessibility are needed. However, the proof of Theorem 2.14 makes clear that the property just defined is *equivalent* to local r-accessibility. Another type of accessibility is suggested by these considerations:

2.16 Definition. In any space S a point p of a point set K will be called *semi-r-accessible* from a subset E of its complement if there exists a fcos $\mathfrak{E}$ of S such that every compact r-cycle of $E \cup p$ of diameter $< \mathfrak{E}$ bounds on a compact subset of $E \cup p$. If all points of K are semi-r-accessible from $S - K$, we say simply that K is semi-r-accessible.

It is clear from the above Remarks that if a boundary point x of a connected open subset D of a locally compact, perfectly normal, lc^r space S is semi-r-

accessible, then it is locally r-accessible. Trivial examples show that the converse fails; for instance:

2.17 EXAMPLE. In cartesian 3-space let $D' = \{(x, y, z) \mid x^2 + y^2 < 1\}$; $M = \{(x, y, z) \mid -1 < x < 1, y = 0, z = 0\}$; and $D = D' - M$. If $p = (0, 1, 0)$, then p is locally 1-accessible from D but not semi-1-accessible.

Also, r-accessibility certainly implies semi-r-accessibility, but the converse fails:

2.18 EXAMPLE. In the cartesian plane, let $D' = \{(x, y) \mid x^2 + y^2 < 1\}$; $M = \{(x, y) \mid 0 \leqq x < 1, y = 0\}$ and $D = D' - M$. If $p = (1, 0)$, then p is semi-1-accessible from D but not 1-accessible from D.

To summarize:

2.19 THEOREM. *For $r > 0$, the properties of being locally r-accessible, semi-r-accessible and r-accessible are in general successively stronger.*

Before considering further applications, we investigate the question regarding the topological invariance of the new types of accessibility. By means of the r-dimensional condensation property, we were able to prove topological invariance of the regular r-accessibility property for closed subsets of an $M^n_{r,r+1}$ (Corollary 1.10). In order to establish an invariance for the local r-accessibility property, we introduce the following definition of a property which the reader might compare with the property that was introduced in Lemma 1.6 as being equivalent to lack of r-dimensional condensation:

2.20 DEFINITION. A space S is called *smooth* at $x \in S$ in dimension r if for arbitrary open set P containing x there exists an open set Q such that $x \in Q \subset P$ and such that every compact cycle Z^r mod $S - P$ of $S - x$ is ≈ 0 in $S - x$ mod $S - Q$.

If a space is smooth at a point x in dimension r, then it has no r-dimensional condensation at x. That the converse fails is shown by the first example in the Remark following Theorem 1.2b, with $r = 0$.

2.21 THEOREM. *In order that a closed subset M of an $M^n_{r,r+1}$, S, should be locally r-accessible at $x \in M$, it is necessary and sufficient that M be smooth at x in dimension $n - r - 1$.*

PROOF OF NECESSITY. Consider first the case $r < n - 1$, and let $s = n - r - 1$. Also let $E = S - M$, and P, Q, R and W be open sets containing x such that $W \subset R \subset Q \subset P$ and such that (1) if Z^r is a compact cycle of $W \cap (E \cup x)$ then $Z^r \sim 0$ on a compact subset of $R \cap (E \cup x)$; (2) if Z^{s-1} is a cycle on $F(Q)$, then $Z^{s-1} \sim 0$ on $S - R$. Let $T = (S - W) \cup (M \cap W)$ and $T' = (S - R) \cup (M \cap R)$.

Now suppose Z^s is a compact cycle mod $S - P$ on $M - x$. If Z^s_1 is the portion of Z^s in Q, then $\partial Z^s_1 \sim 0$ on $S - R$. By Lemma VII 1.6 there exists

a cycle γ^s on T' such that $\gamma^s \sim Z^s$ in $T - x$ mod $S - R$. Now if $\gamma^s \approx 0$ in $T - x$, then we would have $Z^s \approx 0$ in $T - x$ mod $S - R$, and hence in $M - x$ mod $S - W$. Suppose $\gamma^s \not\approx 0$ in $T - x$. Then by Theorem VIII 9.1, there exists a cycle Z^r in $W \cap (E \cup x)$ which is linked with γ^s. But by the choice of W, $Z^r \sim 0$ in $R \cap (E \cup x) \subset S - T'$. As γ^s is on T', this contradicts the fact that γ^s and Z^r are linked.

The case $r = n - 1$ is handled similarly, except that local avoidability of S plays no role, γ^s being obtainable without it.

Proof of sufficiency. Let M be smooth at x in dimension $s = n - r - 1$. Then with P arbitrary, there exists Q such that if Z^s is a compact cycle of $M - x$ mod $S - P$, then $Z^s \approx 0$ in $M - x$ mod $S - Q$. Let Z^r be a compact cycle of $Q \cap (E \cup x)$, and suppose that $Z^r \not\approx 0$ in $P \cap (E \cup x)$. Then by Theorem VIII 9.1, Z^r is linked with a compact cycle Z^s of $S - P \cap (E \cup x)$; the portion of Z^s in P is a compact cycle mod $S - P$ on $M - x$, and hence $Z^s \approx 0$ in $M - x$ mod $M - Q$. By Theorem VI 4.6a, this implies that $Z^s \sim 0$ mod $M - Q$ in every neighborhood of $M - x$. In particular, $Z^s \sim 0$ mod $S - Q$ in a neighborhood of $M - x$ that excludes a carrier of Z^r, contradicting the fact that Z^s and Z^r are linked.

We conclude, then, that $Z^r \approx 0$ in $P \cap (E \cup x)$, and hence, by Theorem VI 4.6, that $Z^r \sim 0$ in every neighborhood of $P \cap (E \cup x)$. In particular, if $R \subset Q$ is an arbitrary neighborhood of x, and $U = [P \cap (E \cup x)] \cup R$, then $Z^r \sim 0$ in U. It follows that there exists a compact set F in U and a cycle C^{r+1} mod K on F, where K is a carrier of Z^r in $Q \cap (E \cup x)$, such that $\partial C^{r+1} \sim Z^r$ on K. Now $K \cap M = x$ and $F \cap M \subset R$, and if C_1^{r+1} is the portion of C^{r+1} in R, then ∂C_1^{r+1} lies on $(K \cap R) \cup [F \cap F(R)]$. That is, ∂C_1^{r+1} is a cycle of $\overline{R} \cap (E \cup x)$ such that $Z^r \sim \partial C_1^{r+1}$ in $P \cap (E \cup x)$. It follows from Lemma 2.13 that x is locally r-accessible.

2.22 Corollary. *The property of being locally r-accessible at a certain point is a topological invariant of closed subsets of an* $M^n_{r,r+1}$.

In order to establish invariance of the r-accessibility property, we have recourse, in addition to the "smoothness" property utilized in the case of local r-accessibility, to the numbers $p^r(S, \approx)$ (Cf. V 20.3).

2.23 Theorem. *In order that a closed subset of an* $M^n_{r,r+1}$, *S, should be r-accessible at* $x \in M$, *it is necessary and sufficient that M be smooth at x in dimension* $n - r - 1$ *and that* $p^{n-r-1}(M - x, \approx) = 0$.

Proof of necessity. If M is r-accessible at $x \in M$, then M is smooth at x in dimension $n - r - 1$ by Theorems 2.14 and 2.21. And were $p^{n-r-1}(M - x, \approx) \neq 0$, there would exist by Theorem VIII 9.1 a cycle Z^r on a compact subset of $(S - M) \cup x$ which fails to bound on the latter set, thus contradicting the assumption of r-accessibility at x.

Proof of sufficiency. Let $E = S - M$ and let Z^r be a cycle on a compact subset of $E \cup x$. Since $p^{n-r-1}(M - x, \approx) = 0$, it follows from Theorem VIII 9.1

that $p^r(E \cup x, \approx) = 0$. Hence $Z^r \approx 0$ in $E \cup x$. Now for any open set P containing x there exists by Theorem 2.21 an open set Q such that $x \in Q \subset P$ and such that every compact cycle of $\bar{Q} \cap (E \cup x)$ bounds on a compact subset of $P \cap (E \cup x)$. Let $U = E \cup Q$. Since $Z^r \approx 0$ in $E \cup x$, it follows from Theorem VI 4.6 that $Z^r \sim 0$ on a compact subset F of U. By the same method as used in the sufficiency proof of Theorem 2.21, we can show that there exists a cycle γ^r of $\bar{Q} \cap (E \cup x)$ such that $Z^r \sim \gamma^r$ in $E \cup x$. By the choice of Q, $\gamma^r \sim 0$ on a compact subset F' of $P \cap (E \cup x)$, and consequently $Z^r \sim 0$ on $F \cup F' \subset E \cup x$.

2.24 Corollary. *The property of being r-accessible at a certain point is a topological invariant for closed subsets of an* $M^n_{r,r+1}$.

For the proof of the invariance of semi-r-accessibility, we introduce the following property:

2.25 Definition. A space S is called *r-declinable* at $x \in S$ if there exists an open set P containing x such that every compact r-cycle of S is homologous to a compact r-cycle of $S - P$. If S is r-declinable at every point, we say simply that S is r-declinable. (Compare P. Alexandroff [f; 20].)

2.26 Theorem. *In order that a closed subset M of an $M^n_{r,r+1}$, S, should be semi-r-accessible at $x \in M$, it is necessary and sufficient that $M - x$ be $(n - r - 1)$-declinable and that M be smooth at x in dimension $n - r - 1$.*

Proof of necessity. That M is smooth at x in dimension $n - r - 1$ follows from Theorems 2.19 and 2.21. Let $y \in M - x$. Let U be an open set such that $y \in U \Subset S - x$. Let V be an open set such that $S - U \subset V \Subset S - y$. Denote the covering of S consisting of U and V by $\mathfrak{U}$. Since M is semi-r-accessible at x, there exists an $\mathfrak{E} > \mathfrak{U}$ such that every compact cycle of $(S - M) \cup x$ of diameter $< \mathfrak{E}$ bounds on a compact subset of $(S - M) \cup x$. Let P be an element of $\mathfrak{E}$ such that $y \in P$; obviously $P \subset U$.

Suppose there exists a cycle Z^{n-r-1} on a compact subset F of $M - x$, which is not homologous on a compact subset of $M - x$ to a compact cycle of $M - x - U$. Let $T = (S - P) \cup (M \cap P)$. Then $Z^{n-r-1} \not\sim 0$ on T. Consequently there exists in $P - M$ a cycle Z^r which is linked with Z^{n-r-1}. But $Z^r \sim 0$ on a compact subset F' of $(S - M) \cup x \subset S - F$. We conclude that no cycle such as Z^{n-r-1} exists and hence that $M - x$ is $(n - r - 1)$-declinable.

Proof of sufficiency. Let $E = S - M$ and $s = n - r - 1$. As M is smooth at x in dimension s, there exists by Theorem 2.21 an open set U_x containing x such that every compact r-cycle of $U_x \cap (E \cup x)$ bounds on a compact subset of $E \cup x$. Since $M - x$ is s-declinable and S is locally s-avoidable (Lemma IX 3.1, Corollary IX 2.2), for each $y \in M - U_x$ there exist open sets U_y and V_y such that $x \notin U_y$, every compact s-cycle of $M - x$ is homologous on a compact subset of $M - x$ to a compact cycle of $M - U_y$, and every s-cycle of $S - U_y$ bounds on $S - V_y$. For each $z \in E$, there exists an open set

U_z such that $z \in U_z \subset E$ and such that every r-cycle in U_z bounds in E (since S is s-lc). Let $\mathfrak{U}$ be a fcos of S consisting of elements of the collection $\{U_x\} \cup \{V_y\} \cup \{U_z\}$. Evidently $U_x \in \mathfrak{U}$, and if U is an element of $\mathfrak{U}$ different from U_x such that $U \cap M \neq 0$, then $U \in \{V_y\}$.

We assert that if Z^r is a compact cycle of $E \cup x$ of diameter $< \mathfrak{U}$, then $Z^r \sim 0$ on a compact subset of $E \cup x$. If Z^r lies in some element of $\mathfrak{U}$ that is a U_z, this is trivial. And if Z^r lies in U_x, it follows from the definition of U_x. Suppose Z^r lies in a V_y. Let F be a compact subset of V_y carrying Z^r. Then $Z^r \approx 0$ in $E \cup x$. For if not, then by Theorem VIII 9.1, there exists a compact cycle Z^s of $M - x$ which is linked with Z^r. But Z^s is homologous on a compact subset of $M - x$ to a cycle γ^s of $M - x - U_y$, and in turn $\gamma^s \sim 0$ on $S - V_y$; that is, $Z^s \sim 0$ in $S - F$, which is impossible since Z^r and Z^s are linked. We conclude then that $Z^r \approx 0$ in $E \cup x$. Then if V is an open set such that $x \in V \Subset U_x$, Z^r is homologous in $E \cup U_x$ to a compact cycle of $\overline{V} \cap (E \cup x)$ (see latter part of the procf of Theorem 2.21), and hence $Z^r \sim 0$ on a compact subset of $E \cup x$.

2.27 Corollary. *The property of being semi-r-accessible at a certain point is a topological invariant for closed subsets of an* $M^n_{r,r+1}$.

3. Applications to recognition of submanifolds of a manifold. In Theorem 2.9 and Corollary 2.10, we saw that the spherelike k-gcm is completely characterized in the 2-gcm by the property of r-accessibility. In the present section we develop like theorems for the general case.

By an argument similar to that used in the proof of Theorem 1.2b, we may show:

3.1 Lemma. *If D is a point set and x is a point of countable character such that x is regularly r-accessible, $r \geqq 1$, from D, and D is $(r - 1)$-ulc at x rel. bounding cycles of D, then x is locally r-accessible from D.*

The second example in the Remark following Theorem 1.2b exhibits a case where the hypothesis of the above Lemma applies for $r = 1$.

3.2 Theorem. *If M is a k-gcm in an n-gcm S, then M is locally r-accessible from $S - M$ for $r = n - k + 1, \cdots, n - 1$; and if S is an $M^n_{k,k+1}$, and M is orientable, then M is locally $(n - k)$-accessible.*

Proof. By Corollary X 1.8, $S - M$ is r-ulc for $r = n - k, \cdots, n - 1$. Hence by Lemma 3.1, M is locally r-accessible for all $r \geqq n - k + 1$, inasmuch as r-ulc is stronger than regular r-accessibility. By Theorem X 4.5, $S - M$ is $(n - k - 1)$-ulc rel. bounding cycles of $S - M$ when M is orientable and S is an $M^n_{k,k+1}$.

3.3 Theorem. *If M is a spherelike k-gcm in an $M^n_{1,k+1}$, S, then M is r-accessible from $S - M$ for $r = n - k, \cdots, n - 1$.*

Proof. If M is a k-gcm and $x \in M$, it follows from Lemma IX 3.1, Corollary

IX 2.2 and Theorem VI 4.6 that $p^s(M - x, =) = 0$ for $s = 0, 1, \cdots, k - 1$. By Theorem 3.2 and Theorem 2.21, M is smooth at every point in the dimensions $0, 1, \cdots, k - 1$. Hence by Theorem 2.23, M is r-accessible in the dimensions $n - k, \cdots, n - 1$.

3.4 Corollary. *If M is an $(n - 1)$-gcm in a spherelike n-gcm S, and D is a domain complementary to M, then M is locally r-accessible from D for $r = 0, 1, \cdots, n - 1$; and if M is spherelike, it is r-accessible from D for $r = 0, 1, \cdots, n - 1$.*

Proof. The case $r = 0$ follows from Corollary 2.8, inasmuch as D is 0-ulc and this implies regular accessibility. By Theorem 3.2, M is locally r-accessible from $S - M$ for $r = 1, \cdots, n - 1$, and it is easily shown that this implies local accessibility from D alone. A similar remark holds in the case where M is spherelike.

3.5 Lemma. *Let x be a point of a space S such that $p^r(S, x) = 0$. Then S is r-declinable at x.*

Proof. Since $p^r(S, x) = 0$, there exist open sets P and Q such that $x \in Q \subset P$ and such that if Z^r is a cycle mod $S - P$, then $Z^r \sim 0$ mod $S - Q$. Now let Z^r be any compact cycle of S. Then $Z^r \sim 0$ mod $S - Q$, and by Lemma VII 1.9, there exists a cycle γ^r on $S - Q$ such that $Z^r \sim \gamma^r$.

Remark. That r-declinability does not imply $p^r(S, x) = 0$ is shown by the simple example of the straight line interval: Let S be the set of all real numbers x such that $0 \leqq x \leqq 1$, and let $x = 1/2$. Then S is 1-declinable at x, but $p^r(S, x) = 1$.

3.6 Corollary. *If S is an n-gm, then S is r-declinable for all $r < n$.*

3.7 Lemma. *If D is an open subset of a normal space S, G a subgroup of the group of bounding compact cycles of D, and $x \in F(D)$ which is regularly r-accessible from D rel. cycles of G and $(r + 1)$-accessible from D, then $q^r(D, x; G) = 0$. In particular, if all cycles of D in some neighborhood U of x bound in D, then $q^r(D, x) = 0$.*

Proof. If P is an arbitrary open set containing x (which, in the case of the second sentence of the statement of the lemma we may assume to be a subset of U), then there exists an open set Q such that $x \in Q \subset P$ and such that if $\gamma^r \in G$ is a cycle on a compact subset K of $D \cap Q$, then $\gamma^r \sim 0$ on a compact set $K_1 \subset P \cap (D \cup x)$. And such a cycle γ^r also bounds on a compact subset K_2 of D. Referring to Theorem VII 9.1, and letting $A_1 = [(S - D) - P] \cup F(P)$, $A_2 = (S - D) \cap \overline{P}$, the cycle γ^{r+1} of Theorem VII 7.1 bounds on $D \cup x$, since x is $(r + 1)$-accessible from D, and hence $\gamma^{r+1} \sim 0$ in $S - (A_1 \cap A_2)$. We may conclude that $\gamma^r \sim 0$ in $D \cap P$.

3.8 Lemma. *If D is an open subset of an n-gcm S such that $F(D)$ contains at least three points and is $(n - 1)$-accessible, then $p^{n-1}(D) = 0$.*

Proof. Denote $F(D)$ by M. Suppose Z^{n-1} is a cycle on a compact subset K of D such that $Z^{n-1} \not\sim 0$ in D. If $x_1, x_2 \in M$, $x_1 \neq x_2$, there exist closed subsets K_i of $D \cup x_i$, $i = 1, 2$, such that $Z^{n-1} \sim 0$ on K_i, and K_1 is minimal rel. to the property of containing K and carrying this homology. Hence by Lemma VII 2.19, $K_1 \cup K_2$ carries a cycle Γ^n such that $\Gamma^n \not\sim 0$ on $K_1 \cup K_2$. It follows that $K_1 \cup K_2 = S$ (cf. VIII 3). But this is impossible, since $(K_1 \cup K_2) \cap M = x_1 \cup x_2$ and M contains at least three points.

3.9 Theorem. *In order that an open subset D of a spherelike n-gcm S, $n > 1$, should have a spherelike $(n - 1)$-gcm as boundary, it is necessary and sufficient that* (1) *D be semi-r-connected for $r = 1, 2, \cdots, n - 2$, and* (2) *$F(D)$ be not n-dimensional, contain at least three points, and be r-accessible from D for $r = 0, 1, \cdots, n - 1$.*

The necessity follows from Theorem X 3.2 and Corollary 3.4.

Proof of sufficiency. By Lemma 3.7 and Corollary X 2.6, D is ulc_1^{n-2}. By Theorem 2.3, the set $M = F(D)$ is the common boundary of all components of D, and if D' is one of these components, then D' is 0-ulc by Lemma 3.7; and consequently D' is ulc^{n-2}. By Lemma 3.8, $p^{n-1}(D) = 0$; hence $p^{n-1}(D') = 0$. And then by Theorem X 6.8, M is an orientable $(n - 1)$-gcm.

Suppose $p^r(M) \neq 0$ for some r such that $0 < r < n - 1$. Then by Theorem VIII 6.4, $p^s(S - M) \neq 0$, $s = n - r - 1$, and from Theorem X 5.14 it follows that $p^r(D') \neq 0$ or $p^s(D') \neq 0$; suppose the latter. Let Z^s be a compact cycle of D' such that $Z^s \not\sim 0$ in D'. Then $Z^s \not\sim 0$ in $S - M$, and by Corollary VIII 8.6, Z^s is linked with a cycle Z^r of M. But by Corollary 3.6, there exists a cycle γ^r of M on a closed proper subset K of M such that $\gamma^r \sim Z^r$ on M. Then γ^r and Z^s must be linked. But this is impossible since if $x \in M - K$, then $Z^s \sim 0$ on $(S - M) \cup x$. We conclude, then, that M is spherelike.

3.10 Corollary. *In order that an open subset D, with nondegenerate, not n-dimensional boundary M, of a spherelike n-gcm S, should have a spherelike $(n - 1)$-gcm as boundary, it is necessary and sufficient that $p^r(D) = 0$ for all r such that $0 < r \leqq n - 1$, and that M be r-accessible from D for $r = 0, 1, \cdots, n - 1$.*

[Since $p^{n-1}(D) = 0$, and M is nondegenerate, M contains more than two points.]

3.11 Corollary. *If the boundary, M, of an open subset D of the 3-sphere S^3 is nondegenerate and is r-accessible from D for $r = 0, 1, 2$, and $p^r(D) = 0$ for $r = 1, 2$, then M is an S^2.*

Remark. In S^3, consider the examples of (1) an arc A and (2) a point set T consisting of a plane triangle together with its plane interior. In case (1), the hypothesis of Corollary 3.11 fails to be satisfied for the open set $D = S^3 - A$ because A is not 2-accessible; and in case (2) fails to be satisfied for the set

$D = S^3 - T$ because T is not 1-accessible. The 2-sphere with a radius attached also shows the necessity for including 2-accessibility above.

It is a direct corollary of Theorem VIII 8.6 that if K is a closed subset of an $M^n_{1,1}$, S, and K carries a nonbounding $(n - 1)$-cycle, then K separates S; and conversely. However, suppose that a set M is given in $S - K$, and a cycle Z^{n-1} is given on K which does not bound in $S - M$; what can be said about separation of M by K? (If M is closed, Lemma VIII 8.6 applies to show that Z^{n-1} is linked with a 0-cycle of M and it easily follows that K separates M.) The following lemma, needed in the sequel, provides an answer.

3.12 LEMMA. *Let Z^{n-1} be a cycle on a closed subset K of an $M^n_{1,1}$, S. Let M be a subset of $S - K$ such that $Z^{n-1} \not\sim 0$ in $S - M$. Then M is separated in S by K.*

PROOF. Let K_1 be a subset of S minimal with respect to containing K and carrying a homology $Z^{n-1} \sim 0$ (Lemma VII 2.8). Let $x \in K_1 \cap M$. Since S is locally $(n - 1)$-avoidable (Corollary IX 2.2), $Z^{n-1} \sim 0$ in $S - x$, and therefore there exists a closed set K_2 in $S - x$ that is also minimal with respect to containing K and carrying a homology $Z^{n-1} \sim 0$. By Lemma VII 2.19, there is a cycle Γ^n on $K_1 \cup K_2$ such that $\Gamma^n \not\sim 0$ on $K_1 \cup K_2$. Hence $K_1 \cup K_2 = S$ (VIII 3).

Now let C be a component of $S - K$. If $C \cap K_1 \neq 0$, then $C \subset K_1$. For suppose not. Then $C = (C \cap K_1) \cup (C - K_1)$, and as $C \cap K_1$ is closed rel. C, it must contain a point z which is a limit point of $C - K_1$. But for the same reason given for the existence of K_2 in $S - x$, there exists a K_3 in $S - z$ such that $Z^{n-1} \sim 0$ on K_3, and again by Lemma VII 2.19, there exists a cycle γ^n on $K_1 \cup K_3$ such that $\gamma^n \not\sim 0$ on $K_1 \cup K_3$, leading to the conclusion that $K_1 \cup K_3 = S$. But this is impossible, since in a neighborhood of z which does not meet K_3 there exist points of $C - K_1$ which belong to neither K_1 nor K_3. We conclude, then, that if a component of $S - K$ contains a point of K_i, $i = 1, 2$, it lies in K_i.

Then the point x of the first paragraph of the proof is in a component C_1 of $S - K$ which contains no points of K_2. And since K_2 must meet M, there exist points of M in other components of $S - K$ than C_1. Hence M is separated in S by K.

3.13 COROLLARY. *Let M be a subcontinuum of an $M^n_{1,1}$, S, and $x \in M$ such that $M - x$ is connected. Then M is $(n - 1)$-accessible at x.*

PROOF. Let Z^{n-1} be a cycle on a compact subset K of $(S - M) \cup x$. Then $Z^{n-1} \sim 0$ in $(S - M) \cup x = S - (M - x)$, else by Lemma 3.12, K would separate $M - x$.

3.14 LEMMA. *Let D be an open subset of an $M^n_{1,1}$, S, and $x \in F(D) = M$ such that* (1) *$M - x$ is connected,* (2) *x is regularly $(n - 2)$-accessible from D, and* (3) *there exists an open set P containing x such that every $(n - 2)$-cycle in $D \cap P$ bounds in D. Then D is $(n - 2)$-ulc at x.*

PROOF. Given an arbitrary open set U containing x, there exists an open set V such that $x \in V \subset U$, and such that if Z^{n-2} is a cycle of $D \cap V$, then $Z^{n-2} \sim 0$ on a compact subset K_1 of $D \cap (U \cup x)$; and there exists a compact subset K_2 of D such that $Z^{n-2} \sim 0$ on K_2. We need consider only the case where $x \in K_1$.

We may now obtain the cycle γ^{r+1} and the sets A_1 and A_2 of Theorem VII 9.1, where γ^{r+1} becomes a γ^{n-1}, $A_1 = (M - P) \cup F(P)$ and $A_2 = M \cap \bar{P}$. If γ^{n-1} cannot be chosen so that it bounds in $S - (M - x)$, then by Lemma 3.12, $K_1 \cup K_2$ separates points of $M - x$ in S. But this would imply that $M - x$ is not connected. Hence $\gamma^{n-1} \sim 0$ in $S - M \subset S - (A_1 \cap A_2)$, and by Theorem VII 9.1, $Z^{n-2} \sim 0$ in $S - M - F(P)$. It follows that $\gamma^{n-1} \sim 0$ in $D \cap P$.

We can now give a characterization, in terms of accessibility properties, of the spherelike k-gcm in the spherelike n-gcm.

3.15 THEOREM. *In order that a k-dimensional closed subset M of an $M^n_{1,k+1}$, S, should be a spherelike k-gcm, it is necessary and sufficient that* (1) $p^{n-k-1}(S - M) = 1$, *and if F is a proper closed subset of M, then* $p^{n-k-1}(S - F) = 0$; (2) *if $k > 1$, $S - M$ is semi-r-connected and M is r-accessible for $r = n - k, \cdots, n - 2$;* (3) *M is regularly $(n - k - 1)$-accessible rel. bounding cycles.*

PROOF. The necessity follows from Theorem X 4.5 and Theorem 3.3 above.

To prove the sufficiency, we shall first show that the conditions of Theorem X 4.5 are fulfilled. We need consider only the case $k > 0$, since if $k = 0$, condition (1) of the hypothesis implies M is an S^0.

Condition (1) of Theorem X 4.5 is identical with condition (1) of the present theorem. To prove condition (2) for $r = n - k, \cdots, n - 3$, we use the method employed in the second paragraph of the sufficiency proof of Theorem 3.9. To show that $S - M$ is $(n - 2)$-ulc, we apply Lemma 3.14. If $x \in M$, then $M - x$ is connected by Corollary VII 3.3, and with (2) of the hypothesis of the present theorem, the hypothesis of Lemma 3.11 is satisfied. Condition (3) of Theorem X 4.5 is also proved by like methods, using the regular $(n - k - 1)$-accessibility relative bounding cycles. We conclude, then, that M is an orientable k-gcm.

To see that M is spherelike, we have to show that $p^s(M) = 0$ for $s = 1, \cdots, k - 1$; or what is equivalent, that $p^r(S - M) = 0$ for $r = n - k, \cdots, n - 2$. Suppose Z^r, $n - k \leqq r \leqq n - 2$, is a nonbounding cycle of $S - M$. Then by Corollary VIII 8.6, Z^r is linked with a cycle Z^s of M, $1 \leqq s \leqq k - 1$. But this is impossible, since if $x \in M$, there exists by Corollary 3.6 a cycle γ^s of $M - x$ in the same homology class of $H^s(M)$ as Z^s, and $Z^r \sim 0$ on $(S - M) \cup x$.

The following corollaries are of interest since they afford characterizations of the euclidean S^1 and S^2 in the spherelike n-gcm:

3.16 COROLLARY. *If M is a 1-dimensional closed subset of an $M^n_{1,2}$, S, such that* (1) $p^{n-2}(S - M) = 1$ *and if F is a proper closed subset of M,* $p^{n-2}(S - F) = 0$,

(2) *M is regularly $(n - 2)$-accessible rel. bounding cycles; then M is a spherelike 1-gcm. In particular, if S is perfectly separable, or an S^n, then M is an S^1.*

The assumption that M is 1-dimensional may be omitted in the latter cases, since it may be shown that if a cycle Z^{n-2} of $S - M$, where M is compact metric and lc, links M, then Z^{n-2} is linked with an S^1 of M. Cf. Wilder [i]. The case where $S = S^n$ and $k = 1$ was also discussed by P. Alexandroff [f; 19, Theorem IV]. However, Alexandroff assumed regular $(n - 2)$-accessibility instead of regular $(n - 2)$-accessibility rel. bounding cycles alone; it was actually the equivalent of the latter which, in the case $n = 2$, Schoenflies assumed for the characterization of the S^1 by accessibility properties (IV 7).

3.17 Corollary. *In order that a 2-dimensional closed subset M of a perfectly separable $M^n_{1,3}$, S, should be a 2-sphere, it is necessary and sufficient that* (1) *$p^{n-3}(S - M) = 1$ and $p^{n-3}(S - F) = 0$ if F is a proper closed subset of M,* (2) *$S - M$ be semi-$(n - 2)$-connected and M be $(n - 2)$-accessible, and* (3) *M be regularly $(n - 3)$-accessible rel. bounding cycles.*

And the following corollary for the case $k = n - 1$ is interesting in contrast to Theorem 3.9, where the emphasis was placed on a single complementary domain:

3.18 Corollary. *In order that an $(n - 1)$-dimensional closed subset M of a spherelike n-gcm S should be a spherelike $(n - 1)$-gcm, it is necessary and sufficient that M separate S and be r-accessible for $r = 0, 1, \cdots, n - 2$, and that $S - M$ be semi-r-connected for $r = 1, \cdots, n - 2$.*

Proof of sufficiency. Since M is 0-accessible, it is the common boundary of all its complementary domains by Theorem 2.3. Let D be one of the latter. That D is ulc^{n-3} may be proved by a method similar to that used in the proof of Theorem 3.9, and that D is $(n - 2)$-ulc follows from Lemma 3.14. Hence M is an orientable $(n - 1)$-gcm by Theorem X 6.9, and condition (1) of Theorem 3.15 is satisfied for $k = n - 1$.

Now in order to characterize the general k-gcm—that is, the k-gcm which is not necessarily simply-connected in dimensions less than k—we turn, as might be expected, to the weaker forms of accessibility. In particular, we shall give a characterization that utilizes semi-r-accessibility. As the latter was defined (2.16), it is dependent upon the particular point chosen, and is not a uniform property. Consider the following definition:

3.19 Definition. A closed subset M of a space S will be called *uniformly semi-r-accessible* from an open subset D of $S - M$ if there exists a fcos $\mathfrak{E}$ of S such that if $x \in M$ and Z^r is a compact cycle of $D \cup x$ of diameter $< \mathfrak{E}$, then $Z^r \sim 0$ on a compact subset of $D \cup x$. If it is desired to indicate a special fcos $\mathfrak{E}$ for which this property holds, we shall say that M is *uniformly semi-r-accessible of norm $\mathfrak{E}$ from D.*

It is trivial that uniform semi-r-accessibility implies semi-r-accessibility at each point of M. But the converse fails to hold. For in S^3 let M be the homeomorph of a plane circular disc and K the image, under the same homeomorphism, of the boundary of the disc. Then M is semi-1-accessible at every point, but not uniformly semi-1-accessible; no covering of S^3 which serves for an $x \in K$ will serve at the same time for nearby points of $M - K$.

3.20 Lemma. *In order that a closed subset, M, of an $M^n_{r,r+1}$, S, should be uniformly semi-r-accessible from $S - M$, it is necessary and sufficient that* (1) *M be smooth at every point in dimension $n - r - 1$ and* (2) *there exist a fcos $\mathfrak{U}$ of M such that if $x \in M$, $U \in \mathfrak{U}$, and Z^{n-r-1} is a compact cycle of $M - x$, then $Z^{n-r-1} \approx 0 \bmod M - U$ on $M - x$.*

Proof of necessity. (1) follows from Theorems 2.19 and 2.21. Let $\mathfrak{E}$ be a fcos of S such that M is uniformly semi-r-accessible of norm $\mathfrak{E}$ from $S - M$. Denote $n - r - 1$ by s. Let $x \in M$, $U \in \mathfrak{E}$, and Z^s a cycle of $M - x$. Suppose $Z^s \not\approx 0 \bmod M - U$ on $M - x$. Then $Z^s \not\approx 0$ on $(S - U) \cup (M \cap U) - x$, and by Theorem VIII 9.1 there exists a cycle Z^r in $U \cap (S - M) \cup x$ which is linked with Z^s. But Z^r is of diameter $< \mathfrak{E}$ and therefore $Z^r \sim 0$ on $(S - M) \cup x$—hence in the complement of some carrier of Z^s. We conclude, then, that $Z^s \approx 0 \bmod M - U$ on $M - x$. And the desired covering of M consists of the intersections with M of the elements of $\mathfrak{E}$.

Proof of sufficiency. Given $\mathfrak{U}$, we obtain a fcos $\mathfrak{E}$ of S in the following manner: Each $U \in \mathfrak{U}$ may be augmented by $S - M$; the resulting fcos of S we call $\mathfrak{U}'$. Now if $y \in S - M$, there exist open sets V and W such that $y \in W \subset V \subset S - M$ and such that s-cycles on $S - V$ are homologous to zero on $S - W$. If $y \in M$, let V denote the intersection of the elements of $\mathfrak{U}'$ that contain y, and let W be an open set such that $y \in W \subset V$ and again such that s-cycles on $S - V$ bound on $S - W$. Let $\mathfrak{E}$ be a finite collection of the sets W covering S.

Now let $W \in \mathfrak{E}$, $x \in M$, and Z^r a cycle of $(S - M) \cup x$ in W. Suppose $Z^r \not\approx 0$ in $(S - M) \cup x$. Then by Theorem VIII 9.1, Z^r is linked with a Z^s of $M - x$. But let $U' \in \mathfrak{U}'$ such that $W \subset U'$. By hypothesis, $Z^s \approx 0 \bmod M - U$ in $M - x$, and hence by Theorem VI 4.6a, for every open subset P of S that contains $M - x$, $Z^s \sim 0 \bmod M - U$ in P. If we let $P = S - K$, where K is a closed subset of $(S - M) \cup x$ in W carrying Z^r, then $Z^s \sim 0 \bmod M - U$ in $S - K$. But this implies that Z^r is homologous, on a compact subset of $S - K$, to a cycle γ^r on $S - U$, and since $\gamma^r \sim 0$ on $S - W$, this implies that $Z^r \sim 0$ in $S - K$. As this contradicts the fact that Z^r and γ^r are linked, we conclude that $Z^r \approx 0$ in $(S - M) \cup x$.

Now since M is smooth at x, M is locally r-accessible at x. It follows that $Z^r \sim 0$ in $(S - M) \cup x$ by an argument similar to that used in the sufficiency proof for Theorem 2.23.

3.21 Corollary. *For closed subsets of an $M^n_{r,r+1}$, uniform semi-r-accesibility from the complement is a topological invariant.*

The following lemma follows immediately from definitions:

3.22 LEMMA. *If a closed subset M of an n-gcm S is r-accessible from an open set P, then M is uniformly semi-r-accessible from P.*

3.23 THEOREM. *If M is an orientable k-gcm, $k > 0$, in an $M^n_{k,k+1}$, S, then M is $(n - k - 1)$-accessible at every point.*

PROOF. Since for every proper closed subset F of M, $p^k(F) = 0$, it follows that $p^k(M - x) = 0$ and a fortiori that $p^k(M - x, \approx) = 0$.

Now if $x \in M$, and P is an arbitrary open subset of M containing x, there exists an open set Q such that $x \in Q \subset P$ and such that $(k - 1)$-cycles on $F(P)$ bound on $M - Q$ (Corollary IX 2.2). Then if Z^k is a compact cycle mod $S - P$ of $M - x$, there exists a compact cycle γ^k of $M - x$ such that $\gamma^k \sim Z^k$ mod $S - Q$, and since $\gamma^k \sim 0$ on its carrier, it follows that $Z^k \sim 0$ mod $S - Q$ in $M - x$; and a fortiori ≈ 0 mod $S - Q$ in $M - x$. Hence M is by definition smooth at every point in dimension k.

It follows from Theorem 2.23 that M is $(n - k - 1)$-accessible.

3.24 COROLLARY. *If M is an orientable k-gcm in an $M^n_{k,k+1}$, then M is uniformly semi-$(n - k - 1)$-accessible from its complement.*

3.25 THEOREM. *Let M be a k-gcm in an n-gcm S. Then, in case $k > 1$, M is uniformly semi-r-accessible from $S - M$ for $r = n - k + 1, \cdots, n - 2$. In any case, if S is an $M^n_{k,k+1}$ and M is orientable, then M is uniformly semi-$(n - k)$-accessible.*

PROOF. By Corollary X 1.8, $S - M$ is r-ulc for $r = n - k, \cdots, n - 2$. Let $\mathfrak{E}$ be a fcos of S such that r-cycles of $S - M$ of diameter $< \mathfrak{E}$ all bound in $S - M$. Let $x \in M$ and Z^r a cycle of $(S - M) \cup x$ of diameter $< \mathfrak{E}$. If Z^r has a carrier not containing x, it is trivial that $Z^r \sim 0$ in $(S - M) \cup x$. Otherwise, we may show that $Z^r \sim 0$ in $(S - M) \cup x$ by an argument similar to that used in the proof of Theorem 1.2b. (For the case $r = n - k$, condition (3) of Theorem X 4.5 is needed.)

For the general orientable k-gcm we can state the following theorem, whose proof is left to the reader.

3.26 THEOREM. *In order that a closed subset M of an $M^n_{k,k+1}$, S, should be an orientable k-gcm, $k > 1$ (the case $k = 1$ has already been treated in Corollary 3.16), it is necessary and sufficient that (1) $p^{n-k-1}(S - M) = 1$, and if F is a proper closed subset of M, then $p^{n-k-1}(S - F) = 0$; (2) M be regularly $(n - k - 1)$-accessible rel. bounding cycles, and for some fcos $\mathfrak{E}$ of S, M be uniformly semi-r-accessible of norm $\mathfrak{E}$ for $r = n - k, \cdots, n - 1$; and (3) there exist fcos $\mathfrak{D} > \mathfrak{E}' >^* \mathfrak{E}$ of S such that bounding $(n - k - 1)$-cycles of $S - M$ of diameter $< \mathfrak{D}$ bound on compact subsets of $S - M$ of diameter $< \mathfrak{E}'$, and if $r = n - k, \cdots, n - 2$, r-cycles of $S - M$ of diameter $< \mathfrak{D}$ bound on compact subsets of $S - M$ of diameter $< \mathfrak{E}'$.*

For the case $k = n - 1$ of Theorem 3.26, the conditions stated may be considerably weakened, in that (1) may be deleted and (2), (3) applied with reference to a single domain:

3.27 THEOREM. *In order that a domain D, with nonempty, non-n-dimensional boundary, in an $M^n_{1,1}$, S, should have an orientable $(n - 1)$-gcm as boundary, it is necessary and sufficient that* (1) *$F(D)$ be regularly* 0*-accessible from D, and for some fcos $\mathfrak{E}$ of S, $F(D)$ be uniformly semi-r-accessible of norm $\mathfrak{E}$ from D for* $r = 1, 2, \cdots, n - 1$; (2) *there exist fcos $\mathfrak{D} > \mathfrak{E}' >^* \mathfrak{E}$ of S such that for* $r = 0, 1, \cdots, n - 2$, *r-cycles of D of diameter $< \mathfrak{D}$ bound on compact subsets of D of diameter $< \mathfrak{E}'$*; (3) $p^{n-1}(D) = 0$.

BIBLIOGRAPHICAL COMMENT

The contents of this chapter were reported in abstract form in Wilder [A12, A13].

APPENDIX

SOME UNSOLVED PROBLEMS

Below are listed some unsolved problems. Since we have not systematically attempted solving any of these, we can make no predictions regarding how difficult these problems may be. Perhaps some of them will turn out quite simple—may, indeed, be corollaries of results that have escaped our attention in the literature on topology. The discerning reader will perceive problems in the preceding chapters that are not mentioned below. However, this is not intended as a complete list in any sense. Moreover, we do not list any problems of purely algebraic type, problems concerning homotopy, nor problems concerning mappings of manifolds which suggest themselves above.

1. **Point set problems.** In connection with the material on plane point sets, it was mentioned that there remain many unsolved problems concerning nonclosed subsets of the plane. One such is described herewith: If M is a connected set and D a subset of M such that $M - D$ is totally disconnected, then D may be called a dispersion set of M. If no proper subset of D is a dispersion set of M, then let us call D a primitive dispersion set of M.

1.1 *Does the plane, E^2, have a primitive dispersion set?*

If M is a common boundary of (at least) two domains A and B in S^2, and M is lc $=$ lc^0, then M is an S^1 (Theorem IV 6.6); hence by the Jordan Curve Theorem, $S^2 = M \cup A \cup B$.

1.2 *If M is a common boundary of two domains A and B in S^n and M is lc^{n-2}, is $S^n = M \cup A \cup B$?* (Solve by methods valid in an n-gcm?) *If the answer is negative, can there exist a domain C such that $A \neq C \neq B$ and such that M is the boundary of C?* (See 2.3a below.)

1.3 *How liberally are the perfectly normal compact spaces supplied with compact metric subsets?* (See 4.3 below.)

1.4 *What, in general, is the minimal cardinal number always possible for a complete set (basis) of open subsets of a perfectly normal space?*

2. **Problems concerning homology.** It was shown in Theorem VII 4.5 that if A and B are disjoint closed subsets of a compact metric space S, and every cycle Z^r on A bounds on a closed subset of $S - B$, then there exists an open set U containing B such that every Z^r on A bounds on $S - U$.

2.1 *If A and B are disjoint closed subsets of a compact space S and every r-cycle on A bounds on a compact subset of $S - B$, does there exist an open set U containing B such that every r-cycle on A bounds on $S - U$?*

It was shown in Theorem VII 2.23 that if Z^r is a cycle of a compact space S,

then there exists in S a closed set A which is minimal with respect to being a closed carrier of a cycle γ^r such that $Z^r \sim \gamma^r$ on S; the set A is a minimal closed carrier of γ^r.

2.2 *If Z^r is a Čech cycle in a compact space S, under what conditions does there exist a cycle γ^r such that $\gamma^r \sim Z^r$ in S and γ^r has a closed, r-dimensional carrier?*

If an lc^0, compact metric space is minimal closed carrier of a nonbounding 1-cycle, then it is an S^1; in particular, such a space can carry only one lirh 1-cycle.

2.3 *Can an lc^n, n-dimensional, compact space be minimal closed carrier of more than one non-bounding Čech n-cycle?* (*If desirable, add* VIII 3D *to hypothesis*).

2.3a. Same as 2.3 with "lc^n" replaced by "lc^{n-1}".

In Theorem X 5.8 it was shown that a ulc^k open subset of an orientable n-gcm S has an lc^k closure. By virtue of Theorem IV 4.12, every ulc (or 0-ulc) subset of a compact space has lc (or 0-lc) closure.

2.4 *If U is a ulc^k open subset of a compact space S is $\overline{U}$ lc^k?*

The solution of 2.4 would probably not be difficult if the "chain-realization" process employed above were replaced by a process not dependent upon the existence of non-cobounding n-cocycles. In [o] we employed one such process, using Vietoris cycles and chains and a finite coefficient field. In this manner an affirmative solution of 2.4 was obtained in [q] for the metric case.

2.5 *Set up a geometric realization process analogous to that of* [o] *for locally compact non-metric spaces and arbitrary field $\mathfrak{F}$.*

3. **Dimension theory problems.** Among the dimension theory problems that should be solved is of course the problem already pointed out in connection with Theorems X 3.3 and X 6.10:

3.1 *If M is a common boundary of two domains in an n-gcm S, is M $(n - 1)$-dimensional?*

More generally,

3.2 *Are the at most $(n - 1)$-dimensional closed subsets of an n-gm S identical with the closed subsets that contain no interior points of S?*

And in analogy with a well-known theorem for the euclidean case (see Hurewicz-Wallman [H-W; Theorem IV 3]), one can ask:

3.3 *Must an n-dimensional (not necessarily closed) subset of an n-gm S contain interior points of S?*

3.4 *State and prove the analogue of the Menger-Nöbeling imbedding theorem*[1] *for an orientable n-gcm; what is the lowest dimension n for which a compact space of dimension k can be imbedded in an orientable n-gcm?*

4. **Problems concerning generalized manifolds.** The following problem arises in connection with IX 6:

4.1 *Is every n-gm locally orientable?*

In IX 7.1, in connection with the example of a 3-gcm that is not regular, the following question arose:

[1]For references, etc., see Hurewicz-Wallman [H–W; p. 56, footnote 11].

4.2 *Do the points of L in the example of* IX 7.1 *have arbitrarily small neighborhoods whose boundaries do not have infinite* 1-*dimensional connectivity numbers?*

Throughout much of the material on manifolds, it was assumed that the manifolds were perfectly normal. The following question has already been proposed by Alexandroff [f; 33, Problem IV′] in connection with the Čech definition of manifold:

4.3 *Is a perfectly normal generalized manifold necessarily metric?*

Perfect normality was especially needed in establishing duality theorems for the case where the sets in question were not assumed to have finite Betti numbers:

4.4 *Establish the Alexander and Poincaré types of duality for a generalized manifold without use of the perfect normality.*

The following problem was communicated to the author by E. G. Begle:

4.5 *Does there exist a generalized manifold not orientable for any field whatsoever?*

The following problem has already been pointed out in X 9.5:

4.6 *In S^3, let K denote the Alexander "horned 2-sphere" and D the domain complementary to K such that the fundamental group of D does not vanish. Let $K' \cup D'$ be a topological image of $K \cup D$ under a homeomorphism h such that $K' = h(K)$, $D' = h(D)$. Let K and K' be identified so that for $x \in K$, $x = h(x)$. If $K \cup D \cup D'$ is then topologized in suitable manner, is it an S^3?* (It is a spherelike 3-gcm by Theorem X 9.2).

BIBLIOGRAPHY

Only books that are of special significance for the text and papers that are specifically referred to therein are included in this bibliography. For papers on manifolds of the classical type, not referred to herein, the reader is referred to the bibliographies in the Colloquium volumes of Lefschetz and the book of Seifert and Threlfall cited below. For additional citations in the general literature of Topology, reference may be made to the works of Lefschetz and Seifert-Threlfall as well as to the bibliographies in the Colloquium volumes of R. L. Moore and G. T. Whyburn cited below.

Books

ALEXANDROFF, P. and HOPF, H.

[A–H] *Topologie*, Berlin, Springer, 1935 (Die Grundlehren der mathematischen Wissenschaften, bd. 45).

HAUSDORFF, F.

[H] *Grundzüge der Mengenlehre*, Leipzig, von Weit, 1914.

[H_1] *Mengenlehre*, Berlin, de Gruyter, 1927, 1935.

HILBERT, D. and COHN-VOSSEN, S.

[H–C] *Anschauliche Geometrie*, Berlin, Springer, 1932; in reprint, N. Y., Dover, 1944.

HUREWICZ, W. and WALLMAN, H.

[H–W] *Dimension Theory*, Princeton Univ. Pr., 1941 (Princeton Mathematical Series, no. 4).

KERÉKJÁRTÓ, B. VON.

[K] *Vorlesungen über Topologie*, I, *Flächentopologie*, Berlin, Springer, 1923.

KURATOWSKI, K.

[K] *Topologie* I, Warsaw, 1933; 2d ed., 1948.

LEFSCHETZ, S.

[L] *Algebraic Topology*, New York, 1942 (American Mathematical Society Colloquium Publications, vol. 27).

[L_1] *Topics in Topology*, Princeton Univ. Pr. 1942 (Annals of Mathematics Studies, no. 10).

[L_2] *Topology*, New York, 1930 (American Mathematical Society Colloquium Publications, vol. 12).

MENGER, K.

[Me] *Kurventheorie*, Berlin, Teubner, 1932.

[Me_1] *Dimensionstheorie*, Berlin, Teubner, 1928.

MOORE, R. L.

[Mo] *Foundations of Point Set Theory*, New York, 1932 (American Mathematical Society Colloquium Publications, vol. 13).

NEWMAN, M. H. A.

[N] *Elements of the Topology of Plane Sets of Points*, Cambridge Univ. Pr., 1939.

PONTRJAGIN, L.

[P] *Topological Groups*, Princeton Univ. Pr., 1939 (Princeton Mathematical Series, no. 2).

REIDEMEISTER, K.

[R] *Topologie der Polyeder*, Leipzig, Akademische Verlagsgesellschaft, 1938.

SCHOENFLIES, A.
[S] *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, II, Leipzig, Teubner, 1908 (Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsbände, bd. II).
SEIFERT, H. and THRELFALL, W.
[S–T] *Lehrbuch der Topologie*, Leipzig, Teubner, 1934.
SIERPIŃSKI, W.
[S] *Introduction to General Topology*, Univ. of Toronto Pr., 1934.
[Si] *Leçons sur les Nombres Transfinis*, Paris, Gauthier-Villars, 1928.
TUCKER, A. W.
[T] *Elementary Topology*, mimeographed notes by F. W. Burton and J. H. Lewis, Princeton University, 1935–36.
VEBLEN, O.
[V] *Analysis Situs*, 2d ed., New York, 1931 (American Mathematical Society Colloquium Publications, vol. 5, part 2).
WHYBURN, G. T.
[Wh] *Analytic Topology*, New York, 1942 (American Mathematical Society Colloquium Publications, vol. 28).
WILDER, R. L. and AYRES, W. L., editors
[UM] *Lectures in Topology*, The University of Michigan Conference of 1940, Univ. of Mich. Pr., 1941.

PAPERS, ETC.

ALEXANDER, J. W.
[a] *A proof and extension of the Jordan-Brouwer separation theorem*, Transactions of the American Mathematical Society, vol. 23 (1922), pp. 333-349.
[b] *Combinatorial analysis situs*, I, Transactions of the American Mathematical Society, vol. 28 (1926), pp. 301-329.
[c] *An example of a simply connected surface bounding a region which is not simply connected*, Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 8-10.
[d] *Note on Pontrjagin's topological theorem of duality*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 222-225.
[e] *Manifolds*, Encyclopaedia Britannica, 14th ed., vol. 14, pp. 804-805.
ALEXANDROFF, P.
[a_1] *Über stetige Abbildungen kompakter Räume*, Proceedings of the Amsterdam Academy, vol. 28 (1925), pp. 997-999.
[a_2] *Über stetige Abbildungen kompakter Räume*, Mathematische Annalen, vol. 96 (1926-27), pp. 555-571.
[b] *Zur Begründung der n-dimensionalen mengentheoretische Topologie*, Mathematische Annalen, vol. 94 (1925), pp. 296-308.
[c] *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Annals of Mathematics (2), vol. 30 (1928-29), pp. 101-187.
[d] (with Urysohn, P.) *Mémoire sur les espaces topologiques compacts*, Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam, Afdeeling Natuurkunde (Erste Sectie), Deel XIV, No. 1 (1929).
[e] *Zur Homologie-Theorie der Kompakten*, Compositio Mathematica, vol. 4 (1937), pp. 256-270.
[f] *On local properties of closed sets*, Annals of Mathematics (2), vol. 36 (1935), pp. 1-35.
[g] *Einfachste Grundbegriffe der Topologie*, Berlin, Springer, 1932.
[h] *Une généralisation nouvelle du théorème de Phragmén-Brouwer*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 184 (1927), pp. 575-577.

[i] (with Pontrjagin, L.) *Les variétés à n-dimensions généralisées*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 202 (1936), pp. 1327-29.

[j] *General combinatorial topology*, Transactions of the American Mathematical Society, vol. 49 (1941), pp. 41-105.

[k] *On homological situation properties of complexes and closed sets*, Transactions of the American Mathematical Society, vol. 54 (1943), pp. 286-339.

BEGLE, E.

[a] *Locally connected spaces and generalized manifolds*, American Journal of Mathematics, vol. 64 (1940), pp. 553-574.

[b] *Duality theorems for generalized manifolds*, American Journal of Mathematics, vol. 67 (1945), pp. 59-70

[c] *A note on local connectivity*, Bulletin of the American Mathematical Society, vol. 54 (1948), pp. 147-148.

BING, R. H.

[a] *The Kline sphere characterization problem*, Bulletin of the American Mathematical Society, vol. 52 (1946), pp. 644-653.

BROUWER, L. E. J.

[a] *Zur analysis situs*, Mathematische Annalen, vol. 68 (1910), pp. 422-434.

[b] *Beweis des Invarianz der geschlossenen Kurve*, Mathematische Annalen, vol. 72 (1912), pp. 422-425.

[c] *Beweis des Jordanschen Satzes für den n-dimensionalen Raum*, Mathematische Annalen, vol. 71 (1912), pp. 314-319.

[d] *Über Jordanschen Mannigfaltigkeiten*, Mathematische Annalen, vol. 71 (1912), pp. 320-327.

[e] *Beweis des Jordanschen Kurvensatzes*, Mathematische Annalen, vol. 69 (1910), pp. 169-175.

ČECH, E.

[a] *Théorie générale de l'homologie dans un espace quelconque*, Fundamenta Mathematicae, vol. 19 (1932), pp. 149-183.

[b] *Théorie générale des variétés et de leurs théorèmes de dualité*, Annals of Mathematics (2), vol. 34 (1933), pp. 621-730.

[c] *Sur la dimension des espaces parfaitement normaux*, Bulletin international de l'Académie des Sciences de Bohême, 1932, pp. 1-18.

[d] *Sur la connexité local d'ordre supérieur*, Compositio Mathematica, vol. 2 (1935), pp. 1-25.

[e] *On general manifolds*, Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 110-111.

[f] *Sur les nombres de Betti locaux*, Annals of Mathematics (2), vol. 35 (1934), pp. 678-701.

[g] *On pseudomanifolds* (mimeographed), Princeton, N. J., The Institute for Advanced Study, 1935.

[h] *Sur la décomposition d'une pseudovariété par un sousensemble fermé*, Comptes Rendus de l'Académie des Sciences, vol. 198 (1934), pp. 1342-1345.

CHITTENDEN, E. W.

[a] *On the metrization problem and related problems in the theory of abstract sets*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 13-34.

EILENBERG, S.

[a] *Singular homology theory*, Annals of Mathematics, vol. 45 (1944), pp. 407-447.

FRANKL, F.

[a] *Topologische Beziehungen in sich kompakter Teilmengen euklidischer Räume zu ihren Komplementen sowie Anwendung auf die Prim-Enden-Theorie*, Akademie der Wissenschaften, Wien, Math.-Naturw. Kl., Sitzungsberichte, Abt. 2A, vol. 136 (1927), pp. 689-699.

GAWEHN, I.

[a] *Über unberandete zweidimensionale Mannigfaltigkeiten,* Mathematische Annalen, vol. 98 (1927), pp. 321-354.

GEHMAN, H. M.

[a] *Concerning irreducibly connected sets and irreducible continua,* Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 544-547.

[b] *Irreducible continuous curves,* American Journal of Mathematics, vol. 49 (1927), pp. 189-196.

[c] *Concerning irreducible continua,* Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 431-435.

HAHN, H.

[a] *Über die Komponenten offener Mengen,* Fundamenta Mathematica, vol. 2 (1921), pp. 189-192.

JORDAN, C.

[a] *Cours d'Analyse,* 2 ed., Paris, Gauthier-Villars, 1893, vol. 1.

VAN KAMPEN, E. R.

[a] *On some characterizations of 2-dimensional manifolds,* Duke Mathematical Journal, vol. 1 (1935), pp. 74-93.

[b] *Die kombinatorische Topologie und die Dualitätssätze,* The Hague (1929). (Leyden thesis).

KAPLAN, S.

[a] *Homologies in metric separable spaces,* Dissertation, University of Michigan, Ann Arbor, 1942.

[b] *Homology properties of arbitrary subsets of euclidean spaces,* Transactions of the American Mathematical Society, vol. 62 (1947), pp. 248-271.

KNASTER, B. and KURATOWSKI, C.

[a] *Sur les ensembles connexes,* Fundamenta Mathematicae, vol. 2 (1921), pp. 206-255.

[b] *Sur les continus non-bornés,* Fundamenta Mathematicae, vol. 5 (1924), pp. 23-58.

KURATOWSKI, C.

[a] *Une définition topologique de la ligne de Jordan,* Fundamenta Mathematicae, vol. 1 (1920), pp. 40-43.

[b] *Une caractérisation topologique de la surface de la sphère,* Fundamenta Mathematicae, vol. 13 (1929), pp. 307-318.

KLINE, J. R.

[a] *Closed connected sets which remain connected upon the removal of certain connected subsets,* Fundamenta Mathematicae, vol. 5 (1924), pp. 3-10.

[b] *Separation theorems and their relation to recent developments in analysis situs,* Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 155-192.

LEBESGUE, H.

[a] *Sur les correspondances entre les points de deux espaces,* Fundamenta Mathematicae, vol. 2 (1921), pp. 256-285.

LEFSCHETZ, S.

[a] *The residual set of a complex on a manifold and related questions,* Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 614-622, pp. 805-807.

[b] *Closed point-sets on a manifold,* Annals of Mathematics (2), vol. 29 (1928), pp. 232-254.

[c] *On generalized manifolds,* American Journal of Mathematics, vol. 55 (1933), pp. 469-504.

[d] *Chain-deformations in topology,* Duke Mathematical Journal, vol. 1 (1935), pp. 1-18.

LENNES, N. J.

[a] Bulletin of the American Mathematical Society, vol. 12 (1906), p. 284, abstract no. 10.

[b] *Curves in non-metrical analysis situs with applications in the calculus of variations,* American Journal of Mathematics, vol. 33 (1911), pp. 287-326.

MAZURKIEWICZ, S.

[a] *Sur un problème de M. Knaster*, Fundamenta Mathematicae, vol. 13 (1929), pp. 146-150.

MENGER, K.

[a] *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277-306.

MOORE, R. L.

[a] *On the most general class L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true*, Proceedings of the National Academy of Sciences, vol. 5 (1919), pp. 206-210.

[b] *Concerning simple continuous curves*, Transactions of the American Mathematical Society, vol. 21 (1920), pp. 313-320.

[c] *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254-260.

[d] *Concerning connectedness im kleinen and a related property*, Fundamenta Mathematicae, vol. 3 (1922), pp. 232-237.

[e] *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 289-302.

[f] *A characterization of Jordan regions by properties having no reference to their boundaries*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364-370.

[g] *On the relation of a continuous curve to its complementary domains in space of three dimensions*, Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 33-38.

POINCARÉ, H.

[a] *Analysis situs*, Journal de l'École Polytechnique, (2), vol. 1 (1895), pp. 1-121.

PONTRJAGIN, L.

[a] *Über den algebraischen Inhalt topologischer Dualitatssätze*, Mathematische Annalen, vol. 105 (1931), pp. 165-205.

[b] *The general topological theorem of duality for closed sets*, Annals of Mathematics, (2), vol. 35 (1934), pp. 904-914.

[c] *Zum Alexanderschen Dualitätssatz*, Göttinger Nachrichten, Math.-Phys. Kl., 1927, pp. 315-322.

[d] *Zum Alexanderschen Dualitätssatz*, Zweite Mitteilung, Göttinger Nachrichten, Math.-Phys. Kl., 1927, pp. 446-456.

SCHOENFLIES, A.

[a] *Bemerkungen zu dem vorstehenden Aufsatz des Herrn L. E. J. Brouwer*, Mathematische Annalen, vol. 68 (1910), pp. 435-444.

SIERPIŃSKI, W.

[a] *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44-60.

STEENROD, N. E.

[a] *Universal homology groups*, American Journal of Mathematics, vol. 58 (1936), pp. 661-701.

SWINGLE, P. M.

[a] *An unnecessary condition in two theorems of analysis situs*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 607-618.

TIETZE, H.

[a] *Über die topologischen Invarianten mehrdimensionalen Mannigfaltigkeiten*, Monatshefte für Mathematik und Physik, vol. 19 (1908), pp. 1-118.

TORHORST, M.

[a] *Über den Rand der einfach zusammenhängenden ebenen Gebiete*, Mathematische

URYSOHN, P.

[a] *Über die Metrization der kompakten topologischen Räume*, Mathematische Annalen, vol. 92 (1924), pp. 275-293.

[b] *Über die Mächtigkeit der zusammenhängenden Mengen*, Mathematische Annalen, vol. 94 (1925), pp. 262-295.

[c] *Zum Metrizationsproblem*, Mathematische Annalen, vol. 94 (1925), pp. 309-315.

[d] *Über im kleinen zusammenhängenden Kontinua*, Mathematische Annalen, vol. 98 (1927), pp. 296-308.

[e_1] *Mémoire sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 30-137.

[e_2] *Mémoire sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 8 (1926), pp. 225-356.

VAUGHAN, H. E.

[a] *On local Betti numbers*, Duke Mathematical Journal, vol. 2 (1936), pp. 117-137.

VEBLEN, O. and ALEXANDER, J. W.

[a] *Manifolds of N dimensions*, Annals of Mathematics (2), vol. 14 (1913), pp. 163-178.

VIETORIS, L.

[a] *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Mathematische Annalen, vol. 97 (1927), pp. 454-472.

WHITNEY, H.

[a] *On products in a complex*, Annals of Mathematics (2), vol. 39 (1938), pp. 397-432.

[b] *A characterization of the closed 2-cell*, Transactions of the American Mathematical Society, vol. 35 (1933), pp. 261-273.

WHYBURN, G. T.

[a] *Semi-locally connected sets*, American Journal of Mathematics, vol. 61 (1939), pp. 733-749.

[b] *Concerning accessibility in the plane and regular accessibility in n dimensions*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 504-510.

[d] *Concerning regular and connected point sets*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 685-689.

[e] *A generalized notion of accessibility*, Fundamenta Mathematicae, vol. 14 (1929), pp. 311-326.

[f] *On the cyclic connectivity theorem*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 429-433.

[g] *Concerning continuous images of the interval*, American Journal of Mathematics, vol. 53 (1931), pp. 670-674.

[h] *On the construction of simple arcs*, American Journal of Mathematics, vol. 54 (1932), pp. 518-524.

[i] *Continuous curves without local separating points*, American Journal of Mathematics, vol. 53 (1931), pp. 163-166.

WILDER, R. L.

[a] *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340-377.

[b] *On a certain type of connected set which cuts the plane*, Proceedings of the International Mathematical Congress, Toronto, 1924, pp. 423-437.

[c] *A converse of the Jordan-Brouwer separation theorem in three dimensions*, Transactions of the American Mathematical Society, vol. 32 (1930), pp. 233-240.

[d] *Point sets in three and higher dimensions and their investigation by means of a unified analysis situs*, Bulletin of the American Mathematical Society, vol. 38 (1932), pp. 649-692.

[e] *A theorem on connected point sets which are connected im kleinen*, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 338-340.

[f] *A point set which has no true quasi-components, and which becomes connected upon the addition of a single point*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 423-427.

[g] *On connected and regular point sets*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 649-655.

[h] *A characterization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11 (1928), pp. 127-131.

[i] *On the linking of Jordan continua in E_n by $(n - 2)$-cycles*, Annals of Mathematics (2), vol. 34 (1933), pp. 441-449.

[j] *Concerning simple continuous curves and related point sets*, American Journal of Mathematics, vol. 53 (1931), pp. 39-55.

[k] *On the properties of domains and their boundaries*, Mathematische Annalen, vol. 109 (1933), pp. 273-306.

[m] *Concerning irreducibly connected sets and irreducible regular connexes*, American Journal of Mathematics, vol. 56 (1934), pp. 547-557.

[n] *Generalized closed manifolds in n-space*, Annals of Mathematics (2), vol. 35 (1934), pp. 876-903.

[o] *On locally connected spaces*, Duke Mathematical Journal, vol. 1 (1935), pp. 543-555.

[p] *Sets which satisfy certain avoidability conditions*, Časopis pro Pěstování Matematiky a Fysiky, vol. 67 (1938), pp. 185-198.

[q] *Property S_n*, American Journal of Mathematics, vol. 61 (1939), pp. 823-832.

[s] *The sphere in topology*, American Mathematical Society Semicentennial Publications, New York, 1938, vol. 2, pp. 136-184.

[t] *A characterization of manifold boundaries in E_n dependent only on lower dimensional connectivities of the complement*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 436-441.

[A_1]–[A_{13}] These are abstracts of unpublished papers in the Bulletin of the American Mathematical Society with volume, page and abstract no. as follows: A_1—35 (1929), 194, 9; A_2—43 (1937), 335, 272; A_3—52 (1946), 446, 219; A_4—52 (1946), 445, 217; A_5—46 (1940), 57, 134; A_6—52 (1946), 445, 216; A_7—46 (1940), 436, 349; A_8—52 (1946), 446, 218; A_9—47 (1941), 58, 114; A_{10}—47 (1941), 58, 113; A_{11}—47 (1941), 58, 112; A_{12}—42 (1936), 496, 308; A_{13}—53 (1947), 507, 287.

YOUNG, G. S., JR.

[a] *A characterization of 2-manifolds*, Duke Mathematical Journal, vol. 14 (1947), pp. 979-990.

ZIPPIN, L.

[a] *On continuous curves and the Jordan Curve Theorem*, American Journal of Mathematics, vol. 52 (1930), pp. 331-350.

[b] *A characterization of the closed 2-cell*, American Journal of Mathematics, vol. 55 (1933), pp. 207-217.

INDEX OF SYMBOLS

(The symbol "f" refers to a footnote; the symbol "ff" to the following page or pages.)

Chain and Homology Groups

Chain groups are designated by the letter "C" followed by suitable symbols; for example, "$C^r(K)$." Cycle groups and groups of bounding cycles are designated similarly by use of the letters "Z" and "B", respectively. Consequently in the table below only the symbols for the homology groups are given; to obtain the corresponding chain, cycle or bounding cycle groups (where they exist), replace "H" by "C," "Z," or "B." Thus the cycle group corresponding to "$H^r(K)$" is "$Z^r(K)$." Hence to look up a group, as "$Z^r(S; M, L; G)$," instead look up "$H^r(S; M, L; G)$" in the index below; the "Z" group desired will be found defined on the page cited.

Betti numbers are generally designated by "p." Thus, to look up the meaning of "$p^r(S: M, L; A, B)$," turn to the page designated for "$H^r(S: M, L; A, B)$." Similar remarks hold for the cohomology case. However, some Betti and co-Betti numbers are listed below, particularly where special definitions are required or a letter different from "p" (such as "g") is used.

Individual cycles and cocycles are variously denoted in the text by the letters "z," "Z," "γ," and "Γ" with appropriate indices. Open point sets are denoted below by "P," "Q," "U," "V;" closed point sets by "A," "B," "J," "M," "L;" a single point by "x." Also, both below and in the text, the letter "K" denotes a complex, "S" a space, "$\mathfrak{F}$" an algebraic field, and "G" an abelian group.

INDEX

AUTHORS CITED

ERRATA

In each instance, the first number refers to the page. "Line $-n$" refers to the nth line from the bottom of the page.

62 Line 23. Insert "of $| Z^{n-2} |$" after "$(n - r)$-cells"
76 Line -21. Change "Lemma 2.4" to "Lemmas 2.3 and 2.4"
128 Line 17. Insert "as in 6.3" after "K'"
158 Line 20. Insert "and C-cycle z^n" after "$H_q(S)$"
180 Line 15. Insert "$(\mathfrak{B}_n^*)$" after second "z^r" as well as before "P"
191 Lines 5 and 7. After "all" insert "arbitrarily small"
206 Line 1. Before "*then*" insert "*such that for some closed set K containing M*, $\gamma^r \sim 0 \bmod K$,"; and before "*such*" insert "*and contained in K*"
237 Line -10. Before "base" insert "interior (relative to the xy-plane) of the"
283 Line 23. Insert "small enough" before "neighborhood"
292 Line -13. The exponent of "g" should be "$n - r - 1$"
303 Line 8. Insert "*ulc*^{k}" before "*open*"
304 6.1 Lemma. Insert "*compact*" before "*space*"
327 3.1 Definition; 3.2 Definition. Insert "and $\bar{Q}$ is compact" before comma.
368 Line 1. Before the second period insert "and $M_0 = \{(0, 0, z) \mid 0 < z \leqq 1\}$"
389 Between "Moore, R. L." and "Poincaré, H." insert
"MULLIKIN, A.
[a] *Certain theorems relating to plane connected point sets*, Transactions of the American Mathematical Society, vol. 24 (1922), pp. 144–162"